The Distribution Function

As we have seen before the distribution function (or phase-space density) $f(\vec{x},\vec{v},t)\,\mathrm{d}^3\vec{x}\,\mathrm{d}^3\vec{v}$ gives a full description of the state of any collisionless system.

Here $f(\vec{x},\vec{v},t)\,\mathrm{d}^3\vec{x}\,\mathrm{d}^3\vec{v}$ specifies the number of stars having positions in the small volume $\mathrm{d}^3\vec{x}$ centered on $\vec{x}$ and velocities in the small range $\mathrm{d}^3\vec{v}$ centered on $\vec{v}$, or, when properly normalized, expresses the probability that a star is located in $\mathrm{d}^3\vec{x}\mathrm{d}^3\vec{v}$.

Define the 6-dimensional phase-space vector

$$\vec{w} = (\vec{x},\vec{v}) = (w_1, w_2, ..., w_6)$$

The velocity of the flow in phase-space is then

$$\dot{\vec{w}} = (\dot{\vec{x}},\dot{\vec{v}}) = (\vec{v}, -\vec{\nabla}\Phi)$$

Note that $\dot{\vec{w}}$ has the same relationship to $\vec{w}$ as the 3D fluid flow velocity $\vec{u} = \dot{\vec{x}}$ has to $\vec{x}$ in an ordinary fluid.

In the absence of collisions (long-range, short-range, or direct) and under the assumption that stars are neither created nor destroyed, the flow in phase-space must conserve mass.

The Continuity Equation

Consider an ordinary fluid in an arbitrary closed volume V bounded by a surface S.

The mass of fluid within V is $M(t) = \int_V \rho(\vec{x}, t) \mathrm{d}^3\vec{x}$ and

$$\frac{\mathrm{d}M}{\mathrm{d}t} = \int_V \left(\frac{\partial \rho}{\partial t}\right) \mathrm{d}^3\vec{x}$$

The mass flowing out of V through an area element $\mathrm{d}^2 S$ per unit time is given by $\rho\vec{v} \cdot \mathrm{d}^2\vec{S}$ with $\mathrm{d}^2\vec{S}$ an outward pointing vector normal to the surface S. Thus

$$\frac{\mathrm{d}M}{\mathrm{d}t} = -\int_S \rho\,\vec{v} \cdot \mathrm{d}^2\vec{S}$$

so that we obtain

$$\int_V \frac{\partial \rho}{\partial t} \mathrm{d}^3\vec{x} + \int_S \rho\,\vec{v} \cdot \mathrm{d}^2\vec{S} = 0$$

Using the divergence theorem $\int_V \vec{\nabla} \cdot \vec{F} \mathrm{d}^3\vec{x} = \int_S \vec{F} \cdot \mathrm{d}^2\vec{S}$ we obtain

$$\int_V \left[\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho\vec{v})\right] \mathrm{d}^3\vec{x} = 0$$

Since this must hold for any volume V we obtain the continuity equation:

$$\boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho\vec{v}) = \frac{\partial \rho}{\partial t} + \rho\,\vec{\nabla} \cdot \vec{v} + \vec{v} \cdot \vec{\nabla}\rho = 0}$$

The Collisionless Boltzmann Equation I

Similarly, for our 6D flow in phase-space the continuity equation is given by

$$\boxed{\frac{\partial f}{\partial t} + \vec{\nabla} \cdot (f \dot{\vec{w}}) = 0}$$

and is called the Collisionless Boltzmann Equation (hereafter CBE).

To simplify this equation we first write out the second term:

$$\vec{\nabla} \cdot (f \dot{\vec{w}}) = \sum_{i=1}^{6} \frac{\partial (f \dot{w}_i)}{\partial w_i} = f \sum_{i=1}^{3} \left[\frac{\partial v_i}{\partial x_i} + \frac{\partial \dot{v}_i}{\partial v_i} \right] + \sum_{i=1}^{3} \left[v_i \frac{\partial f}{\partial x_i} + \dot{v}_i \frac{\partial f}{\partial v_i} \right]$$

Since $\partial v_i / \partial x_i = 0$ (x_i and v_i are independent phase-space coordinates) and $\partial \dot{v}_i / \partial v_i = \frac{\partial}{\partial v_i} \left(- \frac{\partial \Phi}{\partial x_i} \right) = 0$ (gradient of potential does not depend on velocities), we may (using summation convention rewrite the CBE as

$$\boxed{\frac{\partial f}{\partial t} + v_i \frac{\partial f}{\partial x_i} - \frac{\partial \Phi}{\partial x_i} \frac{\partial f}{\partial v_i} = 0}$$

or in vector notation

$$\boxed{\frac{\partial f}{\partial t} + \vec{v} \cdot \vec{\nabla} f - \vec{\nabla} \Phi \cdot \frac{\partial f}{\partial \vec{v}} = 0}$$

