The Virial Equations I

We can obtain an important tensor equation relating global properties of the system, by multiplying the CBE by both v_j and x_k and then integrating over the entire phase-space.

The first step of this has already been performed in our derivation of the Jeans equations, and yielded the momentum equations

$$\frac{\partial(\rho\langle v_j\rangle)}{\partial t} + \frac{\partial(\rho\langle v_i v_j\rangle)}{\partial x_i} + \rho\frac{\partial\Phi}{\partial x_j} = 0$$

Multiplying all terms with x_k and integrating over real space yields

$$\frac{\partial}{\partial t}\int \rho x_k \langle v_j\rangle \mathrm{d}^3x = -\int x_k \frac{\partial(\rho\langle v_i v_j\rangle)}{\partial x_i}\mathrm{d}^3\vec{x} - \int \rho\, x_k\, \frac{\partial\Phi}{\partial x_j}\mathrm{d}^3\vec{x}$$

Using integration by parts the first term on the r.h.s. becomes

$$\begin{aligned}
\int x_k \frac{\partial(\rho\langle v_i v_j\rangle)}{\partial x_i}\mathrm{d}^3\vec{x} &= \int \frac{\partial(\rho x_k\langle v_i v_j\rangle)}{\partial x_i}\mathrm{d}^3\vec{x} - \int \rho\langle v_i v_j\rangle \frac{\partial x_k}{\partial x_i}\mathrm{d}^3\vec{x} \\
&= -\int \delta_{ki}\rho\langle v_i v_j\rangle \mathrm{d}^3\vec{x} \\
&= -\int \rho\langle v_k v_j\rangle \mathrm{d}^3\vec{x} \\
&= -2\mathcal{K}_{kj}
\end{aligned}$$

where we have defined the kinetic energy tensor

$$\mathcal{K}_{ij} = \frac{1}{2}\int \rho\langle v_i v_j\rangle \mathrm{d}^3\vec{x}$$

The Virial Equations II

It is customary to split the kinetic energy tensor into contributions from ordered and random motions:

$$\mathcal{K}_{ij} \equiv \mathcal{T}_{ij} + \tfrac{1}{2}\Pi_{ij}$$

where

$$\mathcal{T}_{ij} \equiv \tfrac{1}{2}\int \rho\,\langle v_i\rangle\,\langle v_j\rangle\,\mathrm{d}^3\vec{x} \qquad\qquad \Pi_{ij} \equiv \int \rho\,\sigma^2_{ij}\,\mathrm{d}^3\vec{x}$$

In addition to the $\mathcal{K}$ we also define the potential energy tensor

$$\mathcal{W}_{ij} \equiv -\int \rho x_i \frac{\partial\Phi}{\partial x_j}\mathrm{d}^3\vec{x}$$

Combining the above we obtain

$$\frac{\partial}{\partial t}\int \rho\, x_k \langle v_j\rangle \mathrm{d}^3x = 2\mathcal{K}_{kj} + \mathcal{W}_{kj}$$

which allows us to write

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \rho\,[x_k\,\langle v_j\rangle + x_j\,\langle v_k\rangle] = 2\mathcal{K}_{jk} + \mathcal{W}_{jk}$$

where we have used that $\mathcal{K}$ and $\mathcal{W}$ are symmetric.

The Virial Equations III

Finally we also define the moment of inertia tensor

$$\mathcal{I}_{ij} \equiv \int \rho \, x_i \, x_j \, \mathrm{d}^3\vec{x}$$

Differentiating with respect to time, and using the continuity equation (i.e., the zeroth moment equation of the CBE) yields

$$\begin{aligned}
\frac{\mathrm{d}I_{jk}}{\mathrm{d}t} &= \int \frac{\partial \rho}{\partial t} x_j \, x_k \, \mathrm{d}^3\vec{x} \\
&= -\int \frac{\partial \rho \langle v_i \rangle}{\partial x_i} x_j \, x_k \, \mathrm{d}^3\vec{x} \\
&= -\int \frac{\partial(\rho \langle v_i \rangle x_j x_k)}{\partial x_i} \mathrm{d}^3\vec{x} + \int \rho \langle v_i \rangle \frac{\partial(x_j x_k)}{\partial x_i} \mathrm{d}^3\vec{x} \\
&= \int \rho \langle v_i \rangle \left[x_j \delta_{ik} + x_k \delta_{ij} \right] \mathrm{d}^3\vec{x} \\
&= \int \rho \left[x_j \langle v_k \rangle + x_k \langle v_j \rangle \right] \mathrm{d}^3\vec{x}
\end{aligned}$$

so that

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int \rho \left[x_k \, \langle v_j \rangle + x_j \, \langle v_k \rangle \right] = \frac{1}{2} \frac{\mathrm{d}^2 \mathcal{I}_{jk}}{\mathrm{d}t^2}$$

which allows us to write the Tensor Virial Theorem as

$$\frac{1}{2} \frac{\mathrm{d}^2 \mathcal{I}_{jk}}{\mathrm{d}t^2} = 2\mathcal{T}_{jk} + \Pi_{jk} + \mathcal{W}_{jk}$$

which relates the gross kinematic and structural properties of gravitational systems.

The Virial Equations IV

If the system is in a steady-state the moment of inertia tensor is stationary, and the Tensor Virial Theorem reduces to $2\mathcal{K}_{ij} + \mathcal{W}_{ij} = 0$.

