Orbits in Axisymmetric Potentials I

Axisymmetric potentials (oblate or prolate) are far more realistic examples to consider in astronomy. Elliptical galaxies might well be spheroidal (but could also be ellipsoidal), while disk galaxies almost certainly are axisymmetric (though highly flattened).

For axisymmetric systems the coordinate system of choise are the cylindrical coordinates (R, θ, z), and $\Phi = \Phi(R, z)$.

Solving Newton's equation of motion in cylindrical coordinates yields:

$$\begin{aligned} \ddot{R} - R\dot{\theta}^2 &= -\frac{\partial \Phi}{\partial R} \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(R^2\dot{\theta}\right) &= 0 \\ \ddot{z} &= -\frac{\partial \Phi}{\partial z} \end{aligned}$$

The second of these expresses conservation of the component of angular momentum about the z-axis; $L_z = R^2\dot{\theta}$, while the other two equations describe the coupled oscillations in the R and z-directions.

NOTE: For stars confined to equatorial plane $z = 0$, equations of motion are identical to that of motion in spherical density distribution (not surprising, since in this case the motion is once again central). Therefore, orbits confined to equatorial plane are rosette orbits.

Orbits in Axisymmetric Potentials II

As for the spherical case, we can reduce the equations of motion to

$$\ddot{R} = -\frac{\partial \Phi_{\rm eff}}{\partial R} \qquad \ddot{z} = -\frac{\partial \Phi_{\rm eff}}{\partial z}$$

with $\Phi_{\rm eff}(R, z) = \Phi(R, z) + \frac{L_z^2}{2R^2}$ the effective potential. The L_z^2/R^2-term serves as a centrifugal barrier, only allowing orbits with $L_z = 0$ near the symmetry-axis.

This allows us to reduce the $3D$ motion to $2D$ motion in Meridional Plane (R, z), which rotates non-uniformly around the symmetry axis according to $\dot{\theta} = L_z/R^2$.

In addition to simplifying the problem, it also allows the use of surfaces-of-section to investigate the orbital properties.

For the energy we can write

$$E = \frac{1}{2}\left[\dot{R}^2 + (R\dot{\theta})^2 + \dot{z}^2\right] + \Phi = \frac{1}{2}\left(\dot{R}^2 + \dot{z}^2\right) + \Phi_{\rm eff}$$

so that the orbit is restricted to the area in the meridional plane satisfying $E \geq \Phi_{\rm eff}$. The curve bounding this area is called the zero-velocity curve (ZVC) (since for a point on it $\vec{v} = 0$).

Epicycle Approximation I

We have defined the effective potential $\Phi_{\rm eff} = \Phi + \frac{L_z^2}{2R^2}$. This has a minimum at $(R, z) = (R_g, 0)$, where

$$\frac{\partial \Phi_{\rm eff}}{\partial R} = \frac{\partial \Phi}{\partial R} - \frac{L_z^2}{R^3} = 0$$

The radius $R = R_g$ corresponds to the radius of a circular orbit with energy $E = \Phi(R_g, 0) + \frac{1}{2}v_c^2 = \Phi(R_g, 0) + \frac{L_z^2}{2R_g^2} = \Phi_{\rm eff}$.

If we define $x = R - R_g$ and expand $\Phi_{\rm eff}$ around the point $(x, y) = (0, 0)$ in a Taylor series we obtain

$$\begin{aligned}\Phi_{\rm eff} \quad = \quad & \Phi_{\rm eff}(R_g, 0) + (\Phi_x)x + (\Phi_y)y + (\Phi_{xy})xy + \frac{1}{2}(\Phi_{xx})x^2 + \\ & \frac{1}{2}(\Phi_{yy})y^2 + \mathcal{O}(xz^2) + \mathcal{O}(x^2 z) + \text{etc}\end{aligned}$$

where

$$\Phi_x = \left(\frac{\partial \Phi_{\rm eff}}{\partial x}\right)_{(R_g,0)} \quad \Phi_{xx} = \left(\frac{\partial^2 \Phi_{\rm eff}}{\partial x^2}\right)_{(R_g,0)} \quad \Phi_{xy} = \left(\frac{\partial^2 \Phi_{\rm eff}}{\partial x \partial y}\right)_{(R_g,0)}$$

By definition of R_g, and by symmetry considerations, we have that

$$\Phi_x = \Phi_y = \Phi_{xy} = 0$$

Epicycle Approximation II

In the epicycle approximation only terms up to second order are considered: all terms of order xz^2, $x^2 z$ or higher are considered negligble. Defining

$$\kappa^2 \equiv \Phi_{xx} \qquad \nu^2 \equiv \Phi_{yy}$$

we thus have that, in the epicycle approximation,

$$\Phi_{\rm eff} = \Phi_{\rm eff}(R_g, 0) + \tfrac{1}{2}\kappa^2 x^2 + \tfrac{1}{2}\nu^2 y^2$$

so that the equations of motion in the meridional plane become

$$\ddot{x} = -\kappa^2\, x \qquad \ddot{y} = -\nu^2\, y$$

Thus, the x- and y-motions are simple harmonic oscillations with the epicycle frequency κ and the vertical frequency ν.

In addition, we have the circular frequency

$$\Omega(R) = \frac{v_c(R)}{R} = \sqrt{\frac{1}{R}\left(\frac{\partial \Phi}{\partial R}\right)_{(R,0)}} = \frac{L_z}{R^2}$$

which allows us to write

$$\kappa^2 = \left(\frac{\partial^2 \Phi_{\rm eff}}{\partial R^2}\right)_{(R_g,0)} = \left(R\frac{{\rm d}\Omega^2}{{\rm d}R} + 4\Omega^2\right)_{R_g}$$

Epicycle Approximation III

As we have seen before, for a realistic galactic potential $\Omega < \kappa < 2\Omega$, where the limits correspond to the homogeneous mass distribution ($\kappa = 2\Omega$) and the Kepler potential ($\kappa = \Omega$)

In the epicycle approximation the motion is very simple:

$$\begin{aligned} R(t) &= A\cos(\kappa t + a) + R_g \\ z(t) &= B\cos(\nu t + b) \\ \phi(t) &= \Omega_g t + \phi_0 - \frac{2\Omega_g A}{\kappa R_g}\sin(\kappa t + a) \end{aligned}$$

with A, B, a , b, and ϕ_0 all constants. The ϕ-motion follows from

$$\dot{\phi} = \frac{L_z}{R^2} = \frac{L_z}{R_g^2}\left(1 + \frac{x}{R_g}\right)^{-2} \simeq \Omega_g\left(1 - \frac{2x}{R_g}\right)$$

Note that there are three frequencies (Ω, κ, ν) and also three isolating integrals of motion in involution: (E_R, E_z, L_z) with $E_R = \frac{1}{2}(\dot{x}^2 + \kappa^2 x^2)$ and $E_z = \frac{1}{2}(\dot{z}^2 + \nu^2 z^2)$ ▷ all orbits are regular.