The Collisionless Boltzmann Equation II

Note: Since $f = f(\vec{x}, \vec{v}, t)$ we have that

$$\mathrm{d}f = \frac{\partial f}{\partial t}\mathrm{d}t + \frac{\partial f}{\partial x_i}\mathrm{d}x_i + \frac{\partial f}{\partial v_i}\mathrm{d}v_i$$

and thus

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x_i}v_i - \frac{\partial \Phi}{\partial x_i}\frac{\partial f}{\partial v_i}$$

Using this we can write the CBE in its compact form:

$$\boxed{\frac{\mathrm{d}f}{\mathrm{d}t} = 0}$$

$\mathrm{d}f/\mathrm{d}t$ expresses the Lagrangian derivative along trajectories through phase-space, and the CBE expresses that this flow is incompressible. In other words, the phase-space density f around the phase-point of a give star always remains the same.

In the presence of collisions it is no longer true that $\dot{\vec{v}} = -\vec{\nabla}\Phi$, and the CBE no longer holds. Rather, collisions result in an additional collision term:

$$\boxed{\frac{\mathrm{d}f}{\mathrm{d}t} = \Gamma(t)}$$

This equation is called the Master Equation. If $\Gamma(t)$ describes long-range collisions only, then we call it the Fokker-Planck Equation.

Coarse-Grained Distribution Function

We defined the DF as the phase-space density of stars in a volume $\mathrm{d}^3\vec{x}\,\mathrm{d}^3\vec{v}$. However, in our assumption of a smooth $\rho(\vec{r})$ and $\Phi(\vec{r})$, the only meaningful interpretation of the DF is that of a probability density.

Note that this probability density is also well defined in the discrete case, even though it may vary rapidly. Since it has an infinitely high resolution, it is often called the fine-grained DF.

Just as the wave-functions in quantum mechanics, the fine-grained DF is not measurable. However, we can use it to compute the expectation value of any phase-space function $Q(\vec{x},\vec{v})$.

A measurable DF, one that is actually related to counting objects in a given phase-space volume, is the so-called coarse-grained DF, $\bar{f}$, defined as the average value of the fine-grained DF, f, in some specified small volume:

$$\bar{f}(\vec{x}_0,\vec{v}_0) = \int\int w(\vec{x}-\vec{x}_0,\vec{v}-\vec{v}_0)\, f(\vec{x},\vec{v})\,\mathrm{d}^3\vec{x}\,\mathrm{d}^3\vec{v}$$

with $w(\vec{x},\vec{v})$ some (properly normalized) kernel which rapidly falls to zero for $|\vec{x}| > \epsilon_x$ and $|\vec{v}| > \epsilon_v$.

NOTE: the fine-grained DF does satisfy the CBE.
the coarse-grained DF does not satisfy the CBE.

Moment Equations I

Although the CBE looks very simple ($\mathrm{d}f/\mathrm{d}t = 0$), solving it for the DF is virtually impossible. It is more practical to consider moment equations.

The resulting Stellar-Hydrodynamics Equations are obtained by multiplying the CBE by powers of velocity and then integrating over all of velocity space.

Consider moment equations related to $v_i^l v_j^m v_k^n$ where the indices (i, j, k) refer to one of the three generalized coordinates, and (l, m, n) are integers.

Recall that

$$\rho = \int f \, \mathrm{d}^3\vec{v} \qquad\qquad \rho\langle v_i^l v_j^m v_k^n \rangle = \int v_i^l v_j^m v_k^n \, f \, \mathrm{d}^3\vec{v}$$

The $(l+m+n)^{\mathrm{th}}$ moment equation of the CBE is

$$\int v_i^l v_j^m v_k^n \frac{\partial f}{\partial t} \, \mathrm{d}^3\vec{v} + \int v_i^l v_j^m v_k^n \, v_a \frac{\partial f}{\partial x_a} \, \mathrm{d}^3\vec{v} - \int v_i^l v_j^m v_k^n \frac{\partial \Phi}{\partial x_a} \frac{\partial f}{\partial v_a} \, \mathrm{d}^3\vec{v} = 0$$

$$\Leftrightarrow \frac{\partial}{\partial t} \int v_i^l v_j^m v_k^n \, f \, \mathrm{d}^3\vec{v} + \frac{\partial}{\partial x_i} \int v_i^l v_j^m v_k^n \, v_a f \mathrm{d}^3\vec{v} - \frac{\partial \Phi}{\partial x_a} \int v_i^l v_j^m v_k^n \frac{\partial f}{\partial v_a} \, \mathrm{d}^3\vec{v} = 0$$

$$\Leftrightarrow \frac{\partial}{\partial t} \left[\rho\langle v_i^l v_j^m v_k^n \rangle\right] + \frac{\partial}{\partial x_i} \left[\rho\langle v_i^l v_j^m v_k^n v_a \rangle\right] - \frac{\partial \Phi}{\partial x_a} \int v_i^l v_j^m v_k^n \frac{\partial f}{\partial v_a} \, \mathrm{d}^3\vec{v} = 0$$

1^{st} term: integration range doesn't depend on t so $\frac{\partial}{\partial t}$ may be taken outside

2^{nd} term: $\frac{\partial}{\partial x_i}$ doesn't depend on v_i, so derivative may be taken outside.

