CHAPTER 12

Reynolds Number & Turbulence

**Non-linearity:** The Navier-Stokes equation is non-linear. This non-linearity arises from the convective (material) derivative term

\[ \vec{u} \cdot \nabla \vec{u} = \frac{1}{2} \nabla u^2 - \vec{u} \times \vec{w} \]

which describes the "inertial acceleration" and is ultimately responsible for the origin of the chaotic character of many flows and of turbulence. Because of this non-linearity, we cannot say whether a solution to the Navier-Stokes equation with nice and smooth initial conditions will remain nice and smooth for all time (at least not in 3D).

**Laminar flow:** occurs when a fluid flows in parallel layers, without lateral mixing (no cross currents perpendicular to the direction of flow). It is characterized by high momentum diffusion and low momentum convection.

**Turbulent flow:** is characterized by chaotic and stochastic property changes. This includes low momentum diffusion, high momentum convection, and rapid variation of pressure and velocity in space and time.

**The Reynolds number:** In order to gauge the importance of viscosity for a fluid, it is useful to compare the ratio of the inertial acceleration (\( \vec{u} \cdot \nabla \vec{u} \)) to the viscous acceleration (\( \nu \left[ \nabla^2 \vec{u} + \frac{1}{3} \nabla (\nabla \cdot \vec{u}) \right] \)). This ratio is called the Reynolds’s number, \( R \), and can be expressed in terms of the typical velocity scale \( U \sim |\vec{u}| \) and length scale \( L \sim 1/\nabla \) of the flow, as

\[ R = \frac{\vec{u} \cdot \nabla \vec{u}}{\nu \left[ \nabla^2 \vec{u} + \frac{1}{3} \nabla (\nabla \cdot \vec{u}) \right]} \sim \frac{U^2/L}{\nu U/L^2} = \frac{UL}{\nu} \]

If \( R \gg 1 \) then viscosity can be ignored (and one can use the Euler equations to describe the flow). However, if \( R \ll 1 \) then viscosity is important.
Similarity: Flows with the same Reynold’s number are similar. This is evident from rewriting the Navier-Stokes equation in terms of the following dimensionless variables

\[
\tilde{u} = \frac{\bar{u}}{U}, \quad \tilde{x} = \frac{\bar{x}}{L}, \quad \tilde{t} = t \frac{U}{L}, \quad \tilde{p} = \frac{P}{\rho U^2}, \quad \tilde{\Phi} = \frac{\Phi}{U^2}, \quad \tilde{\nabla} = L \nabla
\]

This yields (after multiplying the Navier-Stokes equation with \(L/U^2\)):

\[
\frac{\partial \tilde{u}}{\partial \tilde{t}} + \tilde{u} \cdot \tilde{\nabla} \tilde{u} + \tilde{\nabla} \tilde{p} + \tilde{\nabla} \tilde{\Phi} = \frac{1}{\mathcal{R}} \tilde{\nabla}^2 \tilde{u}
\]

which shows that the form of the solution depends only on \(\mathcal{R}\).
As a specific example, consider fluid flow past a cylinder of diameter $L$:

- $R \ll 1$: "creeping flow". In this regime the flow is viscously dominated and (nearly) symmetric upstream and downstream. The inertial acceleration $(\vec{u} \cdot \nabla \vec{u})$ can be neglected, and the flow is (nearly) time-reversible.

- $R \sim 1$: Slight asymmetry develops

- $10 \leq R \leq 41$: Separation occurs, resulting in two counter-rotating vortices in the wake of the cylinder. The flow is still steady and laminar, though.

- $41 \leq R \leq 10^3$: "von Kármán vortex street"; unsteady laminar flow with counter-rotating vortices shed periodically from the cylinder. Even at this stage the flow is still ‘predictable’.

- $R > 10^3$: vortices are unstable, resulting in a turbulent wake behind the cylinder that is ‘unpredictable’. 
Figure 8: The image shows the von Kármán Vortex street behind a 6.35 mm diameter circular cylinder in water at Reynolds number of 168. The visualization was done using hydrogen bubble technique. Credit: Sanjay Kumar & George Laughlin, Department of Engineering, The University of Texas at Brownsville.