Of particular interest is the trace of the Tensor Virial Theorem, which relates the total kinetic energy $K = \frac{1}{2}M\langle v^2\rangle$ to the total potential energy $W = \frac{1}{2}\int \rho(\vec{x})\,\Phi(\vec{x})\,\mathrm{d}^3\vec{x}$.

$$\begin{aligned}
\mathrm{tr}(\mathcal{K}) \equiv \sum_{i=1}^{3}\mathcal{K}_{ii} &= \tfrac{1}{2}\int \rho(\vec{x})\left[\langle v_1^2\rangle(\vec{x}) + \langle v_2^2\rangle(\vec{x}) + \langle v_3^2\rangle(\vec{x})\right]\mathrm{d}^3\vec{x} \\
&= \tfrac{1}{2}\int \rho(\vec{x})\,\langle v^2\rangle(\vec{x})\mathrm{d}^3\vec{x} \\
&= \tfrac{1}{2}M\langle v^2\rangle = K
\end{aligned}$$

where we have used that

$$\langle v^2\rangle = \frac{1}{M}\int \rho(\vec{x})\langle v^2\rangle(\vec{x})\mathrm{d}^3\vec{x}$$

Similarly, the trace of the potential energy tensor is equal to the total potential energy (see next page for derivation):

$$\mathrm{tr}(\mathcal{W}) = W = \tfrac{1}{2}\int \rho(\vec{x})\,\Phi(\vec{x})\,\mathrm{d}^3\vec{x}$$

We thus obtain the scalar virial theorem

$$\boxed{2K + W = 0}$$

The Potential Energy Tensor I

We have defined the **potential energy tensor** as

$$\mathcal{W}_{ij} \equiv -\int \rho x_i \frac{\partial \Phi}{\partial x_j} \mathrm{d}^3\vec{x}$$

Using that $\Phi(\vec{x}) = -G \int \frac{\rho(\vec{x})}{|\vec{x}' - \vec{x}|} \mathrm{d}^3\vec{x}$ we obtain

$$\mathcal{W}_{ij} = G \int \int \rho(\vec{x})\, \rho(\vec{x}')\, \frac{x_i (x'_j - x_j)}{|\vec{x}' - \vec{x}|^3} \mathrm{d}^3\vec{x}' \mathrm{d}^3\vec{x}$$

Using that $\vec{x}$ and $\vec{x}'$ are dummy variables, we may relabel them, and write

$$\mathcal{W}_{ij} = G \int \int \rho(\vec{x}')\, \rho(\vec{x})\, \frac{x'_j (x_k - x'_k)}{|\vec{x} - \vec{x}'|^3} \mathrm{d}^3\vec{x} \mathrm{d}^3\vec{x}'$$

Interchanging the order of integration and summing the above two equations yields the **manifestly symmetric expression**

$$\mathcal{W}_{ij} = -\frac{G}{2} \int \int \rho(\vec{x})\, \rho(\vec{x}')\, \frac{(x'_j - x_j)(x'_k - x_k)}{|\vec{x}' - \vec{x}|^3} \mathrm{d}^3\vec{x}' \mathrm{d}^3\vec{x}$$

This expression allows us to write

$$\begin{aligned} \mathrm{tr}(\mathcal{W}) &\equiv \sum_{i=1}^{3} \mathcal{W}_{ii} = -\frac{G}{2} \int \int \rho(\vec{x}) \rho(\vec{x}') \frac{|\vec{x}' - \vec{x}|^2}{|\vec{x}' - \vec{x}|^3} \mathrm{d}^3\vec{x}' \mathrm{d}^3\vec{x} \\ &= -\frac{G}{2} \int \rho(\vec{x}) \int \frac{\rho(\vec{x}')}{|\vec{x}' - \vec{x}|} \mathrm{d}^3\vec{x}' \mathrm{d}^3\vec{x} = \frac{1}{2} \int \rho(\vec{x}) \Phi(\vec{x}) \mathrm{d}^3\vec{x} = W \end{aligned}$$

The Surface Pressure Term

In our derivation on the previous pages we obtained

$$\begin{aligned}\int x_k \frac{\partial(\rho\langle v_i v_j\rangle)}{\partial x_i}\mathrm{d}^3\vec{x} &= \int \frac{\partial(\rho x_k\langle v_i v_j\rangle)}{\partial x_i}\mathrm{d}^3\vec{x} - \int \rho\langle v_i v_j\rangle \frac{\partial x_k}{\partial x_i}\mathrm{d}^3\vec{x} \\ &= -\int \rho\langle v_k v_j\rangle \mathrm{d}^3\vec{x} = -2\mathcal{K}_{kj}\end{aligned}$$

where we have used that

$$\int \frac{\partial(\rho x_k\langle v_i v_j\rangle)}{\partial x_i}\mathrm{d}^3\vec{x} = \int \rho x_k \langle v_k v_j\rangle \mathrm{d}^2 S = 0$$

based on the assumption that $\rho(r) = 0$ when $r \rightarrow \infty$. However, this is only true for an isolated system with 'vacuum' boundary conditions.

In reality, a halo or galaxy is embedded in a cosmological density field, often with ongoing infall. This yields a non-zero surface pressure. In its most general form the scalar virial theorem therefore reads

$$\boxed{2K + W + S_p = 0}$$

with the surface pressure term

$$S_p = -\int \langle v^2\rangle \vec{r}\cdot\vec{n}\,\mathrm{d}^2\vec{S}$$

As long as $S_p \neq 0$ we thus expect that $2K/|W| \neq 1$.
See Shapiro et al. (astro-ph/0409173) for a detailed discussion.

The Virial Equations V

From a simple dimensional analysis one finds that $|W| \propto GM^2/R$ with M the system's mass and R a characteristic radius.

A useful characteristic radius is the so-called gravitational radius defined by

$$r_g \equiv \frac{GM^2}{|W|}$$

One can relate the gravitational radius to the half-mass radius r_h, defined as radius enclosing half the total mass. As shown by Spitzer (1969), typical stellar systems have $r_g \simeq 2.5 r_h$.

Combining this with the scalar virial theorem we can write that

$$M \simeq 2.5 \frac{r_h \langle v^2 \rangle}{G}$$

which is a useful equation to obtain a (rough) estimate of the virial mass from a measure of the half-mass radius and the rms motion

The Virial Equations VI

Using the scalar virial theorem we obtain

$$E = K + W = -K = \frac{1}{2}W$$

Consider the formation of a virialized object. If the system forms by collecting material from large radii, the initial conditions are well approximated by $K_{\rm init} = W_{\rm init} = E_{\rm init} = 0$.

Because of gravity the matter starts to collapse. Since $W = -GM^2/r_g$ this makes W more negative. At the same time K increases. Initially, during the early collapse, $E = T + W = 0$.