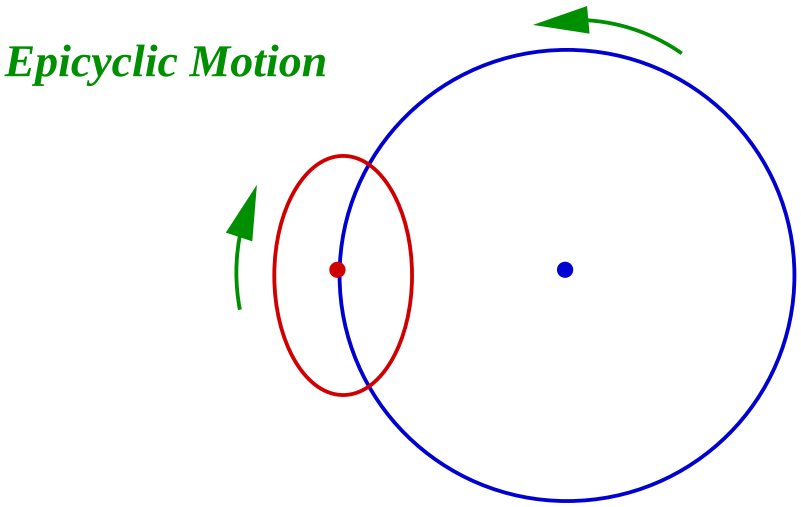
The motion in (R, ϕ) can be described as retrogate motion on an ellipse (the epicycle), whose guiding center (or epicenter) is in prograde motion around the center of the system.

Epicycle Approximation IV

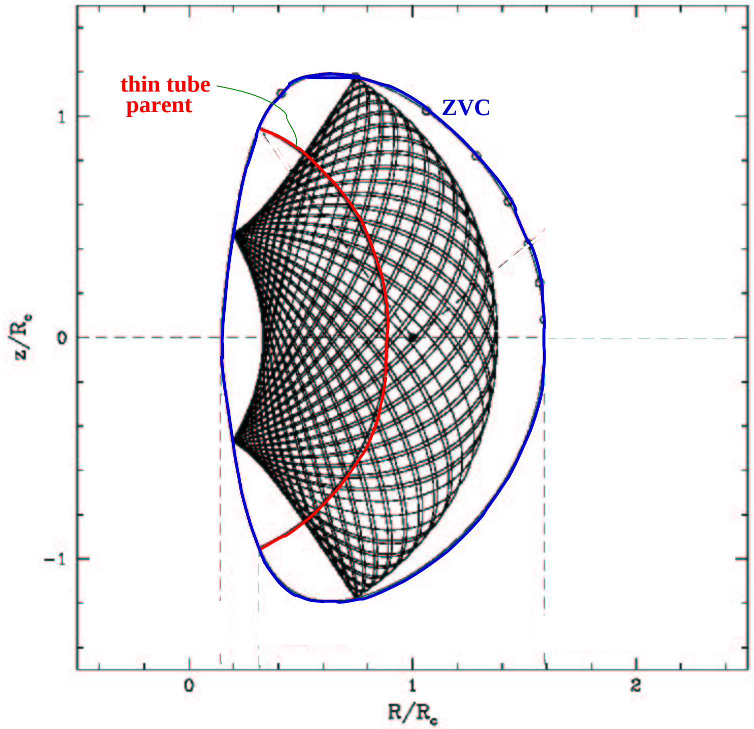
An important question is: "When is the epicycle approximation valid?"

First consider the z-motion: The equation of motion, $\ddot{z} = -\nu^2 z$ implies a constant density in the z-direction. Hence, the epicycle approximation is valid as long as $\rho(z)$ is roughly constant. This is only approximately true very close to equatorial plane. In general, however, epicycle approx. is poor for motion in z-direction.

In the radial direction, we have to realize that the Taylor expansion is only accurate sufficiently close to $R = R_g$. Hence, the epicycle approximation is only valid for small librations around the guiding center; i.e., for orbits with an angular momentum that is close to that of the corresponding circular orbit.



Orbits in Axisymmetric Potentials III

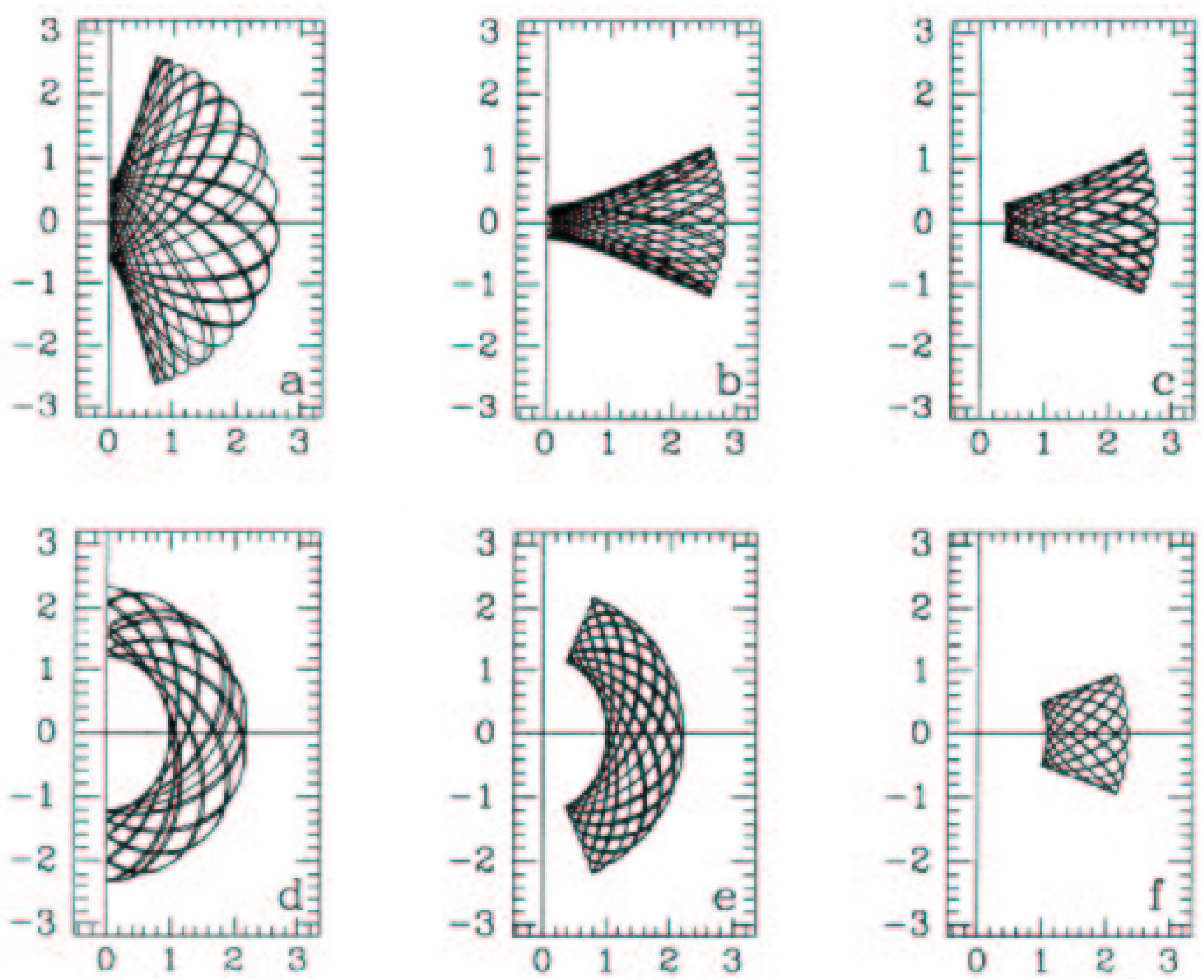


Typical orbit in axisymmetric potential. If orbit admits two isolating integrals of motion, it would (ultimately) fill entire area within ZVC. Rather, orbit is restricted to sub-area within ZVC, indicating that orbit admits a third isolating integral of motion.

