Orbits in Central Force Fields I

Consider the central force field $F(r)$ associated with a spherical density distribution $\rho(r)$.

As we have seen before, the orbits are planar, so that we consider the polar coordinates (r, θ)

The equations of motion are: $\frac{\mathrm{d}^2\vec{r}}{\mathrm{d}t^2} = F(r)\vec{e}_r$

Solving these requires a careful treatment of the unit vectors in polar coordinates:

$$\begin{aligned} \vec{e}_r &= \cos\theta\vec{e}_x + \sin\theta\vec{e}_y \\ \vec{e}_\theta &= -\sin\theta\vec{e}_x + \cos\theta\vec{e}_y \end{aligned}$$

$$\begin{aligned} \frac{\mathrm{d}\vec{r}}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}(r\cos\theta\vec{e}_x + r\sin\theta\vec{e}_y) \\ &= \dot{r}\cos\theta\vec{e}_x - r\dot{\theta}\sin\theta\vec{e}_x + \dot{r}\sin\theta\vec{e}_y + r\dot{\theta}\cos\theta\vec{e}_y \\ &= \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta \end{aligned}$$

and similarly one obtains that

$$\frac{\mathrm{d}^2\vec{r}}{\mathrm{d}t^2} = (\ddot{r} - r\dot{\theta}^2)\vec{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\vec{e}_\theta$$

Orbits in Central Force Fields II

We thus obtain the following set of equations of motions:

$$\begin{aligned} \ddot{r} - r\dot{\theta}^2 &= F(r) = -\frac{\mathrm{d}\Phi}{\mathrm{d}r} \\ 2\dot{r}\dot{\theta} + r\ddot{\theta} &= 0 \end{aligned}$$

Multiplying the second of these equations with r yields, after integration, that $\frac{\mathrm{d}}{\mathrm{d}t}(r^2\dot{\theta}) = 0$. This simply expresses the conservation of the orbit's angular momentum $L = r^2\dot{\theta}$, i.e., the equations of motion can be written as

$$\begin{aligned} \ddot{r} - r\dot{\theta}^2 &= -\frac{\mathrm{d}\Phi}{\mathrm{d}r} \\ r^2\dot{\theta} &= L = \mathrm{constant} \end{aligned}$$

In general these equations have to be solved numerically. Despite the very simple, highly symmetric system, the equations of motion don't provide much insight. As we'll see later, more direct insight is obtained by focussing on the conserved quantities. Note also that the equations of motion are different in different coordinate systems: in Cartesian coordinates (x, y):

$$\begin{aligned} \ddot{x} &= F_x = -\frac{\partial\Phi}{\partial x} \\ \ddot{y} &= F_y = -\frac{\partial\Phi}{\partial y} \end{aligned}$$

Orbits in Central Force Fields III

As shown before, one can use the second equation of motion (in polar coordinates) to eliminate $\dot{\theta}$ in the first, which yields the radial energy equation

$$\frac{1}{2}\dot{r}^2 + \frac{J^2}{2r^2} + \Phi(r) = E$$

which can be rewritten as

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \pm\sqrt{2[E - \Phi(r)] - \frac{J^2}{2r^2}}$$

where the $\pm$ sign is required because r can both increase and decrease. Solving for the turn-around points, where $\mathrm{d}r/\mathrm{d}t = 0$, yields

$$\frac{1}{r^2} = \frac{2[E-\Phi(r)]}{-J^2}$$

which has two solutions: the apocenter r_+ and the pericenter $r_- \leq r_+$. These radii reflect the maximum and minimum radial extent of the orbit.

It is customary to define the orbital eccentricity as

$$e = \frac{r_+ - r_-}{r_+ + r_-}$$

where $e = 0$ and $e = 1$ correspond to circular and radial orbits, resp.

The Lagrangian

The equations of motion as given by Newton's second law depend on the choice of coordinate system

Their derivation involves painful vector calculus when curvi-linear coordinates are involved

In the Lagrangian formulation of dynamics, the equations of motion are valid for any set of so-called generalized coordinates $(q_1, q_2, .., q_n)$, with n the number of degrees of freedom

Generalized coordinates are any set of coordinates that are used to describe the motion of a physical system, and for which the position of every particle in the system is a function of these coordinates and perhaps also time: $\vec{r} = \vec{r}(q_i, t)$. If $\vec{r} = \vec{r}(q_i)$ the system is said to be natural.

Define the Lagrangian function: $\boxed{\mathcal{L} = T - V}$

with T and V the kinetic and potential energy, respectively.

In Cartesian coordinates, and setting the mass $m = 1$, we have

$$\mathcal{L} = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \Phi(x, y, z)$$

In Generalized coordinates we have that $\mathcal{L} = \mathcal{L}(q_i, \dot{q}_i)$.