After the first shell crossing, the system starts to virialize. When virialization is complete, $2T + W = 0$ and $E = W/2$.

Therefore, half the gravitational energy released by collapse is invested in kinetic form. The system somehow disposes of the other half in order to achieve a binding energy $E_b = -E$.

QUESTION Where does the other half of the energy go?

Application: M/L of Spherical Systems

As an application of the Virial Theorem, consider spherical, non-rotating systems (spherical galaxies or globulars)

If the mass-to-light ratio Υ does not depend on radius then

$$K_{xx} = \int \frac{1}{2}\rho(\vec{x})\langle v_x^2\rangle \mathrm{d}^3\vec{x} = \frac{\Upsilon}{2}\int \nu(\vec{x})\langle v_x^2\rangle \mathrm{d}^3\vec{x}$$

where $\nu(\vec{x}) = \rho(\vec{x})/\Upsilon$ is the 3D luminosity distribution, and K_{xx} is the kinetic energy associated with motion in the x-direction

Since a spherical, non-rotating system is isotropic we have that

$$K = K_{xx} + K_{yy} + K_{zz} = 3K_{xx}$$

If one has observationally determined the surface brightness profile $\Sigma(R)$ and the line-of-sight velocity dispersion $\sigma_p^2(R)$ then it is easy to see that

$$K = 3\frac{\Upsilon}{2}\int_0^{2\pi}\mathrm{d}\phi\int_0^{\infty}\mathrm{d}R\,R\,\Sigma(R)\sigma_p^2(R) = 3\pi\Upsilon\int_0^{\infty}\mathrm{d}R\,R\,\Sigma(R)\sigma_p^2(R) \equiv \Upsilon J$$

where we defined the observationally accessible $J = J(\Sigma, \sigma_p^2)$

Application: M/L of Spherical Systems

As seen in exersizes, for spherical system: $W = -\frac{G}{2}\int_0^\infty \frac{M^2(r)}{r^2}\mathrm{d}r$

Using that $M(r) = 4\pi \int_0^r \rho(r')r'^2 \mathrm{d}r'$

where the density profile is related to $\Sigma(R)$ according to

$$\rho(r) = -\frac{\Upsilon}{\pi}\int_r^\infty \frac{\mathrm{d}\Sigma}{\mathrm{d}R}\frac{\mathrm{d}R}{\sqrt{R^2-r^2}}$$

we obtain that

$$W = -8\Upsilon^2 \int_0^\infty \frac{\mathrm{d}r}{r^2}\left[\int_0^r \mathrm{d}r'\, r'^2 \int_{r'}^\infty \frac{\mathrm{d}\Sigma}{\mathrm{d}R}\frac{\mathrm{d}R}{\sqrt{R^2-r^2}}\right]^2 \equiv \Upsilon^2 \tilde{J}$$

where we have defined the observationally accessible integral $\tilde{J} = \tilde{J}(\Sigma)$

According to the virial theorem $2K + W = 0$, and thus $-2K/W = 1$.

Substituting $K = \Upsilon J$ and $W = \Upsilon^2 \tilde{J}$ we thus obtain that

$$\boxed{\Upsilon = -\frac{2J}{\tilde{J}}}$$

Flattening of Oblate Spheroids I

As another application of the virial theorem we relate the flattening of an oblate spheroid to its kinematics.

Consider an oblate system with it's symmetry axis along the z-direction. Because of symmetry considerations we have that

$$\langle v_R \rangle = \langle v_z \rangle = 0 \qquad \langle v_R v_\phi \rangle = \langle v_z v_\phi \rangle = 0$$

If we write that

$$\langle v_x \rangle = \langle v_\phi \rangle \sin\phi \qquad \langle v_y \rangle = \langle v_\phi \rangle \cos\phi$$

we obtain

$$\begin{aligned} \mathcal{T}_{xy} &= \tfrac{1}{2} \int \rho \langle v_x \rangle \langle v_y \rangle \mathrm{d}^3\vec{x} \\ &= \tfrac{1}{2} \int_0^{2\pi} \mathrm{d}\phi \sin\phi \, \cos\phi \int_0^\infty \mathrm{d}R \int_{-\infty}^{\infty} \mathrm{d}z \rho(R,z) \langle v_\phi \rangle^2 (R,z) \\ &= 0 \end{aligned}$$

A similar analysis shows that all other non-diagonal elements of $\mathcal{T}$, Π, and $\mathcal{W}$ have to be zero.

In addition, because of symmetry considerations we must have that $\mathcal{T}_{xx} = \mathcal{T}_{yy}$, $\Pi_{xx} = \Pi_{yy}$, and $\mathcal{W}_{xx} = \mathcal{W}_{yy}$.

Flattening of Oblate Spheroids II

Given these symmetries, the only independent, non-trivial virial equations are

$$2\mathcal{T}_{xx} + \Pi_{xx} + \mathcal{W}_{xx} = 0, \qquad 2\mathcal{T}_{zz} + \Pi_{zz} + \mathcal{W}_{zz} = 0$$

Taking the ratio we find that

$$\frac{2\mathcal{T}_{xx}+\Pi_{xx}}{2\mathcal{T}_{zz}+\Pi_{zz}} = \frac{\mathcal{W}_{xx}}{\mathcal{W}_{zz}}$$

The usefulness of this equation lies in the fact that, for density distributions that are constant on similar concentric spheroids, i.e., $\rho = \rho(m^2)$, the ratio $\mathcal{W}_{xx}/\mathcal{W}_{zz}$ depends only on the axis ratio c/a of the spheroids, and is independent of the density profile! For an oblate body, to good approximation

$$\frac{\mathcal{W}_{xx}}{\mathcal{W}_{zz}} \simeq \left(\frac{c}{a}\right)^{-0.9}$$

Let us start by considering isotropic, oblate rotators.

Then $\Pi_{xx} = \Pi_{zz} = M\tilde{\sigma}^2$, $\mathcal{T}_{zz} = 0$ and $\mathcal{T}_{xx} + \mathcal{T}_{yy} = 2\mathcal{T}_{xx} = \frac{1}{2}M\tilde{v}^2$.

