Summary of the Above

- **Newton's second law:** $$\frac{\mathrm{d}^2\vec{r}}{\mathrm{d}t^2} = -\vec{\nabla}\Phi(\vec{r})$$
  - — Complicated vector arithmetic & coordinate system dependence
- **Lagrangian Formalism:** $$\frac{\partial \mathcal{L}}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = 0$$
  - — n second-order differential equations
- **Hamiltonian Formalism:** $$\frac{\partial \mathcal{H}}{\partial p_i} = \dot{q}_i \qquad \frac{\partial \mathcal{H}}{\partial q_i} = -\dot{p}_i$$
  - — $2n$ first-order differential equations
- **Hamilton-Jacobi equation:** $$\mathcal{H}\left(\frac{\partial S}{\partial q_i}, q_i\right) = E$$

$S(\vec{q}, \vec{p})$ is generator of canonical transformation $(\vec{q}, \vec{p}) \rightarrow (\vec{Q}, \vec{P})$ for which $\mathcal{H}(\vec{q}, \vec{p}) \rightarrow \mathcal{H}'(\vec{P})$. If $S(\vec{q}, \vec{p})$ is separable then the Hamilton-Jacobi equation breaks up in n ordinary differential equations which can be solved by simple quadrature. The resulting equations of motion are:

$$P_i(t) = P_i(0) \qquad Q_i(t) = \left(\frac{\partial \mathcal{H}'}{\partial P_i}\right) t + k_i$$

Constants of Motion

Constants of Motion: any function $C(\vec{q}, \vec{p}, t)$ of the generalized coordinates, conjugate momenta and time that is constant along **every** orbit, i.e., if $\vec{q}(t)$ and $\vec{p}(t)$ are a solution to the equations of motion, then

$$C[\vec{q}(t_1), \vec{p}(t_1), t_1] = C[\vec{q}(t_2), \vec{p}(t_2), t_2]$$

for any t_1 and t_2. The value of the constant of motion depends on the orbit, but different orbits may have the same numerical value of C

A dynamical system with n **degrees of freedom** always has $2n$ independent **constants of motion**. Let $q_i = q_i[\vec{q}_0, \vec{p}_0, t]$ and $p_i = p_i[\vec{q}_0, \vec{p}_0, t]$ describe the solutions to the equations of motion. In principle, these can be inverted to $2n$ relations $q_{i,0} = q_{i,0}[\vec{q}(t), \vec{p}(t), t]$ and $p_{i,0} = p_{i,0}[\vec{q}(t), \vec{p}(t), t]$. By their very construction, these are $2n$ **constants of motion**.

If $\Phi(\vec{x}, t) = \Phi(\vec{x})$, one of these $2n$ relations can be used to eliminate t. This leaves $2n - 1$ non-trivial constants of motion, which restricts the system to a $2n - (2n - 1) = 1$-dimensional surface in **phase-space**, namely the **phase-space trajectory** $\Gamma(t)$

Note that the elimination of time reflects the fact that the physics are **invariant** to time translations $t \longrightarrow t + t_0$, i.e., the time at which we pick our initial conditions can not hold any information regarding our dynamical system.

Integrals of Motion I

Integrals of Motion: any function $I(\vec{x}, \vec{v})$ of the phase-space coordinates $(\vec{x}, \vec{v})$ **alone** that is constant along **every orbit**, i.e.

$$I[\vec{x}(t_1), \vec{v}(t_1)] = I[\vec{x}(t_2), \vec{v}(t_2)]$$

for any t_1 and t_2. The value of the integral of motion can be the same for different orbits. Note that an integral of motion can not depend on time. Thus, all integrals are constants, but not all constants are integrals.

Integrals of motion come in two kinds:

Isolating Integrals of Motion: these reduce the **dimensionality** of the trajectory $\Gamma(t)$ by one. Therefore, a trajectory in a dynamical system with n degrees of freedom and with i **isolating integrals of motion** is restricted to a $2n - i$ dimensional **manifold** in the $2n$-dimensional **phase-space**. Isolating integrals of motion are of great practical and theoretical importance.

Non-Isolating Integrals of Motion: these are integrals of motion that do **not** reduce the dimensionality of $\Gamma(t)$. They are of essentially no practical value for the dynamics of the system.