3^{rd} term: $\frac{\partial \Phi}{\partial x_i}$ doesn't depend on v_i, so may be taken outside.

Moment Equations II

Let's consider the zeroth moment: $l = m = n = 0$

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho\langle v_i\rangle)}{\partial x_i} - \frac{\partial \Phi}{\partial x_i}\int \frac{\partial f}{\partial v_i}\mathrm{d}^3\vec{v} = 0$$

Using the divergence theorem we can write

$$\int \frac{\partial f}{\partial v_i}\mathrm{d}^3\vec{v} = \int f\mathrm{d}^2\vec{S} = 0$$

where the last equality follows from the fact that $f \to 0$ if $|v| \to \infty$. The zeroth moment of the CBE therefore reduces to

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial(\rho\langle v_i\rangle)}{\partial x_i} = 0}$$

Note that this is the continuity equation, identical to that of fluid dynamics.

Just a short remark regarding notation:

$$\langle v_i\rangle = \int v_i f\mathrm{d}^3\vec{v}$$

is used as short-hand for

$$\langle v_i(\vec{x})\rangle = \int v_i(\vec{x}) f(\vec{x},\vec{v})\mathrm{d}^3\vec{v}$$

Thus, $\langle v_i\rangle$ is a local expectation value. For brevity we do not explicitely write the $\vec{x}$-dependence.

Moment Equations III

Next we consider the first-order moment equations $(l, m, n) = (1, 0, 0)$ or $(0, 1, 0)$ or $(0, 0, 1)$

$$\int v_j \frac{\partial f}{\partial t} \mathrm{d}^3\vec{v} + \int v_j v_i \frac{\partial f}{\partial x_i} \mathrm{d}^3\vec{v} - \int v_j \frac{\partial \Phi}{\partial x_i} \frac{\partial f}{\partial v_i} \mathrm{d}^3\vec{v} = 0$$

$$\Leftrightarrow \frac{\partial(\rho\langle v_j\rangle)}{\partial t} + \frac{\partial(\rho\langle v_i v_j\rangle)}{\partial x_i} - \frac{\partial \Phi}{\partial x_i} \int v_j \frac{\partial f}{\partial v_i} \mathrm{d}^3\vec{v} = 0$$

Using integration by parts we write

$$\begin{aligned} \int v_j \frac{\partial f}{\partial v_i} \mathrm{d}^3\vec{v} &= \int \frac{\partial(v_j f)}{\partial v_i} \mathrm{d}^3\vec{v} - \int \frac{\partial v_j}{\partial v_i} f \mathrm{d}^3\vec{v} \\ &= \int v_j f \mathrm{d}^2 S - \int \delta_{ij} f \mathrm{d}^3\vec{v} \\ &= -\delta_{ij}\rho \end{aligned}$$

so that we obtain

$$\boxed{\frac{\partial(\rho\langle v_j\rangle)}{\partial t} + \frac{\partial(\rho\langle v_i v_j\rangle)}{\partial x_i} + \rho \frac{\partial \Phi}{\partial x_j} = 0}$$

These are called the momentum equations. Note that this represents a set of three equations (for $j = 1, 2, 3$), and that a summation over i is implied.

The Jeans Equations I

We can obtain the so-called Jeans Equations by subtracting $\langle v_j \rangle$ times the continuity equation from the momentum equations:

First we write $\langle v_j \rangle$ times the continuity equation:

$$\langle v_j \rangle \frac{\partial \rho}{\partial t} + \langle v_j \rangle \frac{\partial (\rho \langle v_i \rangle)}{\partial x_i} = 0$$

$$\Leftrightarrow \frac{\partial (\rho \langle v_j \rangle)}{\partial t} - \rho \frac{\partial \langle v_j \rangle}{\partial t} + \frac{\partial (\rho \langle v_i \rangle \langle v_j \rangle)}{\partial x_i} - \rho \langle v_i \rangle \frac{\partial \langle v_j \rangle}{\partial x_i} = 0$$

Subtracting this from the momentum equations yields

$$\frac{\partial (\rho \langle v_i v_j \rangle)}{\partial x_i} + \rho \frac{\partial \Phi}{\partial x_j} + \rho \frac{\partial \langle v_j \rangle}{\partial t} - \frac{\partial (\rho \langle v_i \rangle \langle v_j \rangle)}{\partial x_i} + \rho \langle v_i \rangle \frac{\partial \langle v_j \rangle}{\partial x_i} = 0$$