Since this is not a classical integral of motion, and we don't know how to express it in terms of the phase-space coordinates, it is simply called I_3.

Note that the point where orbit touches ZVC can be used to 'label' I_3: The set (E, L_z, I_3) uniquely defines an orbit.

Orbits in Axisymmetric Potentials IV



The orbit shown on the previous page is a so-called short-axis tube orbit. This is the main orbit family in oblate potentials, and is associated with (parented by) the circular orbits in equatorial plane.

Orbits (c), (e) and (f) above are from the same orbit family. Orbits (a), (b) and (d) are special in that $L_z = 0$.

Orbits in Axisymmetric Potentials V

Because of the centrifugal barrier only orbits with $L_z = 0$ will be able to come arbitrarily close to the center.

However, not all orbits with $L_z = 0$ are box orbits. There is another family of zero-angular momentum orbits, namely the two-dimensional loop orbits (e.g., orbit (d) on previous page). Their meriodional plane is stationary (i.e., $\dot{\theta} = 0$) and their angular momentum vector is perpendicular to the z-axis. Hence, $I_3 = L$; note that $[L, L_z] = 0$.

Numerous authors have investigated orbits in axisymmetric potentials using numerical techniques. The main conclusions are:

- Most orbits in axisymmetric potentials designed to model elliptical galaxies are regular and appear to respect an effective third integral I_3.
- The principal orbit family in oblate potentials is the short-axis tube family, while two families of inner and outer long-axis tube orbits dominate in prolate potentials.
- In scale-free or cusped potentials several minor orbits families become important. These are the (boxlets) associated with resonant parents.
- The fraction of phase-space occupied by stochastic, irregular orbits is generally (surprisingly) small.

Orbits in Triaxial Potentials I

Consider a triaxial density distribution with the major, intermediate, and minor axes aligned with the x, y, and z axes, respectively.

In general, triaxial galaxies have four main orbit families: box orbits, and three tube orbits: short axis tubes, inner long-axis tubes, and outer long-axis tubes.

Orbit structure different in cusp, core, main body, and outer part (halo).

In central core, potential is harmonic, and motion is that of a 3D harmonic oscillator. $\triangleright$ all orbits are box orbits, parented by stable long-axial orbit

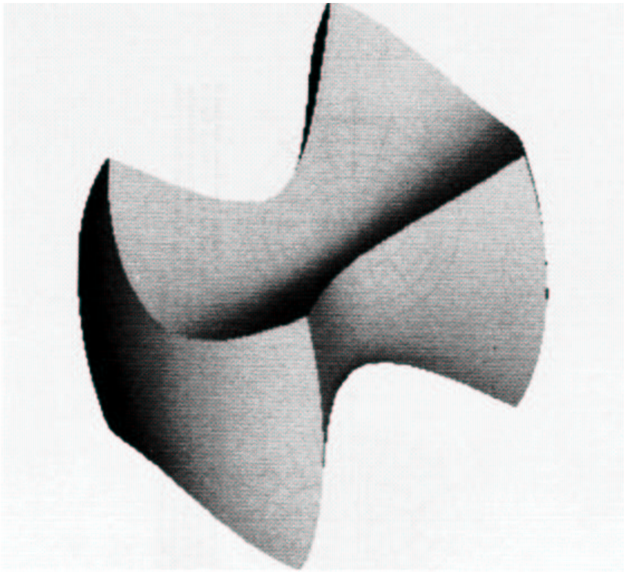
Outside of core region, frequencies become strongly radius (energy) dependent. There comes an energy where $\omega_x = \omega_y$. At this $1:1$-resonance the y-axial orbit becomes unstable and bifurcates into short-axis tube family (two subfamilies with opposite sense of rotation).

At even higher E the $\omega_y : \omega_z = 1:1$ resonance makes z-axial orbit unstable $\longrightarrow$ inner and outer long-axis tube families (each with two subfamilies with opposite sense of rotation).

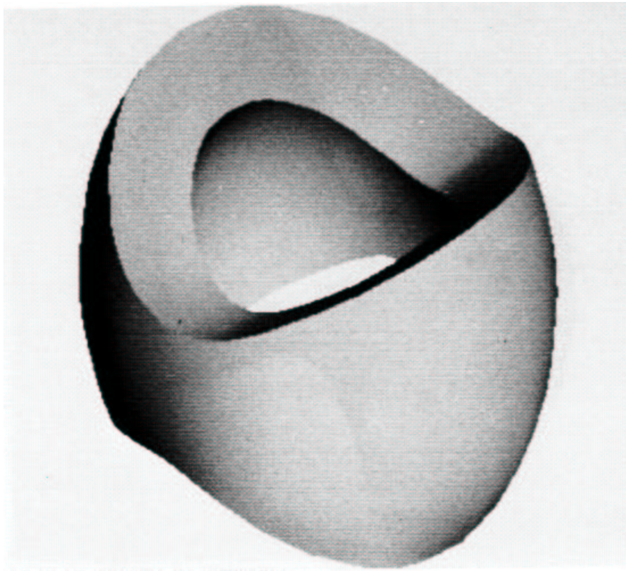
At even larger radii (in 'halo' of triaxial system) the x-axial orbit becomes unstable $\triangleright$ box orbits are replaced by boxlets and stochastic orbits. The three families of tube orbits are also present