Here M is the total mass, $\tilde{\sigma}^2$ is the mass-weighted rms-average of the intrinsic one-dimensional velocity dispersion, and $\tilde{v}^2$ is the mass-weighted rms rotation velocity.

Flattening of Oblate Spheroids III

Thus, for an isotropic, oblate rotators we have that

$$\frac{\frac{1}{2} M \tilde{v}^2 + M \tilde{\sigma}^2}{M \tilde{\sigma}^2} \simeq \left(\frac{c}{a}\right)^{-0.9}$$

which reduces to

$$\boxed{\frac{\tilde{v}}{\tilde{\sigma}} \simeq \sqrt{2[(c/a)^{-0.9} - 1]}}$$

This specifies the relation between the flattening of the spheroid and the ratio of streaming motion to random motion. Note that you need a rather large amount of rotation to achieve only modest flattening: $c/a = 0.7$ requires $\tilde{v} \sim 0.9\tilde{\sigma}$

Next consider a non-rotating, anisotropic, oblate system:

In this case $\Pi_{xx} = M\tilde{\sigma}^2_{xx}$ and $\Pi_{zz} = M\tilde{\sigma}^2_{zz}$, and the virial theorem gives that

$$\boxed{\frac{\tilde{\sigma}_{zz}}{\tilde{\sigma}_{xx}} \simeq \left(\frac{c}{a}\right)^{0.45}}$$

Now a flattening of $c/a = 0.7$ requires only a small anisotropy of $\tilde{\sigma}_{zz}/\tilde{\sigma}_{xx} \simeq 0.85$

Flattening of Oblate Spheroids IV

Finally, consider the general case of rotating, anisotropic, oblate systems

Now we have $\Pi_{zz} = (1-\delta)\Pi_{xx} = (1-\delta)M\tilde{\sigma}^2$, $\mathcal{T}_{zz} = 0$ and $2\mathcal{T}_{xx} = \frac{1}{2}M\tilde{v}^2$, where we have introduced the anistropy parameter $\delta < 1$.

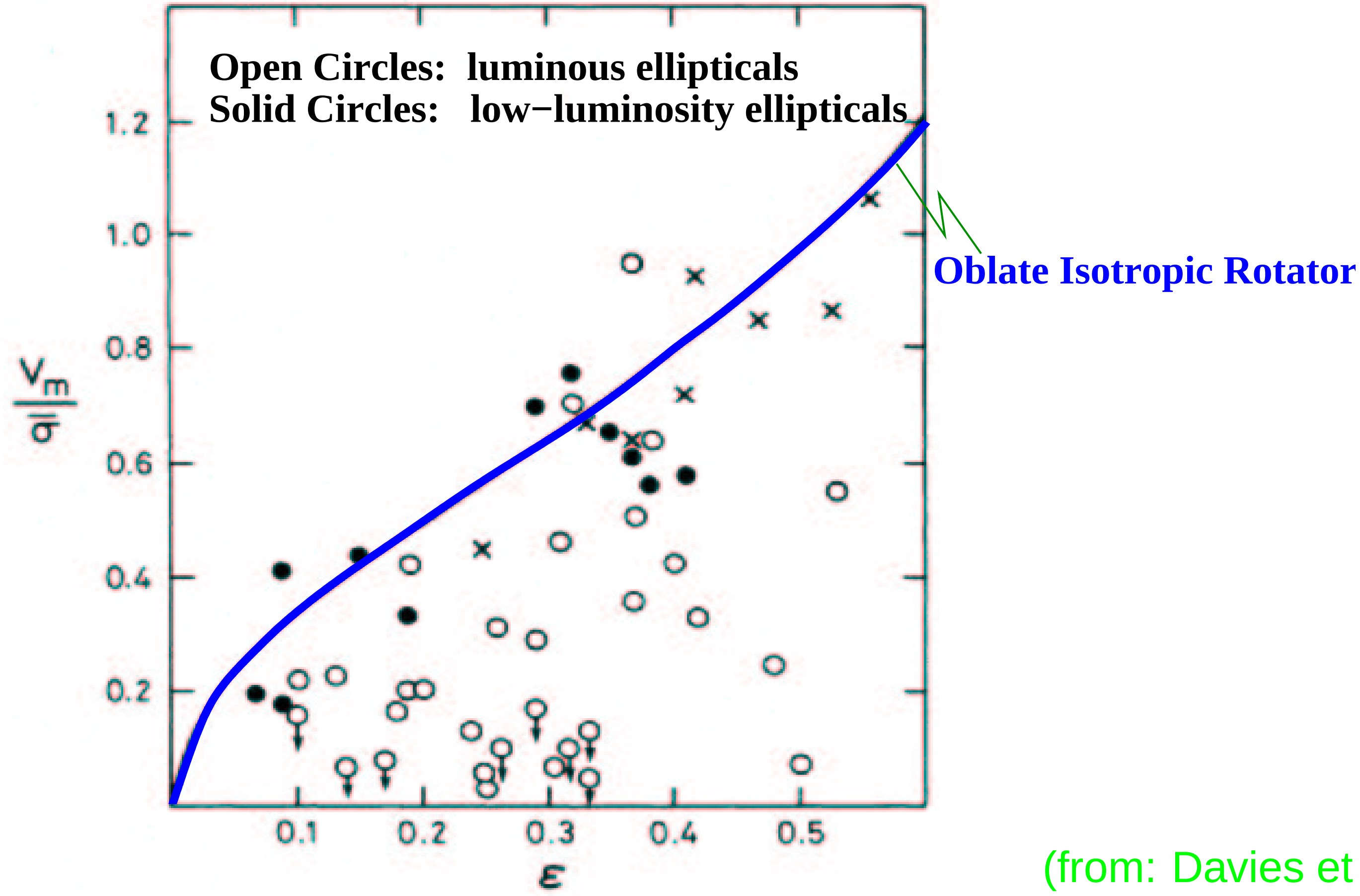
In this case the virial theorem gives

$$\frac{\tilde{v}}{\tilde{\sigma}} \simeq \sqrt{2[(1-\delta)\,(c/a)^{-0.9} - 1]}$$

This shows that observations of $\tilde{v}/\tilde{\sigma}$ and the ellipticity $\varepsilon = 1-(c/a)$ allow us to test whether elliptical galaxies are supported by rotation or by anisotropic presssure.

