Surfaces of Section I

Consider a system with $n = 2$ degrees of freedom (e.g., planar motion), and with a Hamiltonian

$$\mathcal{H}(\vec{x}, \vec{p}) = \frac{1}{2}(p_x^2 + p_y^2) + \Phi(x, y)$$

Conservation of energy, $E = \mathcal{H}$, restricts the motion to a three-dimensional hyper-surface $\mathcal{M}_3$ in four-dimensional phase-space.

To investigate whether the orbits admit any additional (hidden) isolating integrals of motion, Poincaré introduced the surface-of-section (SOS)

Consider the intersection of $\mathcal{M}_3$ with the surface $y = 0$. Integrate the orbit, and everytime it crosses the surface $y = 0$ with $\dot{y} > 0$, record the position in the (x, p_x)-plane. After many orbital periods. the accumulated points begin to show some topology that allows one to discriminate between regular, irregular and resonance orbits.

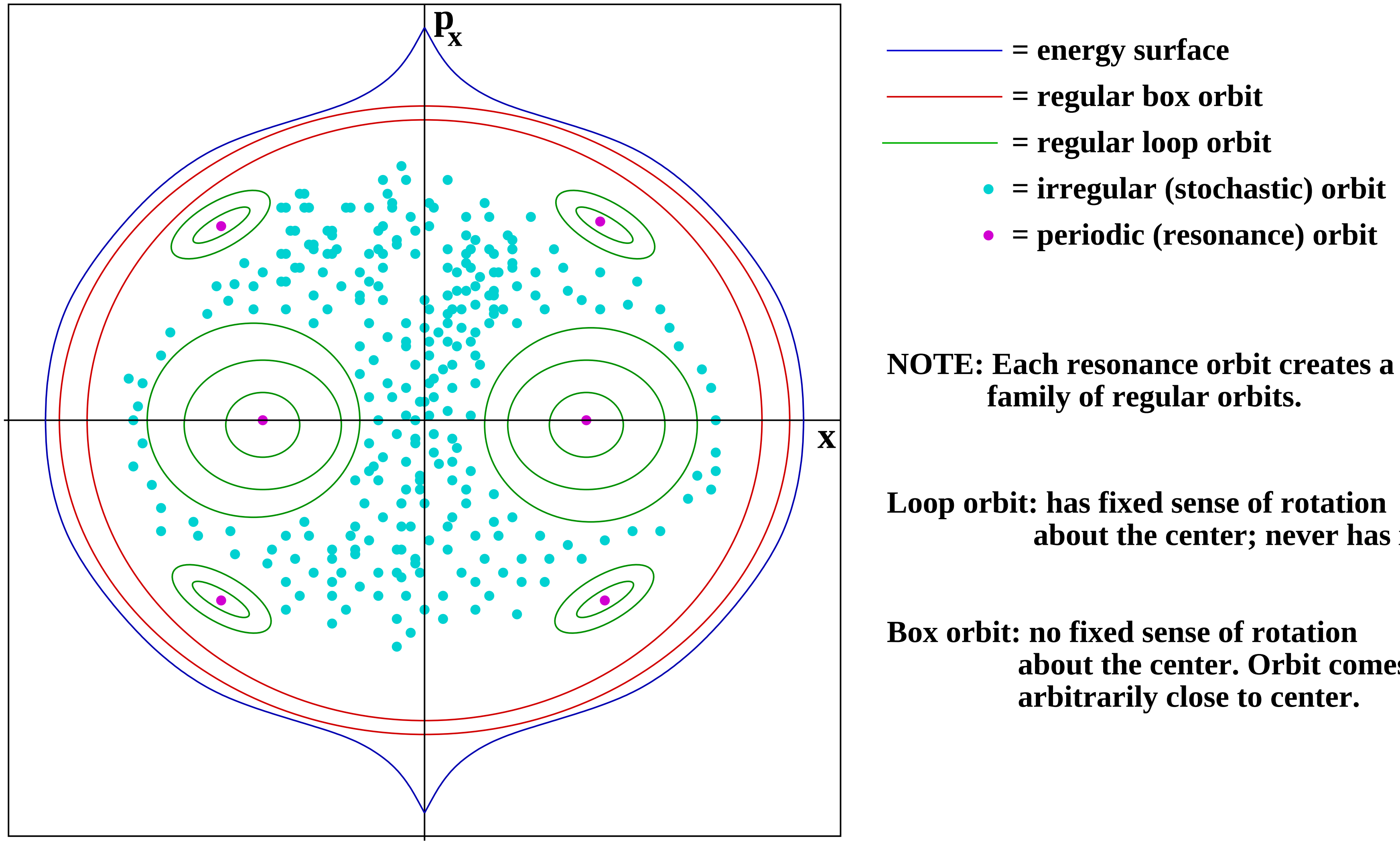
Given (x, p_x) and the condition $y = 0$, we can determine p_y from

$$p_y = +\sqrt{2[E - \Phi(x, 0)] - p_x^2}$$

where the $+$-sign is chosen because $\dot{y} > 0$.

To get insight, and relate orbits to their SOSs, see JAVA-Applet at:
http://burro.astr.cwru.edu/JavaLab/SOSweb/backgrnd.html

Surfaces of Section II



NOTE: Each resonance orbit creates a family of regular orbits.

Loop orbit: has fixed sense of rotation about the center; never has x=

Box orbit: no fixed sense of rotation about the center. Orbit comes arbitrarily close to center.

This figure is only an illustration of the topology of various orbits in a SOS. It does not correspond to an existing Hamiltonian.

Orbits in Spherical Potentials I

A spherical potential has four classical, isolating integrals of motion: Energy E, associated with time-invariance of the Lagrangian, and the three components of the angular momentum vector, L_x, L_y, and L_z, associated with rotational invariance of the potential.