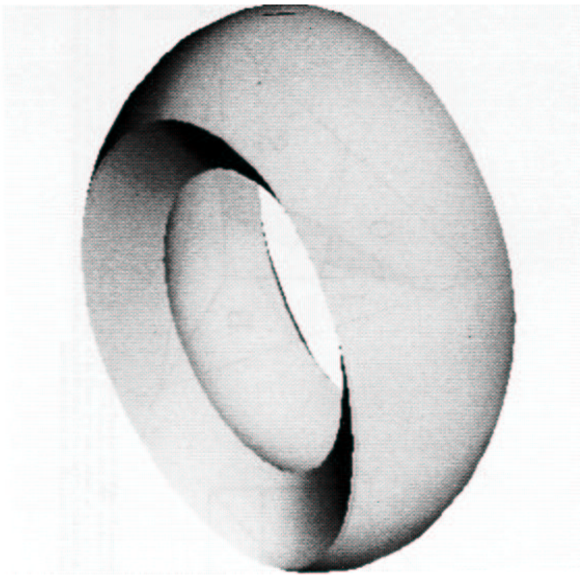
Orbits in Triaxial Potentials II

box orbit

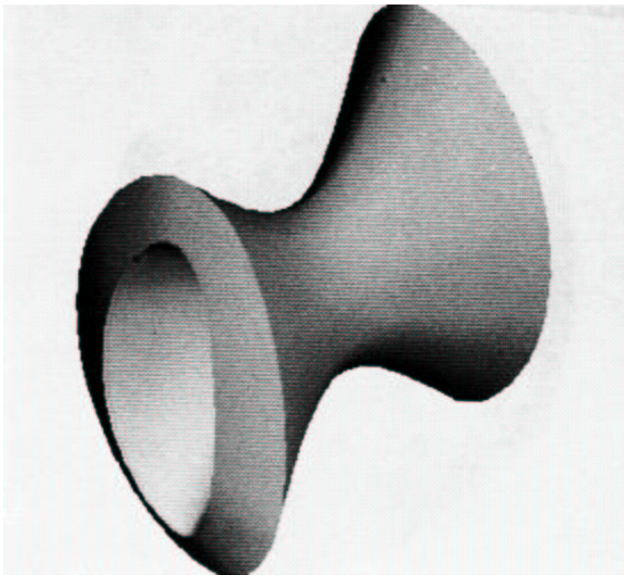


short−axis tube orbit





outer long−axis tube orbit



inner long−axis tube orbit

Orbits in Triaxial Potentials III

If center is cusped rather than cored, resonant orbits families (boxlets) and stochastic orbits take over part of phase-space formerly held by box orbits. The extent to which this happens depends on cusp slope.

Short-axis tubes contribute angular momentum in z-direction; Long-axis tubes contribute angular momentum in x-direction $\triangleright$ total angular momentum vector may point anywhere in plane containing long and short axes. NOTE: this can serve as kinematic signature of triaxiality.

The closed loop orbit around the intermediate y-axis is unstable $\triangleright$ no family of intermediate-axis tubes.

Gas moves on closed, non-intersecting orbits. The only orbits with these properties are the stable loop orbits around x- and z-axes. Consequently, gas and/or dust disks in triaxial galaxies can exist in xy-plane and yz-plane, but not in xz-plane. NOTE: these disks must be ellipsoidal rather than circular, and the velocity varies along ellipsoids.

Stäckel Potentials I

Useful insight may be obtained from separable Stäckel models. These are the only known triaxial potentials that are completely integrable.

In Stäckel potentials all orbits are regular and part of one of the four main families.

Stäckel potentials are separable in ellipsoidal coordinates (λ, μ, ν)

(λ, μ, ν) are the roots for τ of

$$\frac{x^2}{\tau+\alpha} + \frac{y^2}{\tau+\beta} + \frac{z^2}{\tau+\gamma} = 1$$

Here $\alpha < \beta < \gamma$ are constants and $-\gamma \leq \nu \leq -\beta \leq \mu \leq -\alpha \leq \lambda$

Surfaces of constant λ are ellipsoids

Surfaces of constant μ are hyperboloids of one sheet

Surfaces of constant ν are hyperboloids of two sheets

Stäckel potentials are of the form:

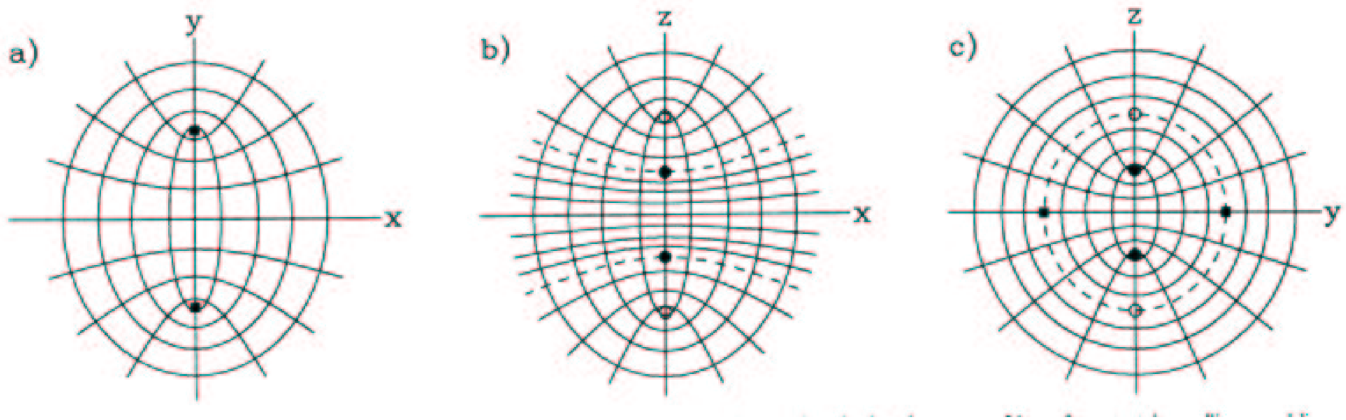
$$\Phi(\vec{r}) = \Phi(\lambda, \mu, \nu) = -\frac{F_1(\lambda)}{(\lambda-\mu)(\lambda-\nu)} - \frac{F_2(\mu)}{(\mu-\nu)(\mu-\lambda)} - \frac{F_3(\nu)}{(\nu-\lambda)(\nu-\mu)}$$

with F_1, F_2 and F_3 arbitrary functions

Stäckel Potentials II

The figure below shows contours of constant (λ, μ, ν) plotted in the three planes (from left to right) xy, xz and yz