Actions and Hamilton's Principle

Define the action integral (also just called the action)

$$I = \int_{t_0}^{t_1} \mathcal{L}\, \mathrm{d}t$$

which is the integral of the Lagrangian along a particle's trajectory as it moves from time t_0 to t_1.

Hamilton's Principle, also called Principle of least action: The equations of motion are such that the action integral is stationary (i.e., $\delta I = 0$) under arbitrary variations δq_i which vanish at the limits of integration t_0 to t_1.

Note that these stationary points are not necessarily minima. They may also be maxima or sadle points.

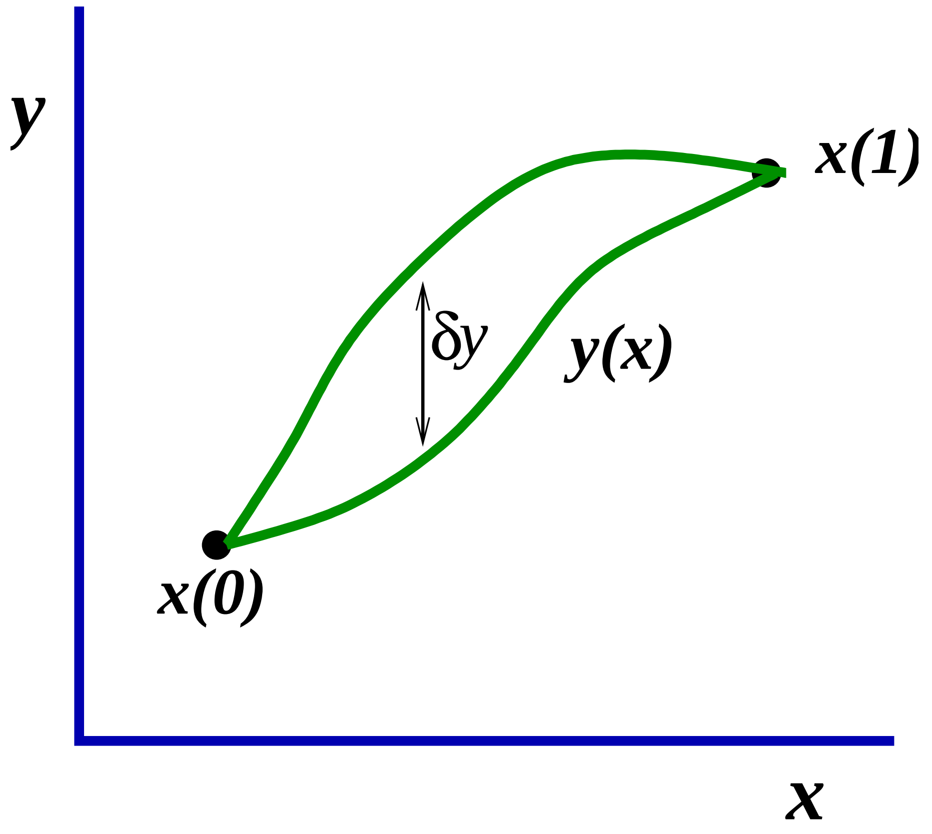
In order to derive these equations of motion, we first familiarize ourselves with the calculus of variations

Calculus of Variations I

We are interested in finding the stationary values of an integral of the form

$$I = \int_{x_0}^{x_1} f(y, \dot{y}) \mathrm{d}x$$

where $f(y, \dot{y})$ is a specified function of $y = y(x)$ and $\dot{y} = \mathrm{d}y/\mathrm{d}x$.



Consider a small variation $\delta y(x)$, which vanishes at the endpoints of the integration interval: $\delta y(x_0) = \delta y(x_1) = 0$

Calculus of Variations II

Using that

$$\delta f = \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial \dot{y}}\delta \dot{y}$$

with $\delta \dot{y} = \frac{\mathrm{d}}{\mathrm{d}x}\delta y(x)$**, the stationary values obey**

$$\delta I = \int_{x_0}^{x_1} \left[\frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial \dot{y}}\frac{\mathrm{d}}{\mathrm{d}x}\delta y\right] \mathrm{d}x = 0$$

Using integration by parts, and $\delta y(x_0) = \delta y(x_1) = 0$**, this reduces to**

$$\delta I = \int_{x_0}^{x_1} \left[\frac{\partial f}{\partial y} + \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial \dot{y}}\right)\right] \delta y \, \mathrm{d}x = 0$$

which yields the so-called Euler-Lagrange equations

$$\boxed{\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial \dot{y}}\right) = 0}$$