NOTE: Since

$$[L_x, L_y] = L_z, \quad [L_y, L_z] = L_x, \quad [L_z, L_x] = L_y$$

the set (L_x, L_y, L_z) is not in involution. However, the absolute value of the angular momentum

$$|L| = \sqrt{L_x^2 + L_y^2 + L_z^2}$$

is in involution with any of its components. We can thus define a set of three isolating integrals of motion in involution, e.g., $(E, |L|, L_z)$. The values for these three integrals uniquely determine the motion, and specify a unique invariant torus.

Orbits in Spherical Potentials II

We have seen before that for motion in a central force field:

$$\frac{dr}{dt} = \pm\sqrt{2[E - \Phi(r)] - \frac{L^2}{r^2}} \qquad \frac{d\theta}{dt} = \frac{L}{r^2}$$

From this we immediately infer the nature of the motion:

- θ-motion is rotation ($d\theta/dt$ is never zero)
- r-motion is libration ($dr/dt = 0$ at apo- and pericenter)

The radial period is

$$T_r = 2\int_{r_-}^{r_+} \frac{dr}{dr/dt} = 2\int_{r_-}^{r_+} \frac{dr}{\sqrt{2[E-\Phi(r)]-L^2/r^2}}$$

In the same period the polar angle θ increases by an amount

$$\Delta\theta = 2\int_{r_-}^{r_+} \frac{d\theta}{dr} dr = 2\int_{r_-}^{r_+} \frac{d\theta}{dt}\frac{dt}{dr} dr = 2\int_{r_-}^{r_+} \frac{L\,dr}{r^2\sqrt{2[E-\Phi(r)]-L^2/r^2}}$$

The azimuthal period can thus be written as $T_\theta = \frac{2\pi}{\Delta\theta} T_r$. From this we see that the orbit will be closed (resonant) if

$$\frac{T_\theta}{T_r} = \frac{2\pi}{\Delta\theta} = \frac{n}{m} \qquad \text{with } n \text{ and } m \text{ integers}$$

Orbits in Spherical Potentials III

In general $\Delta\theta/2\pi$ will not be a rational number. $\triangleright$ orbit not closed.

Instead, a typical orbit resembles a rosette and eventually passes through every point in between the annuli bounded by the apo- and pericenter.

However, there are two special potentials for which all orbits are closed:

- Spherical Harmonic Oscillator Potential: $\Phi(r) = \frac{1}{2}\Omega^2 r^2$

  —Orbits are ellipses centered on center of attraction

  — $T_\theta : T_r = 2 : 1$

- Kepler Potential: $\Phi(r) = -\frac{GM}{r}$

  —Orbits are ellipses with attracting center at one focal point

  — $T_\theta : T_r = 1 : 1$

Since galaxies are less centrally concentrated than point masses and more centrally concentrated than homogeneous spheres, a typical star in a spherical galaxy changes its angular coordinate by $\Delta\theta$ during a radial libration, where $\pi < \Delta\theta < 2\pi$

QUESTION: What is the resonance that corresponds to a circular orbit?

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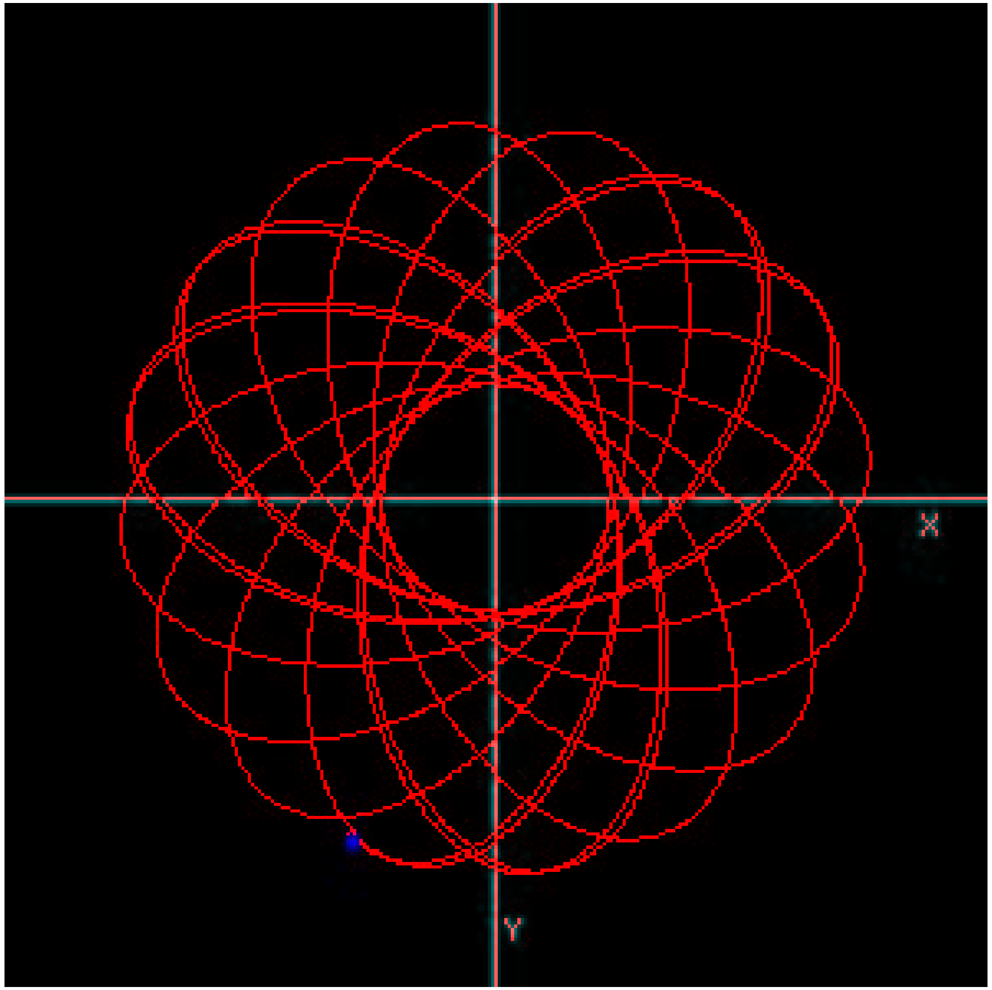
QUESTION: What is the resonance that corresponds to a circular orbit?
ANSWER: Although closed, a circular orbit is NOT a resonance orbit

Orbits in Spherical Potentials IV



An example of a rosette orbit with non-commensurable frequencies. Virtually all orbits in spherical potentials are of this form. The more general name for this type of orbits is loop orbits. They have a net sense of rotation around the center.

