Problem 1: Streamlines

A streamline is defined as a curve that is instantaneously tangent to the velocity vector of a flow. Streamlines show the direction a massless fluid element will travel at any point in time (see Chapter 1 of Lecture Notes).

Consider a two-dimensional fluid with velocity field \( \vec{\mathbf{u}} = (ax, by) \), where \( a, b \) are constants and we adopt Cartesian coordinates. Assume that at time \( t = 0 \) the density everywhere is equal to \( \rho_0 \).

a) Draw streamlines for the following two cases: (i) \( a = b > 0 \), and (ii) \( a = -b > 0 \).

b) Under what condition(s) is the flow incompressible? Under what condition(s) is it curl-free?

c) Consider the location \( A = (2, 2) \). How does the density at \( A \) change as a function of time in cases (i) and (ii) above?

Now consider a two-dimensional velocity field \( \vec{\mathbf{u}} = (-ay, bx) \). Again, assume that at time \( t = 0 \) the density everywhere is equal to \( \rho_0 \).

d) Draw streamlines for the following two cases: (i) \( a = b > 0 \), and (ii) \( a = -b > 0 \).

e) Under what condition(s) is the flow incompressible? Under what condition(s) is it curl-free?

f) How does the density at \( A \) change as function of time at \( A \) in cases (i) and (ii)?
Problem 2: Navier-Stokes; from index form to vector form
The Navier-Stokes equation in index form, as derived in class, is given by
\[
\frac{\rho}{dt} \frac{du_i}{dt} = -\frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j} \left[ \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \delta_{ij} \frac{\partial u_k}{\partial x_k} \right) \right] + \frac{\partial}{\partial x_i} \left( \eta \frac{\partial u_k}{\partial x_k} \right) - \rho \frac{\partial \Phi}{\partial x_i}
\]
Show clearly, step-by-step and using text were needed, that this can be written in vector form as
\[
\rho \frac{d\vec{u}}{dt} = -\nabla P + \mu \nabla^2 \vec{u} + \left( \eta + \frac{1}{3} \mu \right) \nabla (\nabla \cdot \vec{u}) - \rho \nabla \Phi
\]

Problem 3: The Stress Tensor
Consider a fluid in a 2-dimensional, Cartesian coordinate system \((x_1, x_2)\), with stress tensor
\[
\sigma_{ij} = \begin{pmatrix}
\sigma_{11} & \sigma_{12} \\
\sigma_{21} & \sigma_{22}
\end{pmatrix}
\]
and let \(\hat{n}\) and \(\hat{t}\) be the unit normal and unit tangent vectors of a surface \(S\), for which \(\hat{n}\) is rotated by angle \(\theta\) with respect to the \(x_1\) axis.

a) Express \(\hat{n}\) and \(\hat{t}\) in terms of \(\theta\), i.e., what are \(n_1, n_2, t_1\) and \(t_2\) in \(\hat{n} = (n_1, n_2)\) and \(\hat{t} = (t_1, t_2)\)?

b) Show that the normal stress, \(\Sigma_n\), can be written as
\[
\Sigma_n = \sigma_{11} \cos^2 \theta + \sigma_{22} \sin^2 \theta + \sigma_{12} \sin 2\theta
\]
and derive a similar expression for the shear stress, \(\Sigma_s\).

c) Consider the case \(\sigma_{11} = \sigma_{12} = \sigma_{21} = 0\). Under what angle \(\theta\) is the shear stress on \(S\) maximal? For what angle \(\theta\) does the shear on \(S\) vanish? What are the normal stresses in both cases?

d) Answer the same questions as under c), but now for the case with \(\sigma_{11} = \sigma_{22} = 0\).

e) Under what condition are both the normal and shear stress independent of \(\theta\), and what are their values in that case?