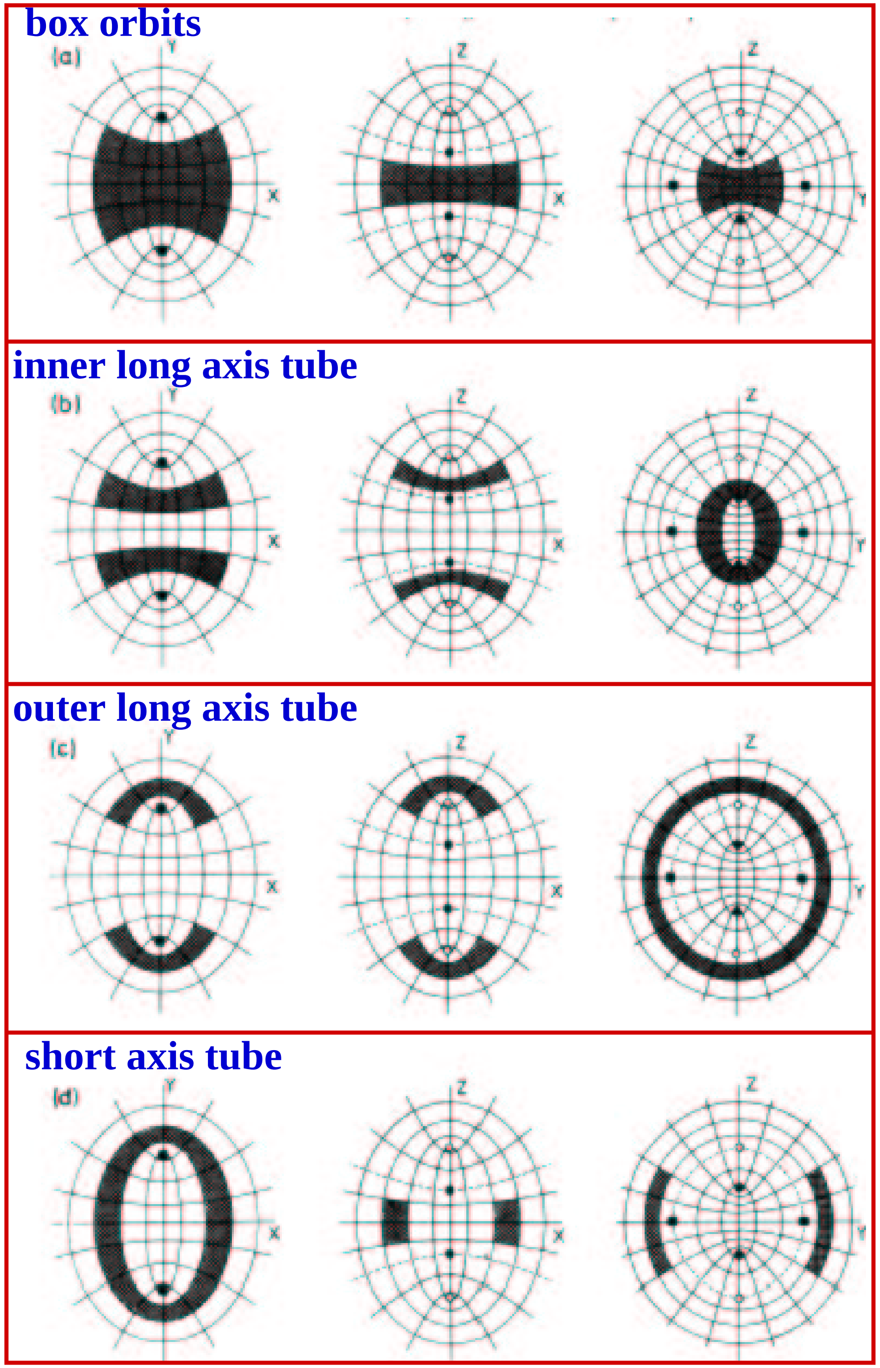
At large distances, the ellipsoidal coordinates become close to spherical. Near the origin they are close to cartesian. For more details, see de Zeeuw, 1985, MNRAS, 216, 273



In triaxial Stäckel potentials all three integrals (E, I_2, I_3) are analytical, and the orbits are confined by contours of constant ellipsoidal coordinates (see next page).

Although Stäckel are a very special class, the fact that they are separable makes them ideally suited to get insight. Most triaxial potentials that do not have a Stäckel form have orbital structures that are similar to that of Stäckel potentials.

Stäckel Potentials III



Orbits in Ellipsoid Land; Summary

System	Dim	Orbit Families
Oblate	$3D$	S
Prolate	$3D$	$I + O$
Triaxial	$3D$	$S + I + O + B$
Elliptic Disk	$2D$	$S + B$

B = **box orbits**

S = **short-axis tubes**

I = **inner long-axis tubes**

O = **outer long-axis tubes**

Libration versus Rotation

Three-dimensional orbits

All tube orbits are build up from $\mathbf{2}$ librations and $\mathbf{1}$ rotation.

All box orbits are build up from $\mathbf{3}$ librations.

All boxlets are build up from $\mathbf{2}$ librations and $\mathbf{1}$ rotation.

Two-dimensional orbits

All loop orbits are build up from $\mathbf{1}$ libration and $\mathbf{1}$ rotation.

All box orbits are build up from $\mathbf{2}$ librations.

All boxlets are build up from $\mathbf{2}$ librations.

Rotating Potentials I

The figures of non-axisymmetric potentials may rotate with respect to inertial space.

The example of interest for astronomy are barred potentials, which are rotating with a certain pattern speed.

We express the pattern speed in angular velocity $\vec{\Omega}_p = \Omega_p \vec{e}_z$

In what follows we denote by $\mathrm{d}\vec{a}/\mathrm{d}t$ **the rate of change of a vector** $\vec{a}$ **as measured by an inertial observer, and by** $\dot{\vec{a}}$ **the rate of change as measured by an observer corotating with the figure.**

It is straightforward to show that

$$\boxed{\frac{\mathrm{d}\vec{a}}{\mathrm{d}t} = \dot{\vec{a}} + \vec{\Omega}_p \times \vec{a}}$$

Applying this twice to the position vector $\vec{r}$**, we obtain**

$$\begin{aligned}
\frac{\mathrm{d}^2\vec{r}}{\mathrm{d}t^2} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{\vec{r}} + \vec{\Omega}_p \times \vec{r}\right) \\
&= \ddot{\vec{r}} + \vec{\Omega}_p \times \dot{\vec{r}} + \vec{\Omega}_p \times \frac{\mathrm{d}\vec{r}}{\mathrm{d}t} \\
&= \ddot{\vec{r}} + \vec{\Omega}_p \times \dot{\vec{r}} + \vec{\Omega}_p \times \left(\dot{\vec{r}} + \vec{\Omega}_p \times \vec{r}\right) \\
&= \ddot{\vec{r}} + 2\left(\vec{\Omega}_p \times \dot{\vec{r}}\right) + \vec{\Omega}_p \times \left(\vec{\Omega}_p \times \vec{r}\right)
\end{aligned}$$

Rotating Potentials II

Since Newton's laws apply to inertial frames we have that

$$\ddot{\vec{r}} = -\vec{\nabla}\Phi - 2\left(\vec{\Omega}_p \times \dot{\vec{r}}\right) - \vec{\Omega}_p \times \left(\vec{\Omega}_p \times \vec{r}\right)$$

Note the two extra terms: $-2\left(\vec{\Omega}_p \times \dot{\vec{r}}\right)$ represents the Coriolis force and $-\vec{\Omega}_p \times \left(\vec{\Omega}_p \times \vec{r}\right)$ the centrifugal force.