Orbits in Planar Potentials I

Before we discuss orbits in less symmetric, three-dimensional potentials, we first focus our attention on Planar Potentials $\Phi(x, y)$. This is useful for the following reasons:

- **There are cases in which the symmetry of the potential allows a reduction of the number of degrees of freedom by means of the effective potential $\Phi_{\rm eff}$. Examples are axisymmetric potentials where $\Phi_{\rm eff}$ allows a study of motion in the so-called meridional plane.**
- **Motion confined to the symmetry-planes of ellipsoidal, spheroidal and spherical potentials is planar.**
- **There are mass distributions of astronomical interest with potentials that are reasonably well approximated by planar potentials (disks).**
- **To get insight into the various orbit families.**

Orbits in Planar Potentials II

As an example, we consider motion in the planar, logarithmic potential

$$\Phi_L(x,y) = \frac{1}{2}\, v_0^2 \ln\left(R_c^2 + x^2 + \frac{y^2}{q^2}\right) \qquad (q \leq 1)$$

This potential has the following properties:

(i) Equipotentials have constant axial ratio q so that influence of non-axisymmetry is similar at all radii

(ii) For $R = \sqrt{x^2 + y^2} \ll R_c$ a power-series expansion gives

$$\Phi_L \simeq \frac{v_0^2}{2R_c^2}\left(x^2 + \frac{y^2}{q^2}\right)$$

which is similar to that of a two-dimensional harmonic oscillator, which corresponds to a homogeneous density distribution.

(iii) For $R \gg R_c$ and $q = 1$ we have that $\Phi_L = \frac{1}{2} v_0^2 \ln R$. One can easily verify that this corresponds to a circular velocity curve $v_{\rm circ}(R) = v_0$; i.e., at large radii Φ_L yields a flat rotation curve, similar to that of disk galaxies.

Orbits in Planar Potentials III

We start our investigation of orbits in $\Phi_L(x, y)$ with those that are confined to $R \ll R_c$, i.e., those confined to the constant density core.

Using series expansion, we can approximate the potential by

$$\Phi_L \simeq \frac{v_0^2}{2R_c^2}\left(x^2 + \frac{y^2}{q^2}\right) = \Phi_1(x) + \Phi_2(y)$$

Note that we can separate the potential. This allows us to immediately identify two isolating integrals of motion in involution from the Hamiltonian:

$$I_1 = p_x^2 + 2\Phi_1(x) \qquad I_2 = p_y^2 + 2\Phi_2(y)$$

The motion of the system is given by the superposition of the librations along the two axes, which are the solutions of the decoupled system of equations

$$\ddot{x} = -\omega_x^2 x \qquad \ddot{y} = -\omega_y^2 y$$

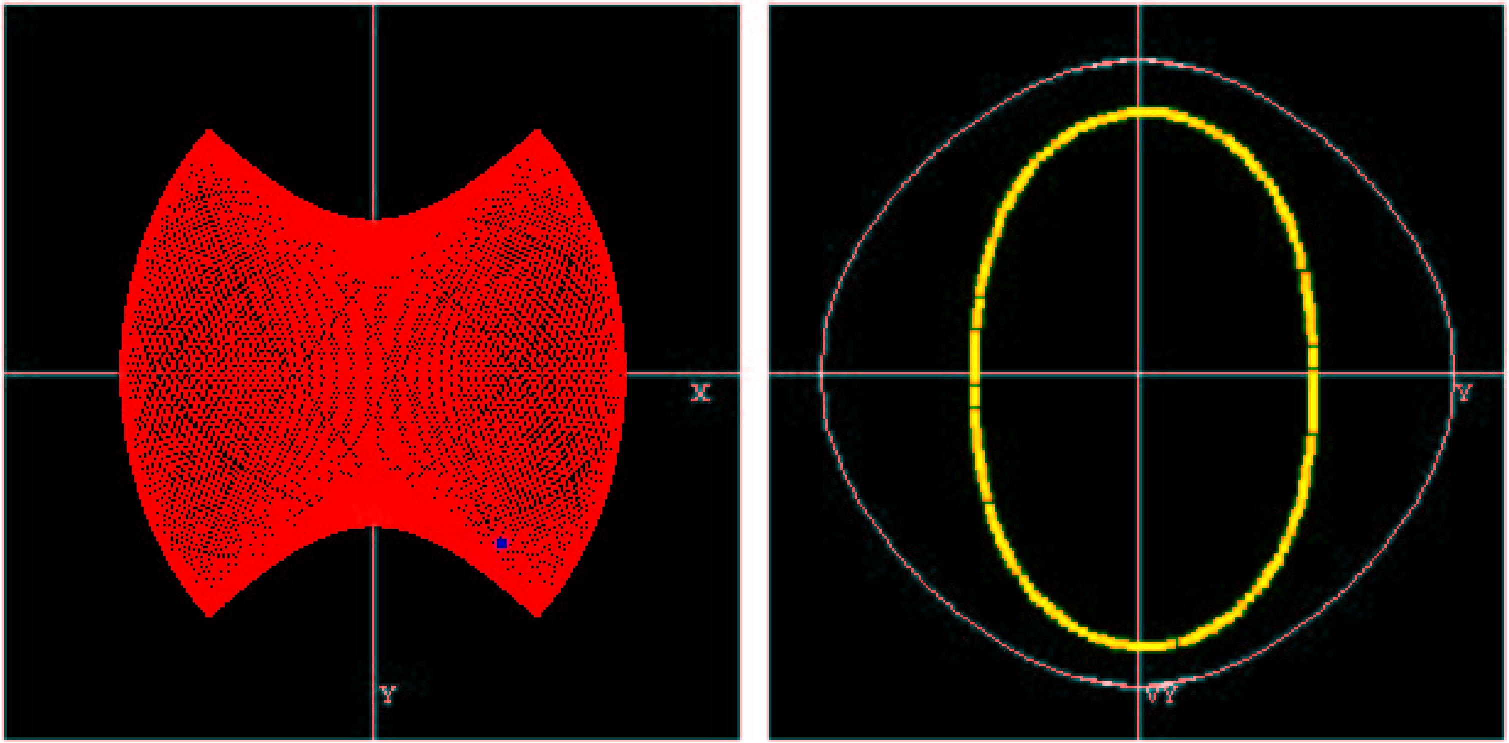
which corresponds to a two-dimensional harmonic oscillator with frequencies $\omega_x = v_0/R_c$ and $\omega_y = v_0/qR_c$. Unless these are incommensurable (i.e., unless $\omega_x/\omega_y = n/m$ for some integers n and m) the star passes close to every point inside a rectangular box.