The following movie shows a $R = 250$ flow past a cylinder. Initially one can witness separation, and the creation of two counter-rotating vortices, which then suddenly become ‘unstable’, resulting in the von Kármán vortex street: http://www.youtube.com/watch?v=IDeGDFZSYo8
Locomotion at Low-Reynolds number: Low Reynolds number corresponds to high kinetic viscosity for a given \( U \) and \( L \). In this regime of ‘creeping flow’ the flow past an object is (nearly) time-reversible. Imagine trying to move (swim) in a highly viscous fluid (take honey as an example). If you try to do so by executing time-symmetric movements, you will not move. Instead, you need to think of a symmetry-breaking solution. Nature has found many solutions for this problem. If we make the simplifying “rule-of-thumb” assumption that an animal of size \( L \) meters moves roughly at a speed of \( U = L \) meters per second (yes, this is very, very rough, but an ant does move close to 1 mm/s, and a human at roughly 1 m/s), then we have that \( \mathcal{R} = UL/\nu \approx L^2/\nu \). Hence, with respect to a fixed substance (say water, for which \( \nu \approx 10^{-2} \text{cm}^2/\text{s} \)), smaller organisms move at lower Reynolds number (effectively in a fluid of higher viscosity). Scaling down from a human to bacteria and single-cell organisms, the motion of the latter in water has \( \mathcal{R} \approx 10^{-5} - 10^{-2} \). Understanding the locomotion of these organisms is a fascinating sub-branch of bio-physics.


**Boundary Layers:** Even when $\mathcal{R} \gg 1$, viscosity always remains important in thin boundary layers adjacent to any solid surface. This boundary layer must exist in order to satisfy the no-slip boundary condition. If the Reynolds number exceeds a critical value, the boundary layer becomes turbulent. Turbulent layers and their associated turbulent wakes exert a much bigger drag on moving bodies than their laminar counterparts.

**Turbulence:** Turbulence is still considered as one of the last "unsolved problems of classical physics" [Richard Feynman]. Indeed, it is an extremely difficult subject. Salmon (1998) nicely sums up the challenge of defining turbulence:

> Every aspect of turbulence is controversial. Even the definition of fluid turbulence is a subject of disagreement. However, nearly everyone would agree with some elements of the following description:

- Turbulence requires the presence of vorticity; irrotational flow is smooth and steady to the extent that the boundary conditions permit.
- Turbulent flow has a complex structure, involving a broad range of space and time scales.
- Turbulent flow fields exhibit a high degree of apparent randomness and disorder. However, close inspection often reveals the presence of embedded coherent flow structures.
- Turbulent flows have a high rate of viscous energy dissipation.
- Advected tracers are rapidly mixed by turbulent flows.

However, one further property of turbulence seems to be more fundamental that all of these because it largely explains why turbulence demands a statistical treatment...turbulence is chaotic.
The following is a brief, qualitative description of turbulence

Turbulence kicks in at sufficiently high Reynolds number (typically $R > 10^3 - 10^4$). Turbulent flow is characterized by irregular and seemingly random motion. Large vortices (called eddies) are created. These contain a large amount of kinetic energy. Due to vortex stretching these eddies are stretched thin until they 'brake up' in smaller eddies. This results in a cascade in which the turbulent energy is transported from large scales to small scales. This cascade is largely inviscid, conserving the total turbulent energy. However, once the length scale of the eddies becomes comparable to the mean free path of the particles, the energy is dissipated; the kinetic energy associated with the eddies is transformed into internal energy. The scale at which this happens is called the Kolmogorov length scale.

Molecular clouds: an example of turbulence in astrophysics are molecular clouds. These are gas clouds of masses $10^5 - 10^6 M_\odot$, densities $n_H \sim 100 - 500$ cm$^{-3}$, and temperatures $T \sim 10$K. They consist mainly of molecular hydrogen and are the main sites of star formation. Observations show that their velocity linewidths are $\sim 6 - 10$ km/s, which is much higher than their sound speed ($c_s \sim 0.2$ km/s). Hence, they are supported against (gravitational) collapse by supersonic turbulence. On small scales, however, the turbulent motions compress the gas to high enough densities that stars can form. A numerical simulation of a molecular cloud with supersonic turbulence is available here:

http://www.youtube.com/watch?v=3z9ZKAkbMhY