These are second-order differential equations for $y(x)$**, whose solutions contain two arbitrary constants that may be determined form the known values of** y **at** x_0 **and** x_1**.**

The Lagrangian Formulation I

Application of the Euler-Lagrange equations to the Lagrangian $\mathcal{L}(q_i, \dot{q}_i)$ yields

$$\frac{\partial \mathcal{L}}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = 0$$

which are the Lagrange equations (one for each degree of freedom), which represent the equations of motion according to Hamilton's principle. Note that they apply to any set of generalized coordinates

In addition to the generalized coordinates we also define the generalized momenta p_i (also called conjugate momenta) and the generalized forces F_i:

$$p_i \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \qquad\qquad F_i \equiv \frac{\partial \mathcal{L}}{\partial q_i}$$

With these definitions the Lagrange equations reduce to

$$\dot{p}_i = \frac{\partial \mathcal{L}}{\partial q_i} = F_i$$

NOTE: in general p_i and F_i are not components of the momentum vector $\vec{p}$ or the force vector $\vec{F}$!!! Whenever q_i is an angle, the conjugate momentum p_i is an angular momentum.

The Lagrangian Formulation II

As an example, let's consider once again motion in a central force field. Our generalized coordinates are the polar coordinates (r, θ), and the Lagrangian is

$$\mathcal{L} = \frac{1}{2}\dot{r}^2 + \frac{1}{2}r^2\dot{\theta}^2 - \Phi(r)$$

The Lagrange equations are

$$\frac{\partial \mathcal{L}}{\partial r} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) = 0 \quad \Rightarrow \quad r\dot{\theta}^2 - \frac{\partial \Phi}{\partial r} - \frac{\mathrm{d}}{\mathrm{d}t}(\dot{r}) = 0 \quad \Rightarrow \quad \ddot{r} - r\dot{\theta}^2 = -\frac{\partial \Phi}{\partial r}$$

$$\frac{\partial \mathcal{L}}{\partial \theta} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = 0 \quad \Rightarrow \quad -\frac{\mathrm{d}}{\mathrm{d}t}(r^2\dot{\theta}) = 0 \quad \Rightarrow \quad r^2\dot{\theta} = L = \mathrm{cst}$$

Note that the Lagrangian formulation allows you to write down the equations of motion much faster than using Newton's second law!

The Hamiltonian Formulation I

The **Hamiltonian** $\mathcal{H}(q_i, p_i)$ is related to the **Lagrangian** $\mathcal{L}(q_i, \dot{q}_i)$ via a **Legendre Transformation**

In general, a **Legendre Transformation** is a transformation of a function $f(x, y)$ to $g(u, y)$, where $u = \frac{\partial f}{\partial x}$ and $\frac{\partial g}{\partial u} = x$

$$g(u, y) = f - u\,x$$

NOTE: You might be familiar with **Legendre Transformations** from **Thermodynamics** where they are used to compute different thermodynamic potentials from the **internal energy** $U = U(S, V)$, such as

enthalpy: $H = H(S, p) = U + p\,V$

Helmholtz free energy: $F = F(T, V) = U - T\,S$

Using a similar Legendre transformation we write the **Hamiltonian** as

$$\mathcal{H}(\vec{q}, \vec{p}, t) = \sum_{i=1}^{n} p_i\,\dot{q}_i(\vec{q}, \vec{p}) - \mathcal{L}(\vec{q}, \dot{\vec{q}}(\vec{q}, \vec{p}), t)$$

To compute $\mathcal{H}(\vec{q}, \vec{p}, t)$, first compute $\mathcal{L}(\vec{q}, \dot{\vec{q}}, t)$, next compute the conjugate momenta $p_i = \partial\mathcal{L}/\partial\dot{q}_i$, compute $\mathcal{H} = \vec{p} \cdot \dot{\vec{q}} - \mathcal{L}(\vec{q}, \dot{\vec{q}}, t)$ and finally express the $\dot{q}_i$ in terms of $\vec{p}$ and $\vec{q}$

The Hamiltonian Formulation II

Differentiating $\mathcal{H}$ with respect to the conjugate momenta yields

$$\frac{\partial \mathcal{H}}{\partial p_j} = \dot{q}_j + \sum_{i=1}^{n} p_i \frac{\partial \dot{q}_i}{\partial p_j} - \sum_{i=1}^{n} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \frac{\partial \dot{q}_i}{\partial p_j}$$

The second and third terms vanish since $p_i = \partial \mathcal{L} / \partial \dot{q}_i$, so that we obtain that $\partial \mathcal{H} / \partial p_j = \dot{q}_j$. Similarly we obtain that

$$\frac{\partial \mathcal{H}}{\partial q_j} = \sum_{i=1}^{n} p_i \frac{\partial \dot{q}_i}{\partial q_j} - \frac{\partial \mathcal{L}}{\partial q_j} - \sum_{i=1}^{n} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \frac{\partial \dot{q}_i}{\partial q_j}$$

Here the first and third terms cancel, and since the Lagrange equations tell us that $\partial \mathcal{L} / \partial q_j = \dot{p}_j$, we obtain that $\partial \mathcal{H} / \partial q_j = -\dot{p}_j$.