These orbits are therefore known as box orbits. Such orbits have no net sense of circulation about the center.

Orbits in Planar Potentials IV

For orbits at larger radii $R \gtrsim R_c$ one has to resort to numerical integration.

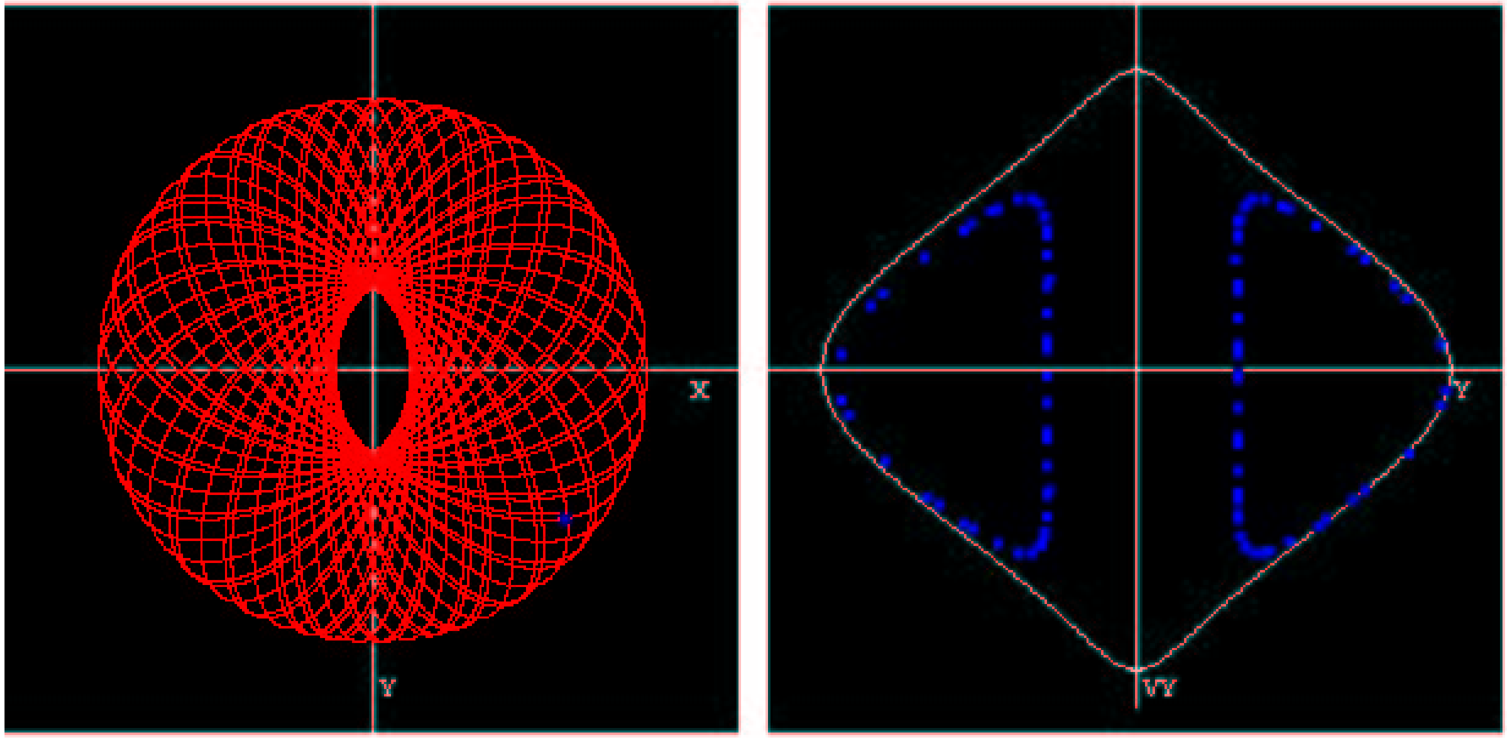
This reveals two major orbit families: The first is the family of box orbits, which have no net sense of cirulation about the center, and which, in the course of time, will pass arbitrarily close to the center of the potential



Note that the orbit completes a filled curve in the SOS, indicating that it admits a second isolating integral of motion, I_2. This is not a classical integral, as it is not associated with a symmetry of the system. We can, in general, not express I_2 in the phase-space coordinates.