The energy is given by

$$\begin{aligned}
E &= \frac{1}{2}\left(\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\right)^2 + \Phi(\vec{r}) \\
&= \frac{1}{2}\left(\dot{\vec{r}} + \vec{\Omega}_p \times \vec{r}\right)^2 + \Phi(\vec{r}) \\
&= \frac{1}{2}\dot{\vec{r}}^2 + \dot{\vec{r}}\cdot\left(\vec{\Omega}_p \times \vec{r}\right) + \frac{1}{2}\left(\vec{\Omega}_p \times \vec{r}\right)^2 + \Phi(\vec{r}) \\
&= \frac{1}{2}\dot{\vec{r}}^2 + \dot{\vec{r}}\cdot\left(\vec{\Omega}_p \times \vec{r}\right) + \frac{1}{2}|\vec{\Omega}_p \times \vec{r}|^2 + \Phi(\vec{r}) \\
&= E_J + \dot{\vec{r}}\cdot\left(\vec{\Omega}_p \times \vec{r}\right) + |\vec{\Omega}_p \times \vec{r}|^2
\end{aligned}$$

Where we have defined Jacobi's Integral

$$E_J \equiv \frac{1}{2}\dot{\vec{r}}^2 + \Phi(\vec{r}) - \frac{1}{2}|\vec{\Omega}_p \times \vec{r}|^2$$

Rotating Potentials III

The importance of E_J becomes apparent from the following:

$$\begin{aligned}
\frac{\mathrm{d}E_J}{\mathrm{d}t} &= \dot{\vec{r}}\frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{\vec{r}}\right) + \frac{\mathrm{d}\Phi}{\mathrm{d}t} - (\vec{\Omega}_p \times \vec{r}) \cdot \frac{\mathrm{d}}{\mathrm{d}t}(\vec{\Omega}_p \times \vec{r}) \\
&= \dot{\vec{r}}\left[\ddot{\vec{r}} + (\vec{\Omega}_p \times \dot{\vec{r}})\right] + \vec{\nabla}\Phi \cdot \dot{\vec{r}} - (\vec{\Omega}_p \times \vec{r}) \cdot (\vec{\Omega}_p \times \dot{\vec{r}}) \\
&= \dot{\vec{r}} \cdot \ddot{\vec{r}} + \dot{\vec{r}} \cdot \vec{\nabla}\Phi - (\vec{\Omega}_p \times \vec{r}) \cdot (\vec{\Omega}_p \times \dot{\vec{r}})
\end{aligned}$$

Here we have used that $\vec{A} \cdot (\vec{A} \times \vec{B}) = 0$ and that $\mathrm{d}\vec{\Omega}_p/\mathrm{d}t = 0$. If we multiply the equation of motion with $\dot{\vec{r}}$ we obtain that

$$\begin{aligned}
& \dot{\vec{r}} \cdot \ddot{\vec{r}} + \dot{\vec{r}} \cdot \vec{\nabla}\Phi + 2\dot{\vec{r}} \cdot (\vec{\Omega}_p \times \dot{\vec{r}}) + \dot{\vec{r}} \cdot \left[\vec{\Omega}_p \times (\vec{\Omega}_p \times \vec{r})\right] = 0 \\
\Leftrightarrow \quad & \dot{\vec{r}} \cdot \ddot{\vec{r}} + \dot{\vec{r}} \cdot \vec{\nabla}\Phi + (\vec{\Omega}_p \times \vec{r}) \cdot (\dot{\vec{r}} \times \vec{\Omega}_p) = 0
\end{aligned}$$

Where we have used that $\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{C} \cdot (\vec{A} \times \vec{B})$. Since $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$ we have that

$$\boxed{\frac{\mathrm{d}E_J}{\mathrm{d}t} = 0}$$

The Jacobi Integral is a conserved quantity, i.e. an integral of motion.

Rotating Potentials IV

For comparison, since $\Phi = \Phi(t)$ the Hamiltonian is explictely time-dependent; consequently, the total energy E is not a conserved quantity (i.e., is not an integral of motion).

The angular momentum is given by

$$\vec{L} = \vec{r} \times \frac{d\vec{r}}{dt} = \vec{r} \times \dot{\vec{r}} + \vec{r} \times (\vec{\Omega}_p \times \vec{r})$$

This allows us to write

$$\begin{aligned} \vec{\Omega}_p \cdot \vec{L} &= \vec{\Omega}_p \cdot (\vec{r} \times \dot{\vec{r}}) + \vec{\Omega}_p \cdot \left[\vec{r} \times (\vec{\Omega}_p \times \vec{r})\right] \\ &= \dot{\vec{r}} \cdot (\vec{\Omega}_p \times \vec{r}) + |\vec{\Omega}_p \times \vec{r}|^2 \end{aligned}$$

from which we obtain that

$$\boxed{E_J = E - \vec{\Omega}_p \cdot \vec{L}}$$

Thus, in a rotating, non-axisymmetric potential neither energy E not angular momentum L are conserved, but the Jacobi integral $E_J = E - \vec{\Omega}_p \cdot \vec{L}$ is.

Note that E_J is the sum of $\frac{1}{2}\dot{\vec{r}}^2 + \Phi$, which would be the energy if the frame were not rotating, and the quantity $-\frac{1}{2}|\vec{\Omega}_p \times \vec{r}|^2 = -\frac{1}{2}\Omega_p^2 R^2$, which can be thought of as the potential energy corresponding to the centrifugal force.

Rotating Potentials V

If we now define the effective potential

$$\Phi_{\rm eff} = \Phi - \tfrac{1}{2}\Omega_p^2 R^2$$

the equation of motion becomes

$$\ddot{\vec{r}} = -\vec{\nabla}\Phi_{\rm eff} - 2(\vec{\Omega}_b \times \dot{\vec{r}})$$

and the Jacobi integral is $E_J = \frac{1}{2}|\dot{\vec{r}}|^2 + \Phi_{\rm eff}$

An orbit with a given value for it's Jacobi Integral is restricted in its motion to regions in which $E_J \leq \Phi_{\rm eff}$. **The surface** $\Phi_{\rm eff} = E_J$ **is therefore often called the zero-velocity surface.**

The effective potential has five points at which both $\partial\Phi_{\rm eff}/\partial x$ **and** $\partial\Phi_{\rm eff}/\partial y$ **vanish. These points,** L_1 **to** L_5**, are called the Lagrange Points (cf. restricted three-body problem).**

- **Motion around** L_3 **(minimum of** $\Phi_{\rm eff}$**) always stable.**
- **Motion around** L_1 **and** L_2 **(saddle points of** $\Phi_{\rm eff}$**) always unstable.**
- **Motion around** L_4 **and** L_5 **(maxima of** $\Phi_{\rm eff}$**) can be stable or unstable depending on potential.**

NOTE: stable/unstable refers to whether orbits remain close to Lagrange points or not.