Integrals of Motion II

A stationary, Hamiltonian system (i.e., $\mathcal{H}(\vec{q}, \vec{p}, t) = \mathcal{H}(\vec{q}, \vec{p})$) with n degrees of freedom always has $2n - 1$ independent integrals of motion, which restrict the motion to the one-dimensional phase-space trajectory $\Gamma(t)$. The number of isolating integrals of motion can, depending on the Hamiltonian, vary between 1 and $2n - 1$.

DEFINITION: Two functions I_1 and I_2 of the canonical phase-space coordinates $(\vec{q}, \vec{p})$ are said to be in involution if their Poisson bracket vanishes, i.e., if

$$[I_1, I_2] = \frac{\partial I_1}{\partial q_i}\frac{\partial I_2}{\partial p_i} - \frac{\partial I_1}{\partial p_i}\frac{\partial I_2}{\partial q_i} = 0$$

A set of k integrals of motion that are in involution forms a set of k isolating integrals of motion.

Liouville's Theorem for Integrable Hamiltonians

A Hamiltonian system with n degrees of freedom which posseses n integrals of motion in involution, (and thus n isolating integrals of motion) is integrable by quadrature.

Integrable Hamiltonians I

LEMMA: If a system with n degrees of freedom has n constants of motion $P_i(\vec{q}, \vec{p}, t)$ [or integrals of motion $P_i(\vec{q}, \vec{p})$] that are in involution, then there will also be a set of n functions $Q_i(\vec{q}, \vec{p}, t)$ [or $Q_i(\vec{q}, \vec{p})$] which together with the P_i constitute a set of canonical variables.

Thus, given n isolating integrals of motion $I_i(\vec{q}, \vec{p})$ we can make a canonical transformation $(\vec{q}, \vec{p}) \longrightarrow (\vec{Q}, \vec{P})$ with $P_i = I_i(\vec{q}, \vec{p}) =$constant and with $Q_i(t) = \Omega_i t + k_i$

An integrable, Hamiltonian system with n degrees of freedom always has a set of n isolating integrals of motion in involution. Consequently, the trajectory $\Gamma(t)$ is confined to a $2n - n = n$-dimensional manifold phase-space.

The surfaces specified by $(I_1, I_2, .., I_n) =$constant are topologically equivalent to n-dimensional tori. These are called invariant tori, because any orbit originating on one of them remains there indefinitely.

In an integrable, Hamiltonian system phase-space is completely filled (one says ‘foliated’) with invariant tori.

Integrable Hamiltonians II

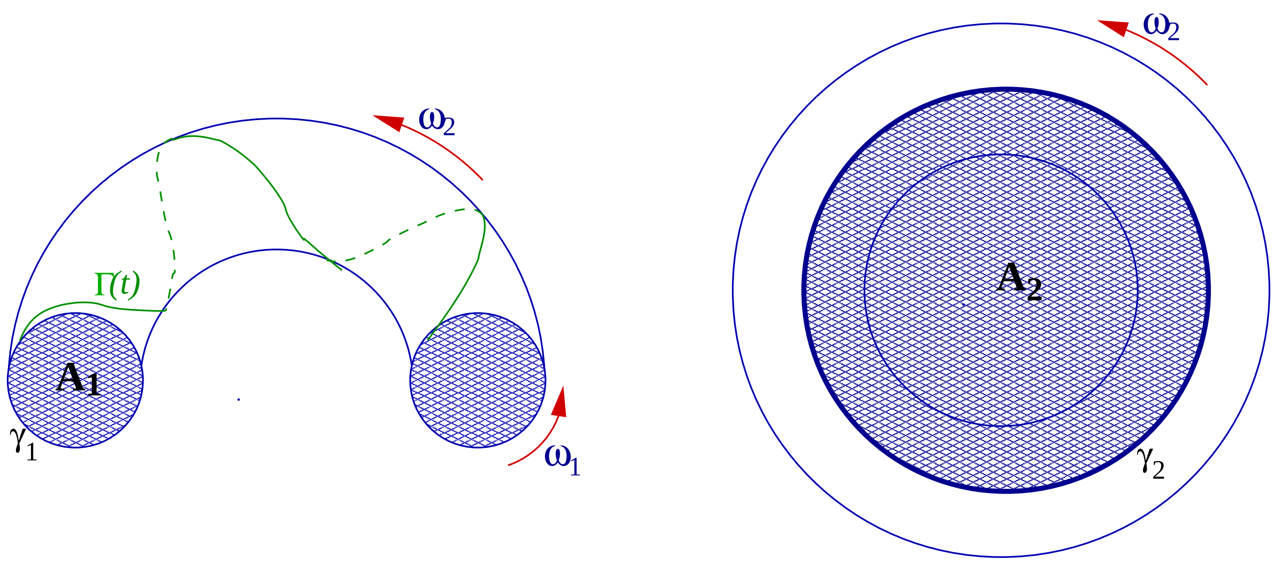
To summarize: if, for a system with n degrees of freedom, the Hamilton-Jacobi equation is separable, the Hamiltonian is integrable and there exist n isolating integrals of motion I_i in involution. In this case there exist canonical transformations $(\vec{q}, \vec{p}) \longrightarrow (\vec{Q}, \vec{P})$ such that equations of motion reduce to:

$$\begin{aligned} P_i(t) &= P_i(0) \\ Q_i(t) &= \left(\frac{\partial \mathcal{H}'}{\partial P_i}\right) t + k_i \end{aligned}$$

One might think at this point, that one has to take $P_i = I_i$. However, this choice is not unique. Consider an integrable Hamiltonian with $n = 2$ degrees of freedom and let I_1 and I_2 be two isolating integrals of motion in involution. Now define $I_a = \frac{1}{2}(I_1 + I_2)$ and $I_b = \frac{1}{2}(I_1 - I_2)$, then it is straightforward to proof that $[I_a, I_b] = 0$, and thus that (I_a, I_b) is also a set of isolating integrals of motion in involution. In fact, one can construct infinitelly many sets of isolating integral of motion in involution. Which one should we choose, and in particular, which one yields the most meaningful description of the invariant tori?

Answer: The Action-Angle variables

Action-Angle Variables I



Let's be guided by the idea of our invariant tori. The figure illustrates a 2D-torus (in 4D-phase space), with a trajectory $\Gamma(t)$ on its surface. One can specify a location on this torus by the two position angles ω_1 and ω_2. The torus itself is characterized by the areas of the two (hatched) cross sections labelled A_1 and A_2. The action variables J_1 and J_2 are intimitally related to A_1 and A_2, which clearly are two integrals of motion.

The action variables are defined by:

$$J_i = \frac{1}{2\pi} \oint_{\gamma_i} \vec{p} \cdot \mathrm{d}\vec{q}$$

with γ_i the closed loop that bounds cross section A_i.

Action-Angle Variables II

The angle-variables ω_i follow from the canonical transformation rule $\omega_i = \frac{\partial S}{\partial J_i}$ with $S = S(\vec{q}, \vec{J})$ the generator of the canonical transformation $(\vec{q}, \vec{p}) \longrightarrow (\vec{\omega}, \vec{J})$. Since the actions J_i are isolating integrals of motion we have that the corresponding conjugate angle coordinates w_i obey

$$\omega_i(t) = \left(\frac{\partial \mathcal{H}'}{\partial J_i}\right) t + \omega_0$$

with $\mathcal{H}' = \mathcal{H}'(\vec{J})$ the Hamiltonian in action-angle variables $(\vec{\omega}, \vec{J})$.

We now give a detailed description of motion on invariant tori:

Orbits in integrable, Hamiltonian systems with n degrees of freedom are characterized by n constant frequencies

$$\boxed{\Omega_i \equiv \frac{\partial \mathcal{H}'}{\partial J_i}}$$

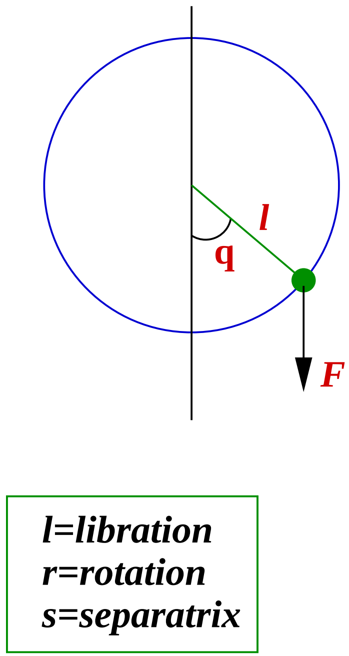
This implies that the motion along each of the n degrees of freedom, q_i, is periodic in time, and this can occur in two ways:

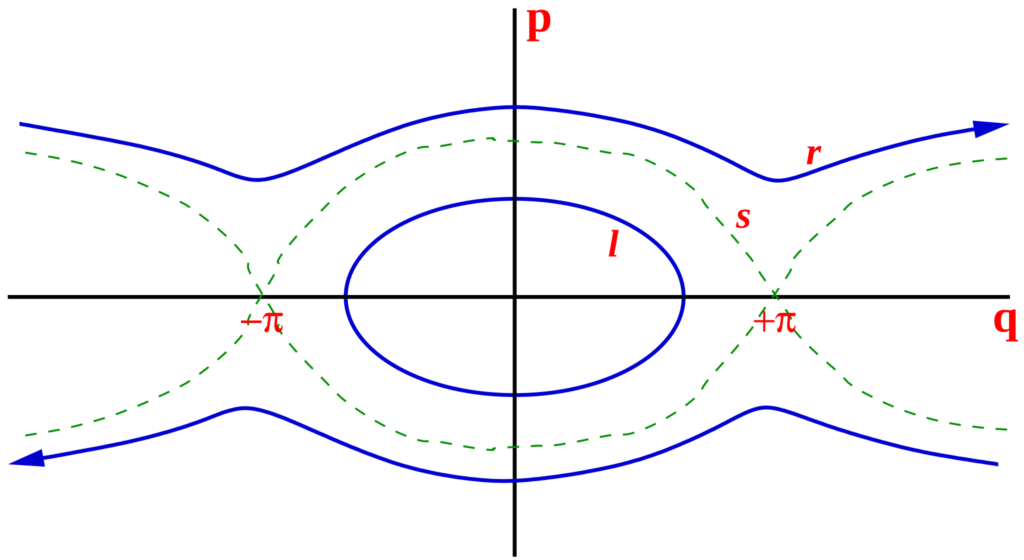
- **Libration:** motion between two states of vanishing kinetic energy
- **Rotation:** motion for which the kinetic energy never vanishes

The Pendulum

To get insight into libration and rotation consider a pendulum, which is a integrable Hamiltonian system with one degree of freedom, the angle q..

The figures below shows the corresponding phase-diagram.





- **Libration:** $q(\omega + 2\pi) = q(\omega)$.
- **Rotation:** $q(\omega + 2\pi) = q(\omega) + 2\pi$

To go from libration to rotation, one needs to cross the separatrix

Action-Angle Variables III

Why are action-angle variables the ideal set of isolating integrals of motion to use?

- **They are are the only conjugate momenta that enjoy the property of adiabatic invariance (to be discussed later)**
- **The angle-variables are the natural coordinates to label points on invariant tori.**
- **They are ideally suited for perturbation analysis, which is used to investigate near-integrable systems (see below)**
- **They are ideally suited to study the (in)-stability of a Hamiltonian system**

Example: Central Force Field

As an example, to get familiar with action-angle variables, let's consider once again motion in a central force field.

As we have seen before, the Hamiltonian is

$$\mathcal{H} = \frac{1}{2}p_r^2 + \frac{1}{2}\frac{p_\theta^2}{r^2} + \Phi(r)$$

where $p_r = \dot{r}$ and $p_\theta = r^2\dot{\theta} = L$.

In our planar description, we have two integrals of motion, namely energy $I_1 = E = \mathcal{H}$ and angular momentum $I_2 = L = p_\theta$.

These are classical integrals of motion, as they are associated with symmetries. Consequently, they are also isolating.

Let's start by checking whether they are in involution

$$[I_1, I_2] = \left[\frac{\partial I_1}{\partial r}\frac{\partial I_2}{\partial p_r} - \frac{\partial I_1}{\partial p_r}\frac{\partial I_2}{\partial r}\right] + \left[\frac{\partial I_1}{\partial \theta}\frac{\partial I_2}{\partial p_\theta} - \frac{\partial I_1}{\partial p_\theta}\frac{\partial I_2}{\partial \theta}\right]$$

Since $\frac{\partial I_2}{\partial p_r} = \frac{\partial I_2}{\partial r} = \frac{\partial I_1}{\partial \theta} = \frac{\partial I_2}{\partial \theta} = 0$ one indeed finds that the two integrals of motion are in involution.