Orbits in Planar Potentials V

The second main family is that of loop orbits. These do have a net sense of circulation, and always maintain a minimum distance from the center of the potential. Any star launched from $R \gg R_c$ in the tangential direction with a speed of the order of v_0 will follow such a loop orbit.



Once again, the fact that the orbit completes a filled curve in the SOS, indicates that it admits a second, (non-classical) isolating integral of motion. Since we don't know what this integral is (in terms of the phase-space coordinates) it is simply called I_2.

Orbits in Planar Potentials VI

In $\Phi_L(x, y)$ there are two main orbit families: loop orbits and box orbits

Each family of orbits is closely associated with a corresponding closed orbit. This closed orbit is called the parent of the orbit family. All closed orbits that are parents to families are said to be stable, in that members of their family that are initially close to them remain close to them at all times. Unstable, closed orbits also exist, but they don't parent an orbit family.

Modulo the irregular orbits, one can obtain a good consensus of the orbits in a system by finding the stable periodic orbits at each energy.

For our planar, logarithmic potential, the parent of the loop orbits is the closed loop orbit (which intersects SOS at a single point on $\dot{x} = 0$ axis).

The parent of the box orbits is the closed long-axis orbit. This is the orbit that is confined to the x-axis with $y = \dot{y} = 0$. In the SOS (x vs. p_x) this is the orbit associated with the boundary curve which corresponds to

$$\frac{1}{2}\dot{x}^2 + \Phi_L(x, 0) = E \qquad \text{i.e. } y = \dot{y} = 0$$

Finally, there is the closed short-axis orbit. This is the orbit that is confined to the y-axis $x = \dot{x} = 0$. In the SOS this is the orbit associated with a single dot exactly at the center of the SOS. Clearly, this orbit is unstable: rather than parenting an orbit family, it marks the transition between loop and box orbits.

Orbits in Planar Potentials VII

If we set the core radius $R_c = 0$ in our planar, logarithmic potential, we remove the homogeneous core and introduce a singular R^{-2} cusp.

This singular logarithmic potential admits a number of new orbit families, which are associated with resonant parents, which we ID by the frequency ratio $\omega_x : \omega_y$.

The family associated with the $2:1$ resonance are called the banana orbits

The family associated with the $3:2$ resonance are called the fish orbits

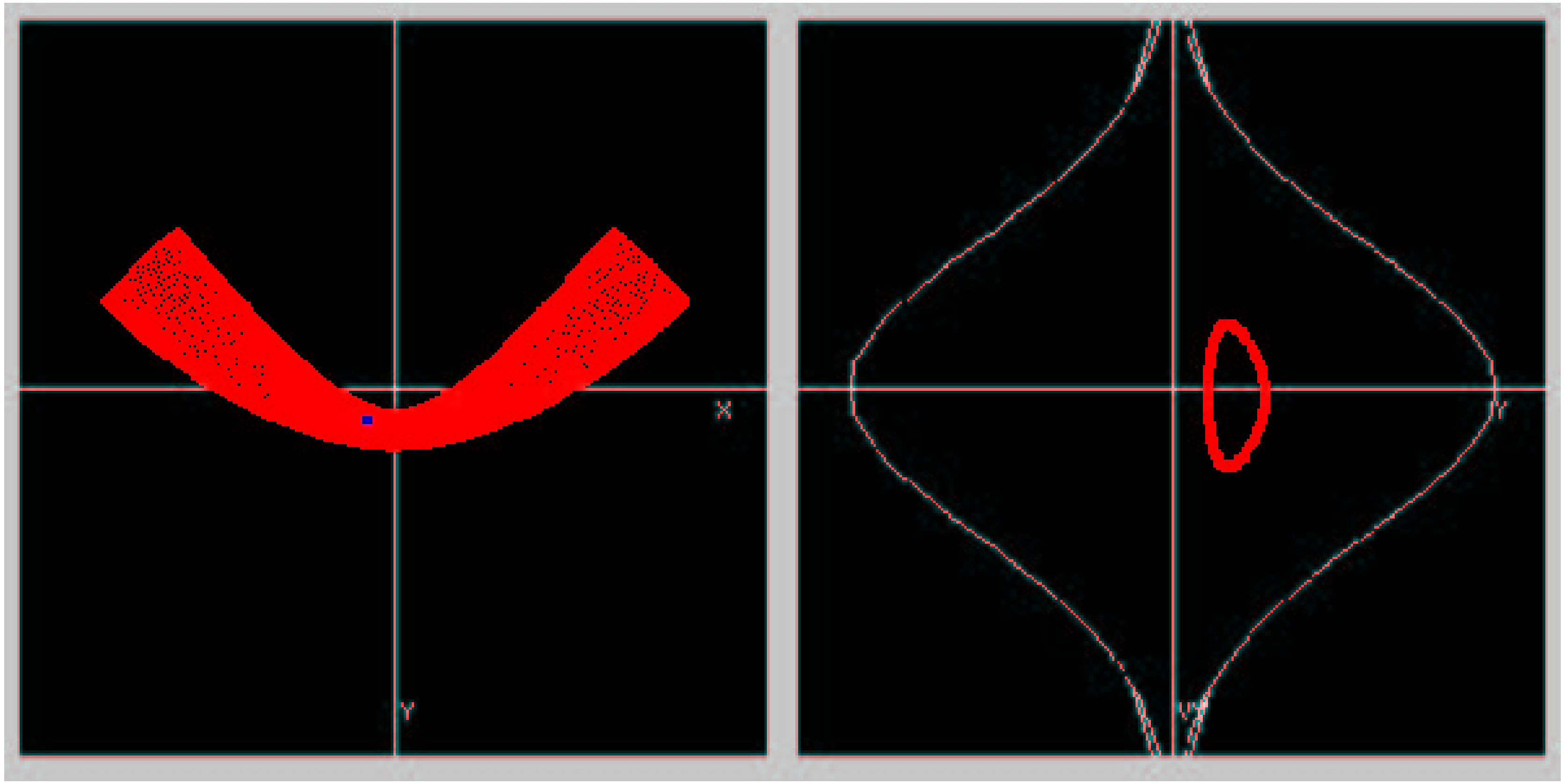
The family associated with the $4:3$ resonance are called the pretzel orbits

All these families together are called boxlet orbits.