Lagrange Points

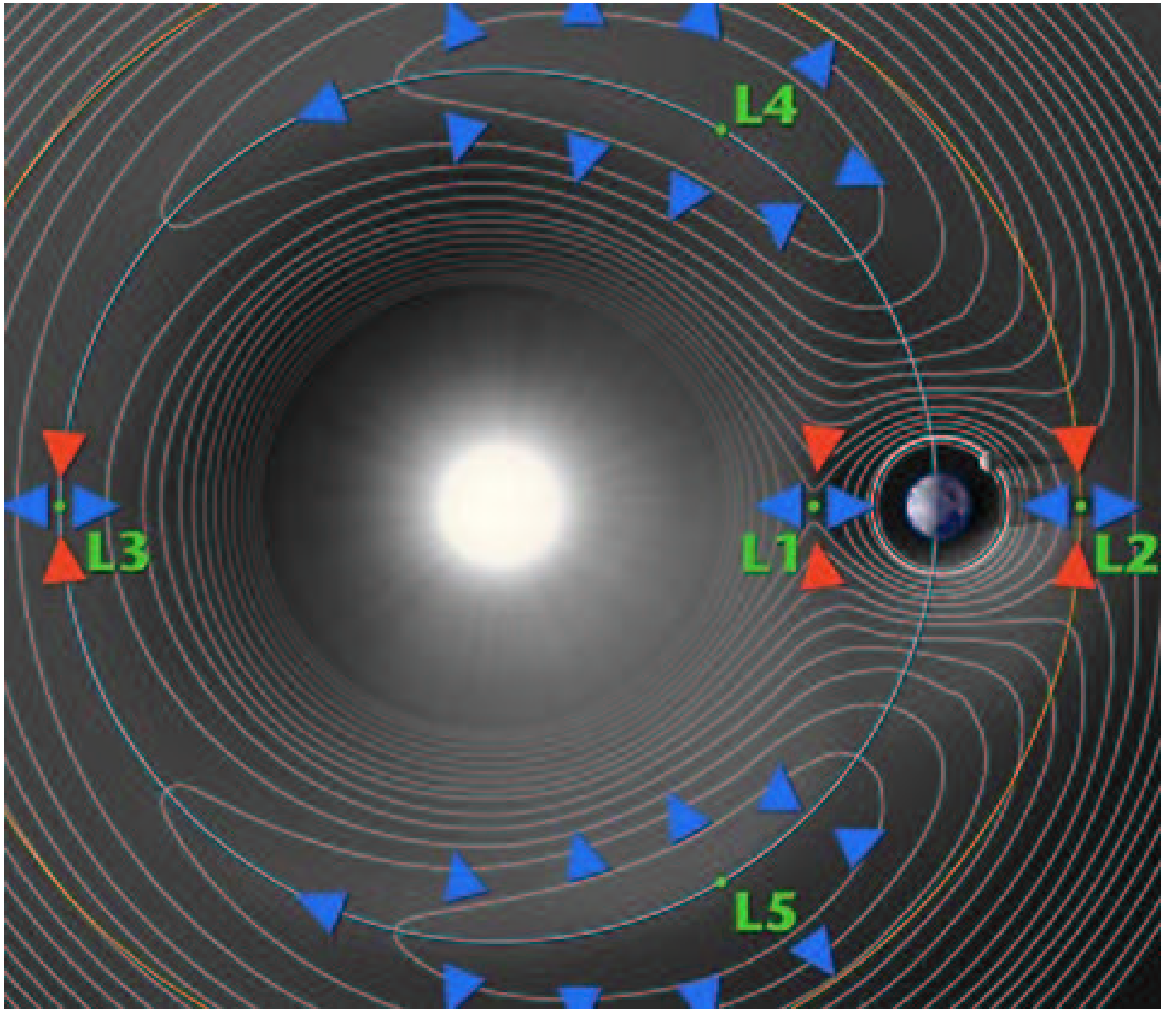


Illustration of Lagrange points (L_1 to L_5) in Sun-Earth-Moon system.

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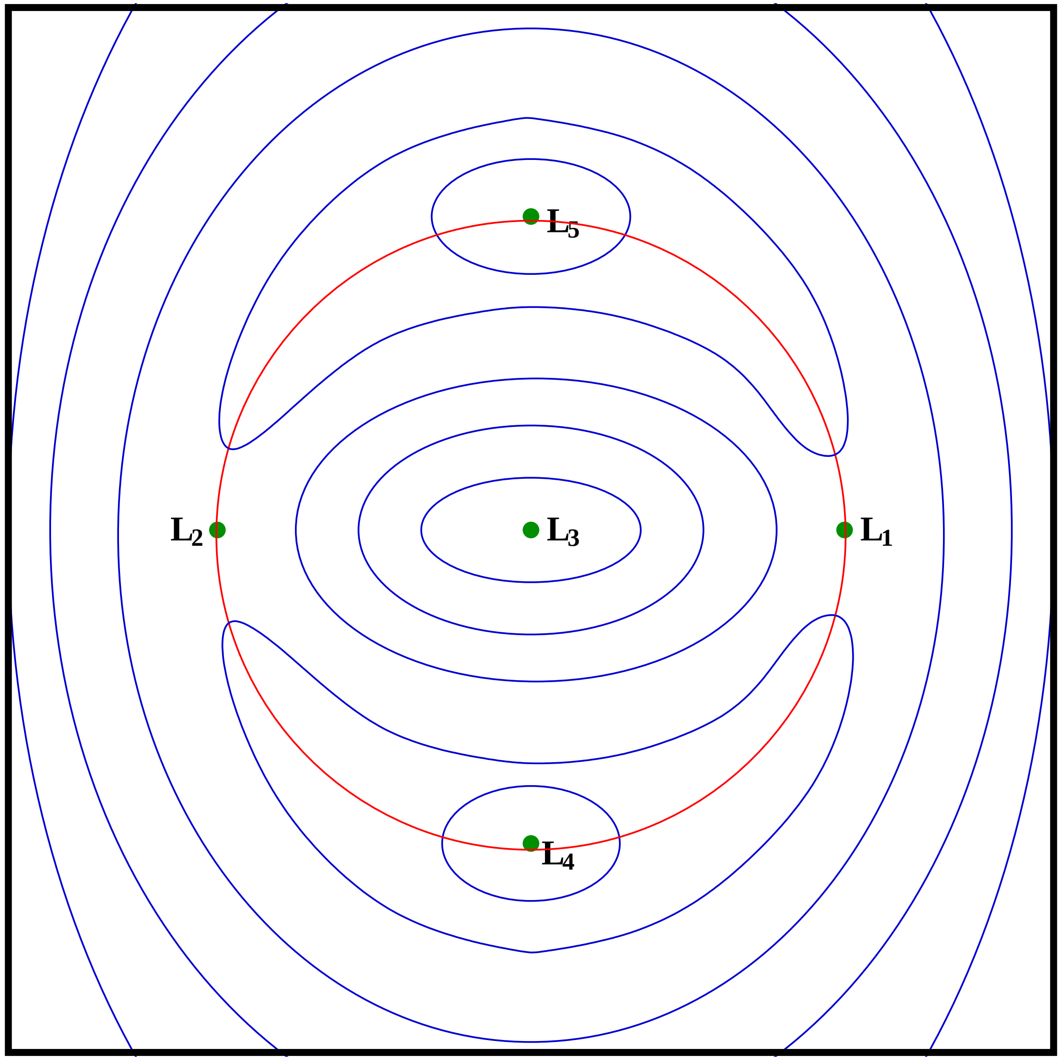


Illustration of Lagrange points (L_1 to L_5) in logarithmic potential. The annulus bounded by circles through L_1, L_2 and L_3, L_4 (depicted as red circle) is called the region of corotation.

Lindblad Resonances I

Let (R, θ) be the polar coordinates that are corotating with the planar potential $\Phi(R, \theta)$. If the non-axisymmetric distortions of the potential, which has a pattern speed Ω_p, is sufficiently small then we may write

$$\Phi(R, \theta) = \Phi_0(R) + \Phi_1(R, \theta) \qquad\qquad |\Phi_1/\Phi_0| \ll 1$$

It is useful to consider the following form for Φ_1

$$\Phi_1(R, \theta) = \Phi_p(R)\cos(m\theta)$$

where $m = 2$ corresponds to a (weak) bar.