A potential problem is that we can not directly measure $\tilde{v}$ nor $\tilde{\sigma}$. Rather, we measure properties that are projected along the line-of-sight. Furthermore, in general we don't see a system edge-on but under some unknown inclination angle i. Note that i also affects the measured v and σ. As shown in Binney & Tremaine, the overall effect is to move a point on the oblate rotator line mainly along that line.

Flattening of Oblate Spheroids V



(from: Davies et al. 1983)

Observations reveal a dichotomy: luminous ellipticals are supported by anisotropic pressure, while fainter ellipticals (and bulges) are consistent with being oblate, isotropic rotators.

NOTE: If luminous ellipticals are anisotropic, there is no good reason why they should be axisymmetric: massive ellipticals are triaxial

The Jeans Theorem I

RECALL: An integral of motion is a function $I(\vec{x}, \vec{v})$ of the phase-space coordinates that is constant along all orbits, i.e.,

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{\partial I}{\partial x_i}\frac{\mathrm{d}x_i}{\mathrm{d}t} + \frac{\partial I}{\partial v_i}\frac{\mathrm{d}v_i}{\mathrm{d}t} = \vec{v} \cdot \vec{\nabla} I - \vec{\nabla}\Phi \cdot \frac{\partial I}{\partial \vec{v}} = 0$$

Compare this to the CBE for a steady-state (static) system:

$$\vec{v} \cdot \vec{\nabla} f - \vec{\nabla}\Phi \cdot \frac{\partial f}{\partial \vec{v}} = 0$$

Thus the condition for I to be an integral of motion is identical with the condition for I to be a steady-state solution of the CBE. Hence:

Jeans Theorem Any steady-state solution of the CBE depends on the phase-space coordinates only through integrals of motion. Any function of these integrals is a steady-state solution of the CBE.

PROOF: Let f be any function of the n integrals of motion $I_1, I_2, \ldots I_n$ then

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \sum_{k=1}^{n} \frac{\partial f}{\partial I_k}\frac{\mathrm{d}I_k}{\mathrm{d}t} = 0$$

which proofs that f satisfies the CBE.

The Jeans Theorem II

More useful than the Jeans Theorem is the Strong Jeans Theorem, which is due to Lynden-Bell (1962).

Strong Jeans Theorem The DF of a steady-state system in which almost all orbits are regular can be written as a function of the independent isolating integrals of motion, or of the action-integrals.

Note that a regular orbit in a system with n degrees of freedom is uniquely, and completely, specified by the values of the n isolating integrals of motion in involution. Thus the DF can be thought of as a function that expresses the probability for finding a star on each of the phase-space tori.

We first consider an application of the Jeans Theorem to Spherical Systems As we have seen, any orbit in a spherical potential admits four isolating integrals of motion: E, L_x, L_y, L_z.

Therefore, according to the Strong Jeans Theorem, the DF of any† steady-state spherical system can be expressed as $f = f(E, \vec{L})$.

---

† except for point masses and uniform spheres, which have five isolating integrals of motion

Jeans Theorem & Spherical Systems

If the system is spherically symmetric in all its properties, then $f = f(E, L^2)$ rather than $f = f(E, \vec{L})$: ie., the DF can only depend on the magnitude of the angular momentum vector, not on its direction.

Contrary to what one might naively expect, this is not true in general. In fact, as beautifully illustrated by Lynden-Bell (1960), a spherical system can rotate without being oblate.

Consider a spherical system with $f(E, \vec{L}) = f(E, -\vec{L})$. In such a system, for each star S on a orbit $\mathcal{O}$, there is exactly one star on the same orbit $\mathcal{O}$ but counterrotating with respect to S. Consequently, this system is perfectly spherically symmetric in all its properties.

Now consider all stars in the $z = 0$-plane, and revert the sense of all those stars with $L_z < 0$. Clearly this does not influence $\rho(r)$, but it does give the system a net sense of rotation around the z-axis.

Thus, although a system with $f = f(E, L^2)$ is not the most general case, systems with $f = f(E, \vec{L})$ are rarely considered in galactic dynamics.

Isotropic Spherical Models I

An even simpler case to consider is the one in which $f = f(E)$.

Since $E = \Phi(\vec{r}) + \frac{1}{2}[v_r^2 + v_\theta^2 + v_\phi^2]$ we have that

$$\langle v_r^2 \rangle = \frac{1}{\rho} \int \mathrm{d}v_r \mathrm{d}v_\theta \mathrm{d}v_\phi \, v_r^2 \, f\left(\Phi + \frac{1}{2}[v_r^2 + v_\theta^2 + v_\phi^2]\right)$$

$$\langle v_\theta^2 \rangle = \frac{1}{\rho} \int \mathrm{d}v_r \mathrm{d}v_\theta \mathrm{d}v_\phi \, v_\theta^2 \, f\left(\Phi + \frac{1}{2}[v_r^2 + v_\theta^2 + v_\phi^2]\right)$$

$$\langle v_\phi^2 \rangle = \frac{1}{\rho} \int \mathrm{d}v_r \mathrm{d}v_\theta \mathrm{d}v_\phi \, v_\phi^2 \, f\left(\Phi + \frac{1}{2}[v_r^2 + v_\theta^2 + v_\phi^2]\right)$$

Since these equations differ only in the labelling of one of the variables of integration, it is immediately evident that $\langle v_r^2 \rangle = \langle v_\theta^2 \rangle = \langle v_\phi^2 \rangle$.