The (singular) logarithmic potential is somewhat special in that it shows a surprisingly regular orbit structure. Virtually the entire phase-space admits two isolating integrals of motion in involution (E and I_2). Clearly, the logarithmic potential must be very near-integrable.

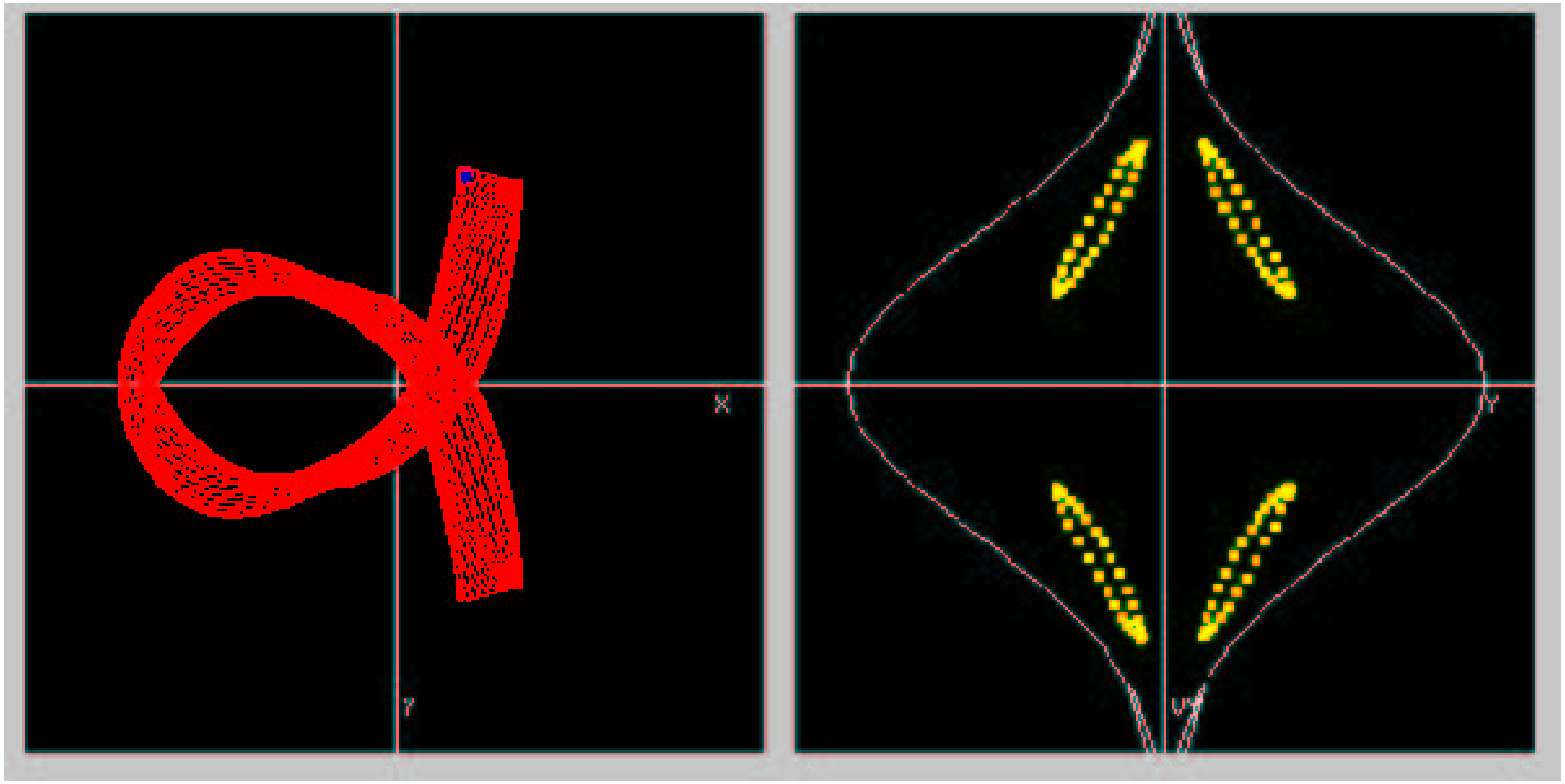
However, upon introducing a massive black hole in the center of the logarithmic potential, many of the box-orbits become stochastic. One says that the BH destroys the box-orbits. Recall that each box orbit comes arbitrarily close to the BH.