Example: Central Force Field

The actions are defined by

$$J_r = \frac{1}{2\pi}\oint_{\gamma_r} p_r \mathrm{d}r \qquad J_\theta = \frac{1}{2\pi}\oint_{\gamma_\theta} p_\theta \mathrm{d}\theta$$

In the case of J_θ the θ-motion is one of rotation. Therefore the closed-line-integral is over an angular interval $[0, 2\pi]$.

$$J_\theta = \frac{1}{2\pi}\int_0^{2\pi} I_2 \mathrm{d}\theta = I_2$$

In the case of J_r, we need to realize that the r-motion is a libration between apocenter r_+ and pericenter r_-. Using that $I_1 = E = \mathcal{H}$ we can write

$$p_r = \sqrt{2[I_1 - \Phi(r)] - I_2^2/r^2}$$

The radial action then becomes

$$J_r = \frac{1}{\pi}\int_{r_-}^{r_+} \sqrt{2[I_1 - \Phi(r)] - I_2^2/r^2}\,\mathrm{d}r$$

Once we make a choise for the potential $\Phi(r)$ then J_r can be solved as function of I_1 and J_θ. Since $I_1 = \mathcal{H}$, this in turn allows us to write the Hamiltonian as function of the actions: $\mathcal{H}(J_r, J_\theta)$.

Example: Central Force Field

As an example, let's consider a potential of the form

$$\Phi(r) = -\frac{\alpha}{r} - \frac{\beta}{r^2}$$

with α and β two constants. Substituting this in the above, one finds:

$$J_r = \alpha \left(\frac{1}{2|I_1|}\right)^{1/2} - \sqrt{J_\theta^2 - 2\beta}$$

Inverting this for $I_1 = \mathcal{H}$ yields

$$\boxed{\mathcal{H}(J_r, J_\theta) = \frac{\alpha^2}{2} \left(J_r + \sqrt{J_\theta^2 - 2\beta}\right)^{-2}}$$

Since the actions are isolating integrals of motion, and we have an expression for the Hamiltonian in terms of these actions, the generalized coordinates that correspond to these actions (the angles w_r and w_θ) evolve as $w_i(t) = \Omega_i\, t + w_{i,0}$

The radial and angular frequencies are

$$\Omega_r = \frac{\partial \mathcal{H}}{\partial J_r} = -\alpha^2 \left(J_r + \sqrt{J_\theta^2 - 2\beta}\right)^{-3}$$

$$\Omega_\theta = \frac{\partial \mathcal{H}}{\partial J_\theta} = -\alpha^2 \left(J_r + \sqrt{J_\theta^2 - 2\beta}\right)^{-3} \frac{J_\theta}{\sqrt{J_\theta^2 - 2\beta}}$$

Example: Central Force Field

The ratio of these frequencies is

$$\frac{\Omega_r}{\Omega_\theta} = \left(1 - \frac{2\beta}{J_\theta^2}\right)^{1/2}$$

Note that for $\beta = 0$, for which $\Phi(r) = -\frac{\alpha}{r}$, and thus the potential is of the Kepler form, we have that $\Omega_r = \Omega_\theta$ independent of the actions (i.e., for each individual orbit).

In this case the orbit is closed, and there is an additional isolating integral of motion (in addition to E and L). We may write this 'third' integral as

$$I_3 = w_r - w_\theta = \Omega_r t + w_{r,0} - \Omega_\theta t - w_{\theta,0} = w_{r,0} - w_{\theta,0}$$

Without loosing generality, we can pick the zero-point of time, such that $w_{r,0} = 0$. This shows that we can think of the third integral in a Kepler potential as the angular phase of the line connecting apo- and peri-center.

Quasi-Periodic Motion

In general, in an integrable Hamiltonian system with the canonical transformation $(\vec{q}, \vec{p}) \longrightarrow (\vec{Q}, \vec{P})$ one has that $q_k = q_k(\omega_1, .., \omega_n)$ with $k = (1, .., n)$. If one changes ω_i by 2π, while keeping the other $\omega_j (j \neq i)$ fixed, then q_i then performs a complete libration or rotation.