If we define $\sigma_{ij}^2 = \langle v_i v_j \rangle - \langle v_i \rangle \cdot \langle v_j \rangle$ then we obtain

$$\rho \frac{\partial \langle v_j \rangle}{\partial t} + \rho \langle v_i \rangle \frac{\partial \langle v_j \rangle}{\partial x_i} = -\rho \frac{\partial \Phi}{\partial x_j} - \frac{\partial (\rho \sigma_{ij}^2)}{\partial x_i}$$

These are called the Jeans Equations. Once again, this represents a set of three equations (for $j = 1, 2, 3$), and a summation over i is implied.

The Jeans Equations II

We can derive a very similar equation for fluid dynamics. The equation of motion of a fluid element in the fluid is

$$\rho \frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = -\vec{\nabla} P - \rho \vec{\nabla} \Phi_{\rm ext}$$

with P the pressure and $\Phi_{\rm ext}$ some external potential.

Since $\vec{v} = \vec{v}(\vec{x}, t)$ we have that $\mathrm{d}\vec{v} = \frac{\partial \vec{v}}{\partial t}\mathrm{d}t + \frac{\partial \vec{v}}{\partial x_i}\mathrm{d}x_i$, and thus

$$\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = \frac{\partial v}{\partial t} + \left(\vec{v} \cdot \vec{\nabla}\right) \vec{v}$$

which allows us to write the equations of motion as

$$\rho \frac{\partial v}{\partial t} + \rho \left(\vec{v} \cdot \vec{\nabla}\right) \vec{v} = -\vec{\nabla} P - \rho \vec{\nabla} \Phi_{\rm ext}$$

These are the so-called Euler Equations. A comparison with the Jeans Equations shows that $\rho\sigma^2_{ij}$ has a similar effect as the pressure. However, now it is not a scalar but a tensor.

$\rho\sigma^2_{ij}$ is called the stress-tensor

The stress-tensor is manifest symmetric ($\sigma_{ij} = \sigma_{ji}$) and there are thus 6 independent terms.

The Jeans Equations III

The stress tensor σ^2_{ij} measures the random motions of the stars around the streaming part $\langle v_i \rangle \langle v_j \rangle$.

Note that the stress-tensor is a local quantity $\sigma^2_{ij} = \sigma^2_{ij}(\vec{x})$. At each point $\vec{x}$ it defines the velocity ellipsoid; an ellipsoid whose principal axes are defined by the orthogonal eigenvectors of σ^2_{ij} with lengths that are proportional to the square roots of the respective eigenvalues.

The incompressible stellar fluid experiences anisotropic pressure-like forces.

Note that the Jeans Equations have 9 unknowns (3 streaming motions $\langle v_i \rangle$ and 6 terms of the stress-tensor). With only three equations, this is not a closed set.

For comparison, in fluid dynamics there are only 4 unknowns (3 streaming motions and the pressure). The 3 Euler Equations combined with the Equation of State forms a closed set.

The Jeans Equations IV

One might think that adding higher-order moment equations of the **CBE** will allow to obtain a closed set of equations. However, adding more equations also adds more unknowns such as $\langle v_i v_j v_k \rangle$, etc. The set of CBE moment equations never closes!

In practice one therefore makes some assumptions, such as assumptions regarding the form of the **stress-tensor**, in order to be able to solve the **Jeans Eequations**.

If, with this approach, a solution is found, the solution may not correspond to a physical (i.e., everywhere positive) DF. Thus, although any real DF obeys the Jeans equations, not every solution to the Jeans equations cprresponds to a physical DF!!!

Cylindrically Symmetric Jeans Equations

As a worked out example we derive the Jeans equations under cylindrical symmetry. We therefore write the Jeans equations in the cylindrical coordinate system (R, ϕ, z).

The first step is to write the CBE in cylindrical coordinates

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \dot{R}\frac{\partial f}{\partial R} + \dot{\phi}\frac{\partial f}{\partial \phi} + \dot{z}\frac{\partial f}{\partial z} + \dot{v}_R\frac{\partial f}{\partial v_R} + \dot{v}_\phi\frac{\partial f}{\partial v_\phi} + \dot{v}_z\frac{\partial f}{\partial v_z}$$

First we recall from vector calculus that

$$\vec{v} = \dot{R}\vec{e}_R + R\dot{\phi}\vec{e}_\phi + \dot{z}\vec{e}_z = v_R\vec{e}_R + v_\phi\vec{e}_\phi + v_z\vec{e}_z$$

from which we obtain that

$$\vec{a} = \frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = \ddot{R}\vec{e}_R + \dot{R}\dot{\vec{e}}_R + \dot{R}\dot{\phi}\vec{e}_\phi + R\ddot{\phi}\vec{e}_\phi + R\dot{\phi}\dot{\vec{e}}_\phi + \ddot{z}\vec{e}_z + \dot{z}\dot{\vec{e}}_z$$