Orbits in Singular Logarithmic Potentials



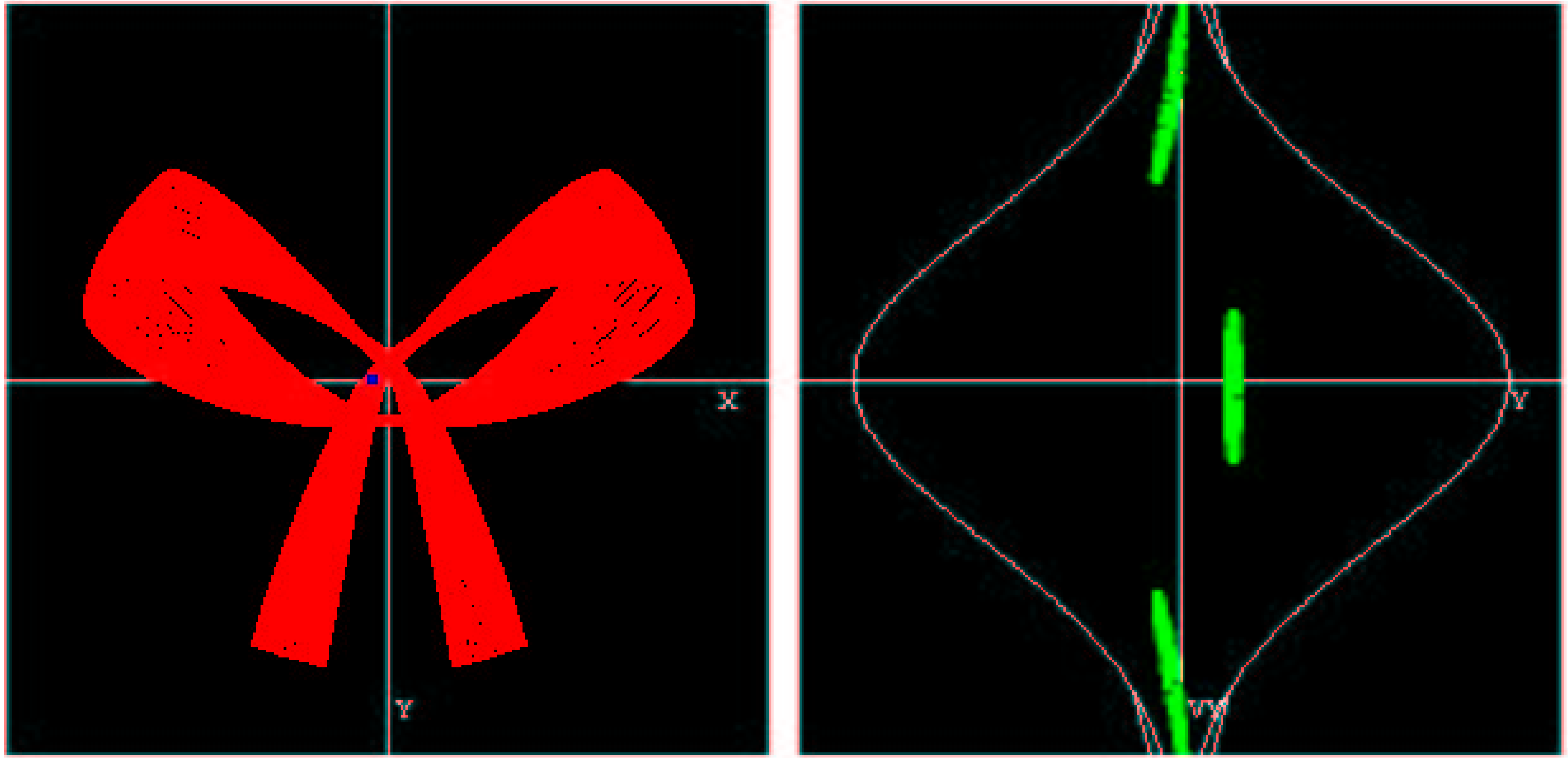
Here is an example of a banana-orbit (member of the $2:1$ resonance family).

Orbits in Singular Logarithmic Potentials



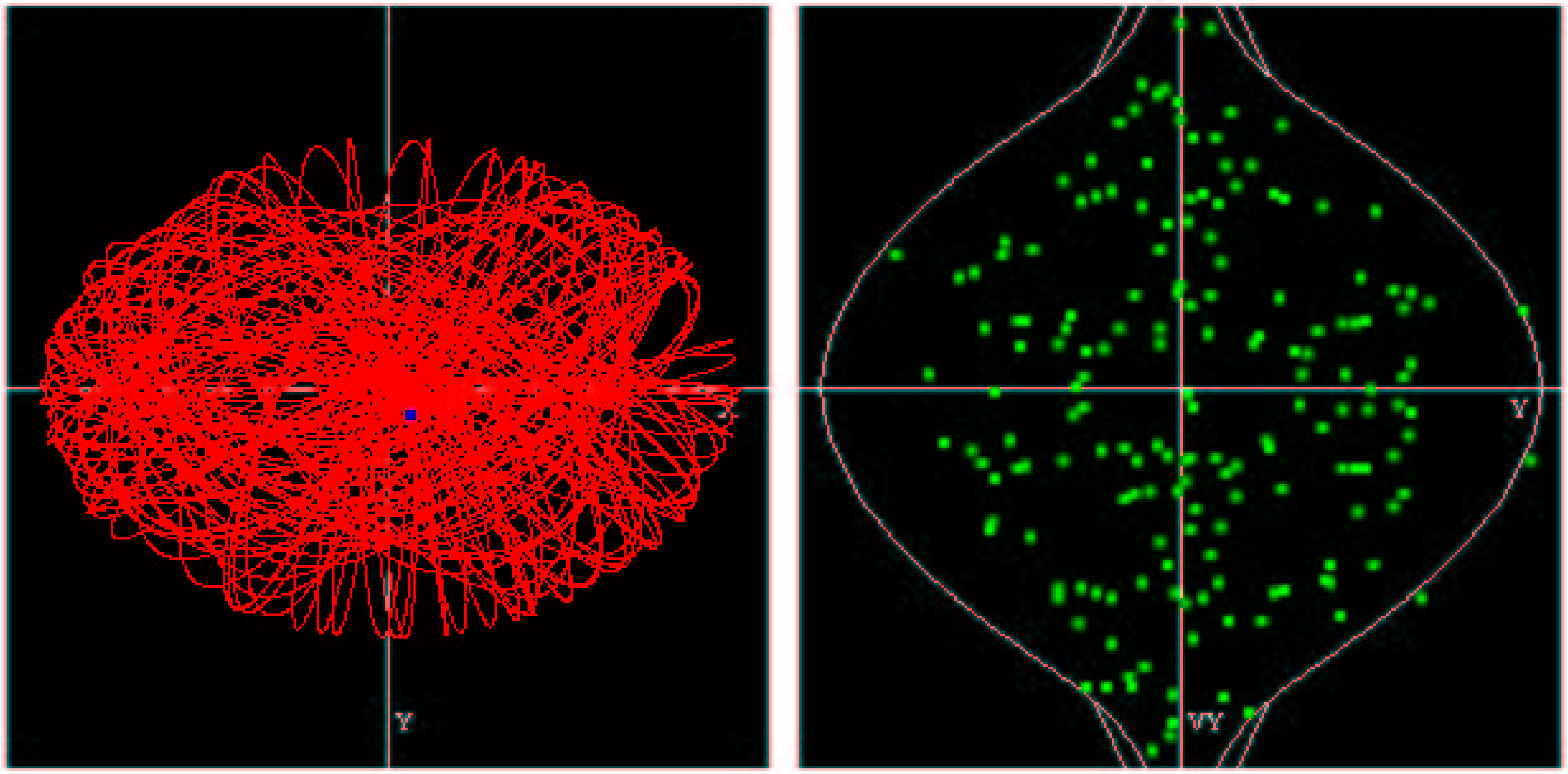
Here is an example of a fish-orbit (member of the $3:2$ resonance family).