In the epicycle approximation the motion in $\Phi_0(R)$ is that of an epicycle, with frequency $\kappa(R)$, around a guiding center which rotates with frequency $\Omega(R) = \sqrt{\frac{1}{R}\frac{\mathrm{d}\Phi_0}{\mathrm{d}R}}$.

In presence of $\Phi_1(R, \theta)$, movement of guiding center is $\theta_0(t) = [\Omega(R) - \Omega_p]\, t$. In addition to natural frequencies $\Omega(R)$ and $\kappa(R)$ there is new frequency Ω_p. Because $\Phi_1(R, \theta)$ has m-fold symmetry, guiding center at R finds itself at effectively same location in (R, θ)-plane with frequency $m\,[\Omega(R) - \Omega_p]$.

Lindblad Resonances II

Motion in R-direction becomes that of harmonic oscillator of natural frequency $\kappa(R)$ that is driven by frequency $m\,[\Omega(R) - \Omega_p]$.

At several R the natural and driving frequencies are in resonance.

(1) Corotation: $\Omega(R) = \Omega_p$
(Guiding center corotates with potential).

(2) Lindblad Resonances: $m\,[\Omega(R) - \Omega_p] = \pm\kappa(R)$
Most important of these are:

$\Omega(R) - \frac{\kappa}{2} = \Omega_p$: Inner Lindblad Resonance

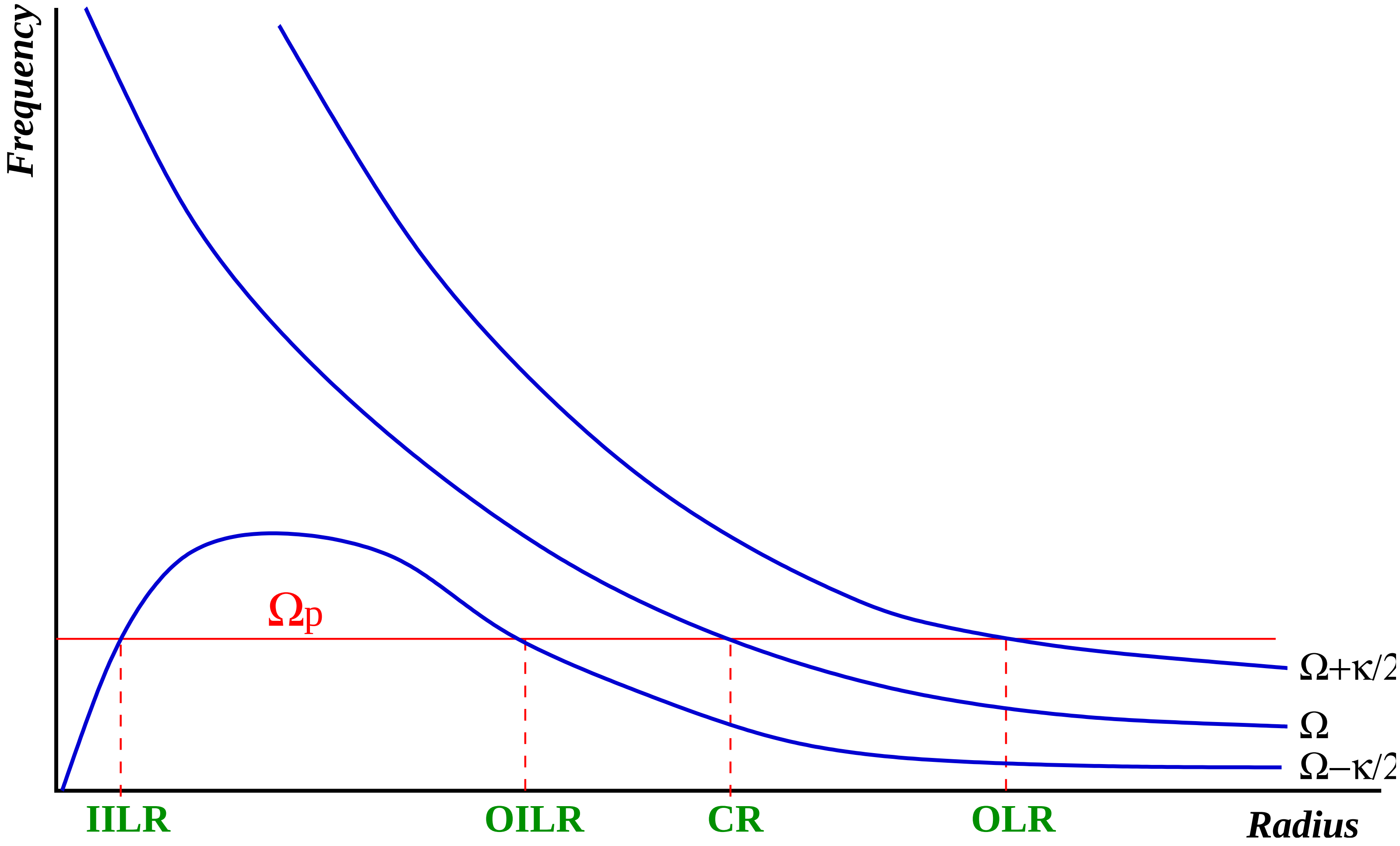
$\Omega(R) + \frac{\kappa}{2} = \Omega_p$: Outer Lindblad Resonance

$\Omega(R) - \frac{\kappa}{4} = \Omega_p$: Ultra Harmonic Resonance

Depending on $\Phi(R,\theta)$ and Ω_p one can have 0, 1, or 2 ILRs. If there are two, we distinguish between Inner Inner Lindblad Resonance (IILR) and Outer Inner Lindblad Resonance (OILR).

If cusp (or BH) is present there is always 1 ILR, because $\Omega(R) - \kappa(R)/2$ increases monotonically with decreasing R.

Lindblad Resonances III



Lindblad Resonances play important role for orbits in barred potentials.

Lindblad Resonances IV

As an example, we discus the orbital families in a planar, rotating, logarithmic potential

(a) Long-axial orbit $\longrightarrow$ stable, oval, prograde, and oriented $\parallel$ to $\Phi_{\rm eff}$. (x_1-family).

(b) Short-axial orbit $\longrightarrow$ stable, oval, retrograde, and oriented $\perp$ to $\Phi_{\rm eff}$.

At $E > E_1$ (at IILR), family (b) becomes unstable and bifurcates into two prograde loop families that are oriented perpendicular to $\Phi_{\rm eff}$. The stable (unstable) family is called the x_2 (x_3) family. At the same energy the x_1-orbits develop self-intersecting loops.

At $E > E_2$ (at OILR) the x_2 and x_3 families dissapear. The x_1 family looses its self-intersecting loops.

In vicinity of corotation annulus there are families of orbits around L_4 and L_5 (if these are stable).

At large radii beyond CR $\Omega_p \gg \Omega(R)$. Consequently, the orbits effectively see a circular potential and the orbits become close to circular rosettes.