Using that $\dot{\vec{e}}_R = \phi\vec{e}_\phi$, $\dot{\vec{e}}_\phi = -\dot{\phi}\vec{e}_R$, and $\dot{\vec{e}}_z = 0$ we have that

$$\vec{a} = \left[\ddot{R} - R\dot{\phi}^2\right]\vec{e}_R + \left[2\dot{R}\dot{\phi} + R\ddot{\phi}\right]\vec{e}_\phi + \ddot{z}\vec{e}_z$$

$$\begin{aligned} v_R &= \dot{R} &\Rightarrow\quad \dot{v}_R &= \ddot{R} \\ v_\phi &= R\dot{\phi} &\Rightarrow\quad \dot{v}_\phi &= \dot{R}\dot{\phi} + R\ddot{\phi} \\ v_z &= \dot{z} &\Rightarrow\quad \dot{v}_z &= \ddot{z} \end{aligned}$$

Cylindrically Symmetric Jeans Equations

This allows us to write

$$\vec{a} = \left[\dot{v}_R - \frac{v_\phi^2}{R}\right]\vec{e}_R + \left[\frac{v_R v_\phi}{R} + \dot{v}_\phi\right]\vec{e}_\phi + \dot{v}_z\vec{e}_z$$

Newton's equation of motion in vector form reads

$$\vec{a} = -\vec{\nabla}\Phi = \frac{\partial\Phi}{\partial R}\vec{e}_R + \frac{1}{R}\frac{\partial\Phi}{\partial\phi}\vec{e}_\phi + \frac{\partial\Phi}{\partial z}\vec{e}_z$$

Combining the above we obtain

$$\begin{aligned}\dot{v}_R &= -\frac{\partial\Phi}{\partial R} + \frac{v_\phi^2}{R}\\ \dot{v}_\phi &= -\frac{1}{R}\frac{\partial\Phi}{\partial R} + \frac{v_R v_\phi}{R}\\ \dot{v}_z &= -\frac{\partial\Phi}{\partial z}\end{aligned}$$

Which allows us to the write the CBE in cylindrical coordinates as

$$\frac{\partial f}{\partial t} + v_R\frac{\partial f}{\partial R} + \frac{v_\phi}{R}\frac{\partial f}{\partial\phi} + v_z\frac{\partial f}{\partial z} + \left[\frac{v_\phi^2}{R} - \frac{\partial\Phi}{\partial R}\right]\frac{\partial f}{\partial v_R} - \frac{1}{R}\left[v_R v_\phi + \frac{\partial\Phi}{\partial\phi}\right]\frac{\partial f}{\partial v_\phi} - \frac{\partial\Phi}{\partial z}\frac{\partial f}{\partial v_z} = 0$$

Cylindrically Symmetric Jeans Equations

The Jeans equations follow from multiplication with v_R, v_ϕ, and v_z and integration over velocity space. Note that the symmetry requires that all derivatives with respect to ϕ must vanish.

The remaining terms are:

$$\int v_R \frac{\partial f}{\partial t} \mathrm{d}^3\vec{v} = \frac{\partial}{\partial t} \int v_R f \mathrm{d}^3\vec{v} = \frac{\partial(\rho\langle v_R \rangle)}{\partial t}$$

$$\int v_R^2 \frac{\partial f}{\partial R} \mathrm{d}^3\vec{v} = \frac{\partial}{\partial R} \int v_R^2 f \mathrm{d}^3\vec{v} = \frac{\partial(\rho\langle v_R^2 \rangle)}{\partial R}$$

$$\int v_R v_z \frac{\partial f}{\partial z} \mathrm{d}^3\vec{v} = \frac{\partial}{\partial z} \int v_R v_z f \mathrm{d}^3\vec{v} = \frac{\partial(\rho\langle v_R v_z \rangle)}{\partial z}$$

$$\int \frac{v_R v_\phi^2}{R} \frac{\partial f}{\partial v_R} \mathrm{d}^3\vec{v} = \frac{1}{R} \left[\int \frac{\partial(v_R v_\phi^2 f)}{\partial v_R} \mathrm{d}^3\vec{v} - \int \frac{\partial(v_R v_\phi^2)}{\partial v_R} f \mathrm{d}^3\vec{v} \right] = -\rho \frac{\langle v_\phi^2 \rangle}{R}$$

$$\int v_R \frac{\partial \Phi}{\partial R} \frac{\partial f}{\partial v_R} \mathrm{d}^3\vec{v} = \frac{\partial \Phi}{\partial R} \left[\int \frac{\partial(v_R f)}{\partial v_R} \mathrm{d}^3\vec{v} - \int \frac{\partial v_R}{\partial v_R} f \mathrm{d}^3\vec{v} \right] = -\rho \frac{\partial \Phi}{\partial R}$$