Assuming that $f = f(E)$ is identical to assuming that the system is isotropic

Note that from

$$\langle v_i \rangle = \frac{1}{\rho} \int \mathrm{d}v_r \mathrm{d}v_\theta \mathrm{d}v_\phi \, v_i \, f\left(\Phi + \frac{1}{2}[v_r^2 + v_\theta^2 + v_\phi^2]\right)$$

it is also immediately evident that $\langle v_r \rangle = \langle v_\theta \rangle = \langle v_\phi \rangle = 0$. Thus, similar as for a system with $f = f(E, L^2)$ a system with $f = f(E)$ has no net sense of rotation.

Isotropic Spherical Models II

In what follows we define the relative potential $\Psi \equiv -\Phi + \Phi_0$ and relative energy $\mathcal{E} = -E + \Phi_0 = \Psi - \frac{1}{2}v^2$. In general one chooses Φ_0 such that $f > 0$ for $\mathcal{E} > 0$ and $f = 0$ for $\mathcal{E} \leq 0$

Now consider a self-consistent, spherically symmetric system with $f = f(\mathcal{E})$. Here self-consistent means that the potential is due to the system itself, i.e.,

$$\nabla^2\Psi = -4\pi G\rho = -4\pi G \int f(\mathcal{E}) \mathrm{d}^3\vec{v}$$

(note the minus sign in the Poisson equation), which can be written as

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\Psi}{\mathrm{d}r}\right) = -16\pi^2 G \int_0^{\Psi} f(\mathcal{E})\,\sqrt{2(\Psi-\mathcal{E})}\,\mathrm{d}\mathcal{E}$$

Note: Here we have chosen Φ_0 so that $\Psi(r\to\infty) = 0$. In systems with infinite total mass, such as the logarithmic potential or the isothermal sphere, the system is more conveniently normalized such that $\Psi(r\to\infty) = -\infty$. In that case $\int_0^{\Psi} \mathrm{d}\mathcal{E}$ needs to be replace by $\int_{-\infty}^{\Psi} \mathrm{d}\mathcal{E}$.

Isotropic Spherical Models III

This relation may be regarded either as non-linear equation for $\Psi(r)$ given $f(\mathcal{E})$, or as linear equation for $f(\mathcal{E})$ given $\Psi(r)$.

"from ρ to f"

As an example, consider a stellar-dynamical system with a DF

$$f(\mathcal{E}) = \frac{\rho_1}{(2\pi\sigma_0^2)^{3/2}} \exp\left(\frac{\Psi - \frac{1}{2}v^2}{\sigma_0^2}\right)$$

The corresponding density is

$$\rho(\Psi) = 4\pi \int_0^{\sqrt{2\Psi}} f(\mathcal{E}) v^2 \mathrm{d}v = 4\pi \int_0^{\Psi} f(\mathcal{E}) \sqrt{2(\Psi - \mathcal{E})} \mathrm{d}\mathcal{E} = \rho_1 \mathrm{e}^{\Psi/\sigma_0^2}$$

The Poisson equation reads

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 \frac{\mathrm{d}\Psi}{\mathrm{d}r}\right) = -4\pi G \rho_1 \mathrm{e}^{\Psi/\sigma_0^2} \quad \Rightarrow \quad \frac{\mathrm{d}\Psi}{\mathrm{d}r} = -\frac{4\pi G \rho_1}{r^2} \int_0^r r^2 \mathrm{e}^{\Psi/\sigma_0^2} \mathrm{d}r$$

Inspection shows that the solution for $\Psi(r)$ and the corresponding $\rho(r)$ are

$$\Psi(r) = -2\sigma_0^2 \ln r \qquad \rho(r) = \frac{\sigma_0^2}{2\pi G r^2}$$

which is the potential-density pair of a singular isothermal sphere.

Isotropic Spherical Models IV

Note that the DF of the singular isothermal sphere implies that

$$f(v) \propto \mathrm{e}^{-\frac{v^2}{2\sigma_0^2}}$$

which is identical to a Maxwell-Boltzmann distribution, if we set $\sigma_0^2 = \frac{k_B T}{m}$. Therefore, the structure of a singular isothermal sphere is identical to that of an isothermal self-gravitating sphere of gas.

The isothermal nature of this system becomes evident if we consider the Jeans equation. For a system with $f = f(\mathcal{E})$ there is only one non-trivial Jeans equation:

$$\frac{1}{\rho}\frac{\mathrm{d}\rho\sigma^2}{\mathrm{d}r} = \frac{\mathrm{d}\Psi}{\mathrm{d}r}$$

where $\sigma^2 \equiv \langle v_r^2 \rangle = \langle v_\theta^2 \rangle = \langle v_\phi^2 \rangle$.

Substituting the expressions for ρ and Ψ this yields

$$\boxed{\sigma^2(r) = \sigma_0^2}$$

thus the local velocity dispersion, which is related to the "temperature", is independent of r.

Eddington's Formula

"from f to ρ"

Using that Ψ is a monotonic function of r, so that ρ can be regarded as a function of Ψ, we have

$$\rho(\Psi) = \int f \mathrm{d}^3\vec{v} = 4\pi \int_0^{\Psi} f(\mathcal{E})\sqrt{2(\Psi - \mathcal{E})}\mathrm{d}\mathcal{E}$$

differentiating both sides with respect to Ψ yields

$$\frac{1}{\sqrt{8}\pi}\frac{\mathrm{d}\rho}{\mathrm{d}\Psi} = \int_0^{\Psi} \frac{f(\mathcal{E})\,\mathrm{d}\mathcal{E}}{\sqrt{\Psi - \mathcal{E}}}$$

which is an Abel integral equation, whose solution is

$$f(\mathcal{E}) = \frac{1}{\sqrt{8}\pi^2}\frac{\mathrm{d}}{\mathrm{d}\mathcal{E}}\int_0^{\mathcal{E}} \frac{\mathrm{d}\rho}{\mathrm{d}\Psi}\frac{\mathrm{d}\Psi}{\sqrt{\mathcal{E} - \Psi}}$$

This is called Eddington's formula, which may also be written in the form

$$f(\mathcal{E}) = \frac{1}{\sqrt{8}\pi^2}\left[\int_0^{\mathcal{E}} \frac{\mathrm{d}^2\rho}{\mathrm{d}\Psi^2}\frac{\mathrm{d}\Psi}{\sqrt{\mathcal{E} - \Psi}} + \frac{1}{\sqrt{\mathcal{E}}}\left(\frac{\mathrm{d}\rho}{\mathrm{d}\Psi}\right)_{\Psi=0}\right]$$

Eddington's Formula

Given a spherically symmetric density distribution, which can be written as $\rho = \rho(\Psi)$ (which is not always possible), Eddington's formula yields a corresponding DF $f = f(\mathcal{E})$.