Orbits in Singular Logarithmic Potentials



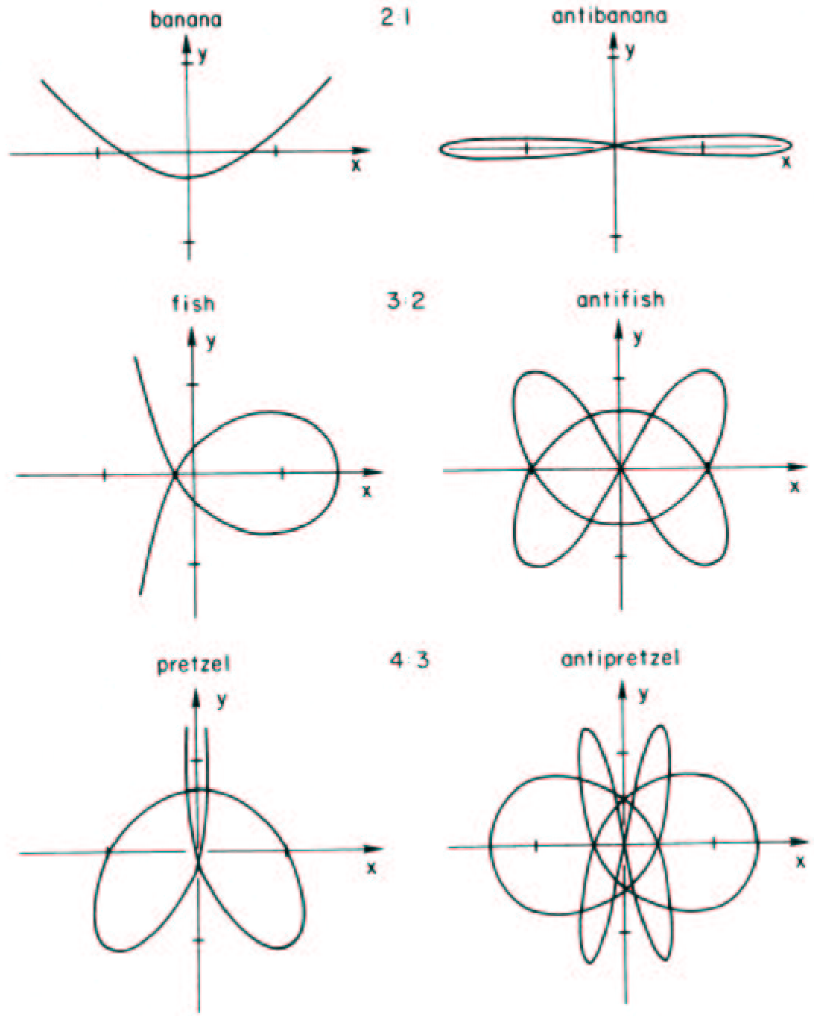
Here is an example of a pretzel-orbit (member of the $4 : 3$ resonance family).

Orbits in Logarithmic Potentials with BH



Here is an example of a stochastic orbit in a (cored) logarithmic potential with a central black hole.

Centrophobic versus Centrophilic



The closed boxlets comes in two kinds: centrophobic, which avoid the center and are stable, and centrophilic, which go through the center and are unstable. The centrophilic versions of the banana and fish orbits are called the antibanana and antifish orbits, etc. Because they are unstable, they don't parent any families.

Orbits in Planar Potentials: Summary

If $\Phi(x, y)$ has a homogeneous core, at sufficiently small E potential is indistinguishable from that of 2D harmonic oscillator

- **x-axial and y-axial closed orbits are stable**
- **they parent a family of box orbits (filling a rectangular box)**

At larger radii, for orbits with larger E:

- **y-axial orbit becomes unstable and bifurcates into two families of loop orbits (with opposite sense of rotation)**
- **close to this unstable, closed y-axial orbit a small layer of stochastic orbits is present**
- **x-axial orbit still stable, and parents family of box orbits**

If $\Phi(x, y)$ is scale-free and/or has central cusp that is sufficiently steep

- **x-axial orbit may become unstable. Its family of box orbits then becomes family of boxlets associated with (higher-order) resonances.**

If central BH is present, discrete scattering events can turn box orbits into stochastic orbits