$$\int \frac{v_R^2 v_\phi}{R} \frac{\partial f}{\partial v_\phi} \mathrm{d}^3\vec{v} = \frac{1}{R} \left[\int \frac{\partial(v_R^2 v_\phi f)}{\partial v_\phi} \mathrm{d}^3\vec{v} - \int \frac{\partial(v_R^2 v_\phi)}{\partial v_\phi} f \mathrm{d}^3\vec{v} \right] = -\rho \frac{\langle v_R^2 \rangle}{R}$$

$$\int v_R \frac{\partial \Phi}{\partial z} \frac{\partial f}{\partial v_z} \mathrm{d}^3\vec{v} = \frac{\partial \Phi}{\partial z} \left[\int \frac{\partial(v_R f)}{\partial v_z} \mathrm{d}^3\vec{v} - \int \frac{\partial v_z}{\partial v_R} f \mathrm{d}^3\vec{v} \right] = 0$$

Cylindrically Symmetric Jeans Equations

Working out the similar terms for the other Jeans equations we obtain the Jeans Equations in Cylindrical Coordinates

$$\frac{\partial(\rho\langle v_R\rangle)}{\partial t} + \frac{\partial(\rho\langle v_R^2\rangle)}{\partial R} + \frac{\partial(\rho\langle v_R v_z\rangle)}{\partial z} + \rho\left[\frac{\langle v_R^2\rangle - \langle v_\phi^2\rangle}{R} + \frac{\partial\Phi}{\partial R}\right] = 0$$
$$\frac{\partial(\rho\langle v_\phi\rangle)}{\partial t} + \frac{\partial(\rho\langle v_R v_\phi\rangle)}{\partial R} + \frac{\partial(\rho\langle v_\phi v_z\rangle)}{\partial z} + 2\rho\frac{\langle v_R v_\phi\rangle}{R} = 0$$
$$\frac{\partial(\rho\langle v_z\rangle)}{\partial t} + \frac{\partial(\rho\langle v_R v_z\rangle)}{\partial R} + \frac{\partial(\rho\langle v_z^2\rangle)}{\partial z} + \rho\left[\frac{\langle v_R v_z\rangle}{R} + \frac{\partial\Phi}{\partial z}\right] = 0$$

Note that there are indeed 9 unknowns in these 3 equations. Only if we make additional assumptions can we solve these equations. In particular, one often makes the following assumptions:

(1) **System is static** $\Rightarrow \frac{\partial}{\partial t}$**-terms are zero and** $\langle v_R\rangle = \langle v_z\rangle = 0$

(2) **Stress Tensor is diagonal** $\Rightarrow \langle v_i v_j\rangle = 0$ **if** $i \neq j$

(3) **Meridional Isotropy** $\Rightarrow \langle v_R^2\rangle = \langle v_z^2\rangle = \sigma_R^2 = \sigma_z^2 \equiv \sigma^2$

Under these assumptions we have 3 unknowns left: $\langle v_\phi\rangle$, $\langle v_\phi^2\rangle$**, and** σ^2**.**

Cylindrically Symmetric Jeans Equations

Under the assumptions listed on the previous page the Jeans equation reduce to

$$\frac{\partial(\rho\sigma^2)}{\partial R} + \rho\left[\frac{\sigma^2 - \langle v_\phi^2\rangle}{R} + \frac{\partial\Phi}{\partial R}\right] = 0$$
$$\frac{\partial(\rho\sigma^2)}{\partial z} + \rho\frac{\partial\Phi}{\partial z} = 0$$

Note that we have only 2 equations left: the system is still not closed.

If from the surface brightness we can estimate the mass density $\rho(R, z)$ and hence the potential $\Phi(R, z)$, we can solve the second of these Jeans equations for the meridional velocity dispersion

$$\sigma^2(R, z) = \frac{1}{\rho}\int_z^\infty \rho\frac{\partial\Phi}{\partial z}\mathrm{d}z$$

and the first Jeans equation then gives the mean square azimuthal velocity $\langle v_\phi^2\rangle = \langle v_\phi\rangle^2 + \sigma_\phi^2$

$$\langle v_\phi^2\rangle(R, z) = \sigma^2(R, z) + R\frac{\partial\Phi}{\partial R} + \frac{R}{\rho}\frac{\partial(\rho\sigma^2)}{\partial R}$$

Thus, although $\langle v_\phi^2\rangle$ is uniquely specified by the Jeans equations, we don't know how it splits in the actual azimuthal streaming $\langle v_\phi\rangle$ and the azimuthal dispersion σ_ϕ^2. Additional assumptions are needed for this.