The Cartesian phase-space coordinates $(\vec{x}, \vec{v})$ must be periodic functions of the angle variables ω_i with period 2π. Any such function can be expressed as a Fourier series

$$\vec{x}(\vec{\omega}, \vec{J}) = \sum_{l,m,n=-\infty}^{\infty} X_{lmn}(\vec{J}) \exp\left[i(l\omega_1 + m\omega_2 + n\omega_3)\right]$$

Using that $\omega_i(t) = \Omega_i t + k_i$ we thus obtain that

$$\boxed{\vec{x}(t) = \sum_{l,m,n=-\infty}^{\infty} \tilde{X}_{lmn} \exp\left[i(l\Omega_1 + m\Omega_2 + n\Omega_3)t\right]}$$

with $\tilde{X}_{lmn} = X_{lmn} \exp\left[i(lk_1 + mk_2 + nk_3)\right]$

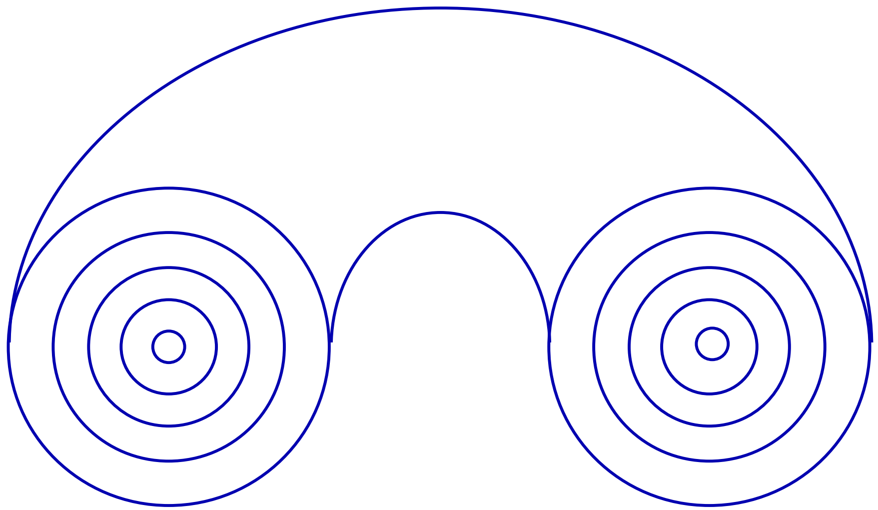
Functions of the form of $\vec{x}(t)$ are said to be quasi-periodic functions of time. Hence, in an integrable systems, all orbits are quasi-periodic, and confined to an invariant torus.

Integrable Hamiltonians III

When one integrates a trajectory $\Gamma(t)$ in an integrable system for sufficiently long, it will come infinitessimally close to any point $\vec{\omega}$ on the surface of its torus. In other words, the trajectory densely fills the entire torus.

Since no two trajectories $\Gamma_1(t)$ and $\Gamma_2(t)$ can intersect the same point in phase-space, we thus immediately infer that two tori are not allowed to intersect.

In an integrable, Hamiltonian system phase-space is completely foliated with non-intersecting, invariant tori



Integrable Hamiltonians IV

In an integrable, Hamiltonian system with n degrees of freedom, all orbits are confined to, and densely fill the surface of n-dimensional invariant tori.

These orbits, which have (at least) as many isolating integrals as spatial dimensions are called regular

Regular orbits have n frequencies Ω_i which are functions of the corresponding actions J_i. This means that one can always find suitable values for J_i such that two of the n frequencies Ω_i are commensurable, i.e. for which

$$\boxed{l\,\Omega_i = m\,\Omega_j}$$

with $i \neq j$ and l, m both integers.

A regular orbit with commensurable frequencies is called a resonant orbit (also called closed or periodic orbit), and has a dimensionality that is one lower than that of the non-resonant, regular orbits. This implies that there is an extra isolating integral of motion, namely

$$I_{n+1} = l\omega_i - m\omega_j$$

Note: since $\omega_i(t) = \Omega_i t + k_i$, one can obtains that $I_{n+1} = lk_i - mk_j$, and thus is constant along the orbit.

Near-Integrable Systems I

Thus far we have focussed our attention on integrable, Hamiltonian systems.

Given a Hamiltonian $\mathcal{H}(\vec{q},\vec{p})$, how can one determine whether the system is integrable, or whether the Hamilton-Jacobi equation is separable?

Unfortunately, there is no real answer to this question: In particular, there is no systematic method for determining if a Hamiltonian is integrable or not!!!

However, if you can show that a system with n degrees of freedom has n independent integrals of motion in involution then the system is integrable.

Unfortunately, the explicit expression of the integrals of motion in terms of the phase-space coordinates is only possible in a very so called classical integrals of motion, those associated with a symmetry of the potential and/or with an invariance of the coordinate system.

In what follows, we only consider the case of orbits in 'external' potentials for which $n = 3$. In addition, we only consider stationary potentials $\Phi(\vec{x})$, so that the Hamiltonian does not explicitely on time and $\mathcal{H}(\vec{q},\vec{p}) = E =$constant. Therefore

Energy is always an isolating integral of motion.