Note, however, that there is no guarantee that the solution for $f(\mathcal{E})$ satisfies the physical requirement that $f \geq 0$ for all $\mathcal{E}$.

Using Eddington's formula

$$f(\mathcal{E}) = \frac{1}{\sqrt{8}\pi^2} \frac{\mathrm{d}}{\mathrm{d}\mathcal{E}} \int_0^{\mathcal{E}} \frac{\mathrm{d}\rho}{\mathrm{d}\Psi} \frac{\mathrm{d}\Psi}{\sqrt{\mathcal{E}-\Psi}}$$

we see that the requirement $f(\mathcal{E}) \geq 0$ is identical to the the requirement that the function

$$\int_0^{\mathcal{E}} \frac{\mathrm{d}\rho}{\mathrm{d}\Psi} \frac{\mathrm{d}\Psi}{\sqrt{\mathcal{E}-\Psi}}$$

is an increasing function of $\mathcal{E}$.

If a density distribution $\rho(r)$ does not satisfy this requirement, then the model obtained by setting the anisotropy parameter $\beta = 0$ [i.e., by assuming that $f = f(\mathcal{E})$] and solving the Jeans Equations is unphysical.

Anisotropic Spherical Models I

In the more general case, spherical systems (with spherical symmetry in all their properties) have $f = f(E, L^2)$.

These models are anisotropic, in that $\langle v_r^2 \rangle \neq \langle v_\theta^2 \rangle = \langle v_\phi^2 \rangle$.

Anisotropic spherical models are non-unique: many different $f(E, L^2)$ can correspond to a given $\rho(r)$ and $\Psi(r)$. These models differ, though, in their dynamic properties. No equivalent of the Edddington Formula thus exists, that allows to compute $f(E, L^2)$ given $\rho(r)$.

Additional assumptions need to be made. For example, Kent & Gunn (1982) discussed models with $f(E, L^2) = g(E) L^{-2\beta}$, which have a constant anisotropy, i.e., $\beta(r) = \beta$.

An other example are the so called Osipkov-Merritt models (Osipkov 1979; Merritt 1985) were the assumption is made that $f(E, L^2) = f(Q)$ with

$$Q = \mathcal{E} - \frac{L^2}{2r_a^2}$$

Here r_a is the so-called anisotropy radius.

Anisotropic Spherical Models II

The usefulness of the Osipkov-Merritt models becomes apparent from

$$\rho_Q(r) \equiv \left(1 + \frac{r^2}{r_a^2}\right) \rho(r) = 4\pi \int_0^{\Psi} f(Q) \sqrt{2(\Psi - Q)} \mathrm{d}Q$$

Thus $[\rho_Q(r), f(Q)]$ are similarly related as $[\rho(r), f(\mathcal{E})]$ so that we may use Eddington's formula to write

$$f(Q) = \frac{1}{\sqrt{8}\pi^2} \frac{\mathrm{d}}{\mathrm{d}Q} \int_0^{Q} \frac{\mathrm{d}\rho_Q}{\mathrm{d}\Psi} \frac{\mathrm{d}\Psi}{\sqrt{Q-\Psi}}$$

For Osipkov-Merritt models one can show that

$$\beta(r) = \frac{r^2}{r^2 + r_a^2}$$

Thus, these models are isotropic for $r \ll r_a$, become radially anisotropic at around r_a, and become competely radial at large r.

Since purely radial orbits contribute density at the center, models with constant density cores can only have DFs of the Osipkov-Merritt form, i.e., $f = f(E, L^2) = f(Q)$, for sufficiently large r_a. Alternatively, if r_a is relatively small, the (self-consistent) $\rho(r)$ needs to have a central cusp.

Anisotropic Spherical Models III

Next we consider the family of Quasi-Separable DFs (Gerhard 1991):

$$f(\mathcal{E}, L^2) = g(\mathcal{E})\, h(x) \qquad\qquad x = \frac{L}{L_0 + L_c(\mathcal{E})}$$

with L_0 a constant, and $L_c(\mathcal{E})$ the angular momentum of the circular orbit with energy $\mathcal{E}$.