Spherically Symmetric Jeans Equations

A similar analysis but for a spherically symmetric system, using the spherical coordinate system (r, θ, ϕ), gives the following set of Jeans equations

$$\frac{\partial(\rho\langle v_r\rangle)}{\partial t} + \frac{\partial(\rho\langle v_r^2\rangle)}{\partial r} + \frac{\rho}{r}\left[2\langle v_r^2\rangle - \langle v_\theta^2\rangle - \langle v_\phi^2\rangle\right] + \rho\frac{\partial\Phi}{\partial r} = 0$$

$$\frac{\partial(\rho\langle v_\theta\rangle)}{\partial t} + \frac{\partial(\rho\langle v_r v_\theta\rangle)}{\partial r} + \frac{\rho}{r}\left[3\langle v_r v_\theta\rangle + \left(\langle v_\theta^2\rangle - \langle v_\phi^2\rangle\right)\cot\theta\right] = 0$$

$$\frac{\partial(\rho\langle v_\phi\rangle)}{\partial t} + \frac{\partial(\rho\langle v_r v_\phi\rangle)}{\partial r} + \frac{\rho}{r}\left[3\langle v_r v_\phi\rangle + 2\langle v_\theta v_\phi\rangle\cot\theta\right] = 0$$

If we now make the additional assumptions that the system is static and that also the kinematic properties of the system are spherical symmetric then there can be no streaming motions and all mixed second-order moments vanish. Consequently, the stress tensor is diagonal with $\sigma_\theta^2 = \sigma_\phi^2$. Under these assumptions only one of the three Jeans equations remains:

$$\frac{\partial(\rho\sigma_r^2)}{\partial r} + \frac{2\rho}{r}\left[\sigma_r^2 - \sigma_\theta^2\right] + \rho\frac{\partial\Phi}{\partial r} = 0$$

Notice once again how the spherical Jeans equation is not sufficient to determine the dynamics: if the density and potential are presumed known, it contains two unknown functions $\sigma_r^2(r)$ and $\sigma_\theta^2(r)$ which can therefore not be determined both.

Spherically Symmetric Jeans Equations

It is useful to define the anisotropy parameter

$$\beta(r) \equiv 1 - \frac{\sigma_\theta^2(r)}{\sigma_r^2(r)}$$

With β thus defined the Jeans equation can be written as

$$\frac{1}{\rho}\frac{\partial(\rho\langle v_r^2\rangle)}{\partial r} + 2\frac{\beta\langle v_r^2\rangle}{r} = -\frac{\mathrm{d}\Phi}{\mathrm{d}r}$$

If we now use that $\mathrm{d}\Phi/\mathrm{d}r = GM(r)/r$ then we obtain

$$M(r) = -\frac{r\langle v_r^2\rangle}{G}\left[\frac{\mathrm{d}\ln\rho}{\mathrm{d}\ln r} + \frac{\mathrm{d}\ln\langle v_r^2\rangle}{\mathrm{d}\ln r} + 2\beta\right]$$

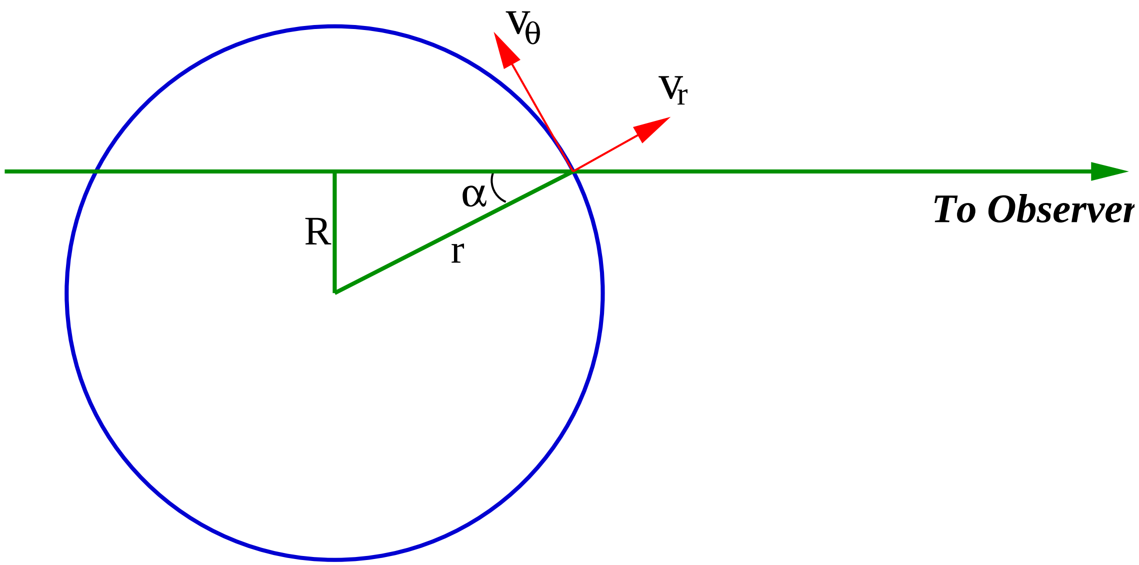
Thus, if we can measure $\rho(r)$, $\langle v_r^2\rangle(r)$, and $\beta(r)$, we can use the Jeans equations to infer the mass profile $M(r)$.