Note: this integral is related to the invariance of the Lagrangian $\mathcal{L}$ under time translation, i.e., to the homogeneity of time.

Near-Integrable Systems II

Integrable Hamiltonians are extremely rare. As a consequence, it is extremely unlikely that the Hamiltonian associated with a typical galaxy potential is integrable.

One can prove that even a slight perturbation away from an integrable potential will almost always destroy any integral of motion other than E.

So why have we spent so must time discussing integrable Hamiltonians?
▷ Because most galaxy-like potentials turn out to be near-integrable.

Definition: A Hamiltonian system is near-integrable if a large fraction of phase-space is still occupied by regular orbits (i.e., by orbits on invariant tori).

The dynamics of near-integrable Hamiltonians is the subject of the Kolmogorov-Arnold-Moser (KAM) Theorem which states:

If $\mathcal{H}_0$ is an integrable Hamiltonian whose phase-space is completely foliated with regular orbits on invariant tori, then in a perturbed Hamiltonian $\mathcal{H} = \mathcal{H}_0 + \varepsilon \mathcal{H}_1$ most orbits will still lie on such tori for sufficiently small ε. The fraction of phase-space covered by these tori $\longrightarrow 1$ for $\varepsilon \longrightarrow 0$ and the perturbed tori are deformed versions of the unperturbed ones.

Near-Integrable Systems III

The stability of the original tori to a perturbation can be proven everywhere except in small regions around the resonant tori of $\mathcal{H}_0$. The width of these regions depends on ε and on the order of the resonance.

According to the Poincaré-Birkhoff Theorem the tori around unstable resonant tori break up and the corresponding regular orbits become irregular and stochastic.

Definition: A resonant orbit is stable if an orbit starting close to it remains close to it. They parent orbit families (see below)

Definition: An irregular orbit is an orbit that is not confined to a n-dimensional torus. In general it can wander through the entire phase-space permitted by conservation of energy.

Consequently, an irregular orbit is restricted to a higher-dimensional manifold than a regular orbit. Irregular orbits are stochastic in that they are extremely sensitive to initial conditions: two stochastic trajectories $\Gamma_1(t)$ and $\Gamma_2(t)$ which at $t = t_0$ are infinitesimally close together will diverge with time.

Increasing ε, increases the widths of the stochastic zones, which may eventually 'eat up' a large fraction of phase-space.

Near-Integrable Systems IV

Note that unperturbed resonant tori form a dense set in phase-space, just like the rational numbers are dense on the real axis.

Just like you can always find a rational number in between two real numbers, in a near-integrable system there will always be a resonant orbit in between any two tori. Since many of these will be unstable, they create many stochastic regions

As long as the resonance is of higher order (i.e, $16 : 23$ rather than $1 : 2$) the corresponding chaotic regions are very small, and tightly bound by their surrounding tori.

Since two trajectories can not cross, an irregular orbit is bounded by its neighbouring regular orbits. The irregular orbit is therefore still (almost) confined to a n-dimensional manifold, and it behaves as if it has n isolating integrals of motion.

▷ It may be very difficult to tell whether an orbit is regular or irregular.

However, iff $n > 2$ an irregular orbit may slip through a 'crack' between two confining tori, a process know as Arnold diffusion.

Because of Arnold diffusion the stochasticity will be larger than 'expected'. However, the time-scale for Arnold diffusion to occus is long, and it is unclear how important it is for Galactic Dynamics.

Near-Integrable Systems V

A few words on nomenclature:

Recall that an (isolating) integral of motion is defined as a function of phase-space coordinates that is constant along every orbit.

A near-integrable systems in principle has only one isolating integral of motion, namely energy E.

Nevertheless, according to the KAM Theorem, many orbits in a near-integrable system are confined to invariant tori.

Although in conflict with the definition, astronomers often say that the regular orbits in near-integrable systems admit n isolating integrals of motion.

Astronomers also often use KAM Theorem to the extreme, by assuming that they can ignore the irregular orbits, and that the Hamiltonians that correspond to their 'galaxy-like' potentials are integrable. Clearly the validity of this approximation depends on the fraction of phase-space that admits three isolating integrals of motion. For most potentials used in Galactic Dynamics, it is still unclear how large this fraction really is, and thus, how reliable the assumption of integrability is.