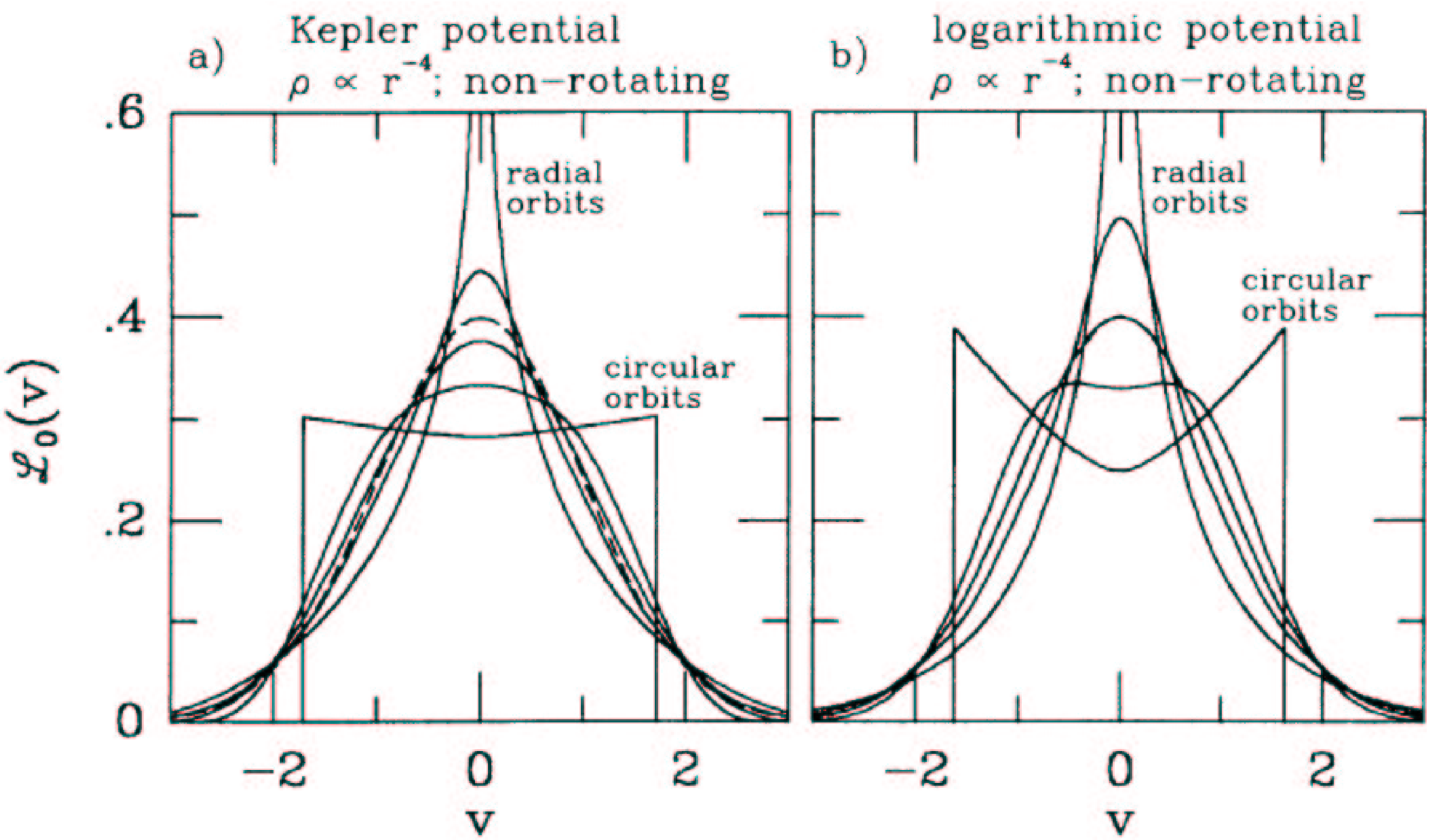
Here $g(\mathcal{E})$ controls the distribution of stars between energy surfaces, while the circularity function $h(x)$ describes the distribution of stars over orbits of different angular momenta on surfaces of constant $\mathcal{E}$.

Depending on the choise for $h(x)$ one can construct models with different anisotropies. If $h(x)$ decreases with increasing x, the model will be radially anisotropic, and vice versa.

Once a choise for $h(x)$ is made, one can (numerically) obtain $g(\mathcal{E})$ for a given $\rho(r)$.

All these various models are useful to explore how different orbital anisotropies impact on observables, such as the line-of-sight velocity distributions (LOSVDs).

Anisotropic Spherical Models IV



Velocity profiles, $\mathcal{L}(v)$, for the outer parts of spherical $f(E, L^2)$ models. Results are shown for $\beta = \infty$ (circular orbits), -1, 0 (isotropic model), 0.5, and 1 (radial orbits). The unit of velocity is the velocity dispersion, which is different for each curve. **(from: van der Marel & Franx 1993)**

Velocity profiles are not expected to be Gaussian

Spherical Models: Summary I

In its most general form, the DF of a static, spherically symmetric model has the form $f = f(E, \vec{L})$. From the symmetry of individual orbits one can see that one always has to have

$$\langle v_r \rangle = \langle v_\theta \rangle = 0 \qquad \langle v_r v_\phi \rangle = \langle v_r v_\theta \rangle = \langle v_\theta v_\phi \rangle = 0$$

This leaves four unknowns: $\langle v_\phi \rangle$, $\langle v_r^2 \rangle$, $\langle v_\theta^2 \rangle$, and $\langle v_\phi^2 \rangle$

If one makes the assumption that the system is spherically symmetric in all its properties then $f(E, \vec{L}) \rightarrow f(E, L^2)$ and

$$\langle v_\phi \rangle = 0 \qquad \langle v_\theta^2 \rangle = \langle v_\phi^2 \rangle$$

In this case the only non-trivial Jeans equation is

$$\frac{1}{\rho}\frac{\partial(\rho\langle v_r^2 \rangle)}{\partial r} + 2\frac{\beta\langle v_r^2 \rangle}{r} = -\frac{\mathrm{d}\Phi}{\mathrm{d}r}$$

with the anisotropy parameter defined by

$$\beta(r) = 1 - \frac{\langle v_r^2 \rangle(r)}{\langle v_r^2 \rangle(r)}$$

Spherical Models: Summary II

Many different models, with different orbital anisotropies, can correspond to the same density distribution. Examples of models are:

- $f(E, L^2) = f(E)$ — **isotropic model, i.e., $\beta = 0$**
- $f(E, L^2) = g(E)\delta(L)$ — **radial orbits only, i.e. $\beta = 1$**
- $f(E, L^2) = g(E)\delta[L - L_c(E)]$ — **circular orbits only, i.e., $\beta = -\infty$**
- $f(E, L^2) = g(E)L^{-2\beta}$ — **constant anisotropy, i.e. $\beta(r) = \beta$**
- $f(E, L^2) = g(E)h(L)$ — **anisotropy depends on circularity function h**
- $f(E, L^2) = f(E + L^2/2r_a^2)$ — **center isotropic, outside radial**

Suppose I have measured the surface brightness profile $\Sigma(R)$ and the line-of-sight velocity dispersion $\sigma_p^2(R)$. Depending on the assumption regarding $\beta(r)$ these data imply very different mass distributions $M(r)$. One can (partially) break this mass-anisotropy degeneracy by using information regarding the LOSVD shapes.