Consider an external, spherical galaxy. Observationally, we can measure the projected surface brightness profile, $\Sigma(R)$, which is related to the luminosity density $\nu(r) = \rho(r)/\Upsilon(r)$ as

$$\Sigma(R) = 2\int_R^\infty \frac{\nu\, r\, \mathrm{d}r}{\sqrt{r^2 - R^2}}$$

with $\Upsilon(r)$ the mass-to-light ratio.

Spherically Symmetric Jeans Equations



Similarly, the line-of-sight velocity dispersion is an observationally accessible quantity. As the figure illustrates, it is related to both $\langle v_r^2\rangle(r)$ and $\beta(r)$ according to

$$\begin{aligned}\Sigma(R)\sigma_p^2(R) &= 2\int_R^\infty \langle(v_r\cos\alpha - v_\theta\sin\alpha)^2\rangle \frac{\nu\, r\, \mathrm{d}r}{\sqrt{r^2-R^2}} \\ &= 2\int_R^\infty \left(\langle v_r^2\rangle\cos^2\alpha + \langle v_\theta^2\rangle \sin^2\alpha\right)\frac{\nu\, r\, \mathrm{d}r}{\sqrt{r^2-R^2}} \\ &= 2\int_R^\infty \left(1-\beta\frac{R^2}{r^2}\right)\frac{\nu\,\langle v_r^2\rangle\, r\,\mathrm{d}r}{\sqrt{r^2-R^2}}\end{aligned}$$

Spherically Symmetric Jeans Equations

The 3D luminosity density is trivially obtained from the observed $\Sigma(R)$:

$$\nu(r) = -\frac{1}{\pi}\int_r^\infty \frac{\mathrm{d}\Sigma}{\mathrm{d}R}\frac{\mathrm{d}R}{\sqrt{R^2-r^2}}$$

In general, we have three unknowns: $M(r)$ (or equivalently $\rho(r)$ or $\Upsilon(r)$), $\langle v_r^2\rangle(r)$ and $\beta(r)$.

With our two observables $\Sigma(R)$ and $\sigma_p^2(R)$ these can only be determined if we make additional assumptions.

EXAMPLE 1: Assume isotropy ($\beta(r) = 0$). In this case we can use the Abel inversion technique to obtain

$$\nu(r)\langle v_r^2\rangle(r) = -\frac{1}{\pi}\int_r^\infty \frac{\mathrm{d}(\Sigma\sigma_p^2)}{\mathrm{d}R}\frac{\mathrm{d}R}{\sqrt{R^2-r^2}}$$

and the enclosed mass follows from the Jeans equation

$$M(r) = -\frac{r\langle v_r^2\rangle}{G}\left[\frac{\mathrm{d}\ln\nu}{\mathrm{d}\ln r} + \frac{\mathrm{d}\ln\langle v_r^2\rangle}{\mathrm{d}\ln r}\right]$$

Note that the first term uses the luminosity density $\nu(r)$ rather than the mass density $\rho(r)$, because σ_p^2 is weighted by light rather than mass.

Spherically Symmetric Jeans Equations

The mass-to-light ratio now follows from

$$\Upsilon(r) = \frac{M(r)}{4\pi \int_0^r \nu(r)\, r^2\, \mathrm{d}r}$$

which can be used to investigate whether system contains dark matter halo or central black hole, but always under assumption that system is isotropic.

EXAMPLE 2: Assume a constant mass-to-light ratio: $\Upsilon(r) = \Upsilon_0$. In this case the luminosity density $\nu(r)$ immediately yields the enclosed mass:

$$M(r) = 4\pi\Upsilon_0 \int_0^r \nu(r)\, r^2\, \mathrm{d}r$$

We can now use the Jeans Equation to write $\beta(r)$ in terms of $M(r)$, $\nu(r)$ and $\langle v_r^2\rangle(r)$. Substituting this in the equation for $\Sigma(R)\sigma_p^2(R)$ allows a solution for $\langle v_r^2\rangle(r)$, and thus for $\beta(r)$. As long as $0 \leq \beta(r) \leq 1$ the model is said to be self-consistent witin the context of the Jeans equations.

Almost always, radically different models (based on radically different assumptions) can be constructed, that are all consistent with the data and the Jeans equations. This is often referred to as the mass-anisotropy degeneracy. Note, however, that none of these models need to be physical: they can still have $f < 0$.