This yields the Hamiltonian equations of motion

$$\frac{\partial \mathcal{H}}{\partial p_i} = \dot{q}_i \qquad\qquad \frac{\partial \mathcal{H}}{\partial q_i} = -\dot{p}_i$$

Note that whereas Lagrange's equations are a set of n second-order differential equations, Hamilton's equations are a set of $2n$ first-order differential equations. Although they are easier to solve, deriving the Hamiltonian itself is more involved.

The Hamiltonian Formulation III

The Hamiltonian description is especially useful for finding conserved quantities, which will play an important role in describing orbits.

If a generalized coordinate, say q_i, does not appear in the Hamiltonian, then the corresponding conjugate momentum p_i is a conserved quantity!!!

In the case of motion in a fixed potential, the Hamiltonian is equal to the total energy, i.e., $\mathcal{H} = E$

DEMONSTRATION: for a time-independent potential $\Phi = \Phi(\vec{x})$ the Lagrangian is equal to $\mathcal{L} = \frac{1}{2}\dot{\vec{x}}^2 - \Phi(\vec{x})$. Since $\vec{p} = \partial\mathcal{L}/\partial\dot{\vec{x}} = \dot{\vec{x}}$ we have that $\mathcal{H} = \dot{\vec{x}} \cdot \dot{\vec{x}} - \frac{1}{2}\dot{\vec{x}}^2 + \Phi(\vec{x}) = \frac{1}{2}\dot{\vec{x}}^2 + \Phi(\vec{x}) = E$

The $2n$-dimensional phase-space of a dynamical system with n degrees of freedom can be described by the generalized coordinates and momenta $(\vec{q}, \vec{p})$. Since Hamilton's equations are first order differential equations, we can determine $\vec{q}(t)$ and $\vec{p}(t)$ at any time t once the initial conditions $(\vec{q}_0, \vec{p}_0)$ are given. Therefore, through each point in phase-space there passes a unique trajectory $\Gamma[\vec{q}(\vec{q}_0, \vec{p}_0, t), \vec{p}(\vec{q}_0, \vec{p}_0, t)]$. No two trajectories Γ_1 and Γ_2 can pass through the same $(\vec{q}_0, \vec{p}_0)$ unless $\Gamma_1 = \Gamma_2$.

The Hamiltonian Formulation IV

As an example, let's consider once more the motion in a central force field. Our generalized coordinates are the polar coordinates (r, θ), and, as we have seen before the Lagrangian is $\mathcal{L} = \frac{1}{2}\dot{r}^2 + \frac{1}{2}r^2\dot{\theta}^2 - \Phi(r)$

The conjugate momenta are $p_r = \frac{\partial \mathcal{L}}{\partial \dot{r}} = \dot{r}$ and $p_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = r^2\dot{\theta}$

so that the Hamiltonian becomes

$$\mathcal{H} = \frac{1}{2}p_r^2 + \frac{1}{2}\frac{p_\theta^2}{r^2} + \Phi(r)$$

Hamilton's equations now become

$$\frac{\partial \mathcal{H}}{\partial r} = -\frac{p_\theta^2}{r^3} + \frac{\partial \Phi}{\partial r} = -\dot{p}_r \qquad \frac{\partial \mathcal{H}}{\partial \theta} = 0 = -\dot{p}_\theta$$

$$\frac{\partial \mathcal{H}}{\partial p_r} = p_r = \dot{r} \qquad \frac{\partial \mathcal{H}}{\partial p_\theta} = \frac{p_\theta}{r^2} = \dot{\theta}$$

which reduce to

$$\ddot{r} - r\dot{\theta}^2 = -\frac{\partial \Phi}{\partial r} \qquad p_\theta = r^2\dot{\theta} = \mathrm{cst}$$

Note that θ does not appear in the Hamiltonian: consequently p_θ is a conserved quantity

Noether's Theorem

In 1915 the German mathematician Emmy Noether proved an important theorem which plays a trully central role in theoretical physics.

Noether's Theorem: If an ordinary Lagrangian posseses some continuous, smooth symmetry, then there will be a conservation law associated with that symmetry.

- **Invariance of $\mathcal{L}$ under time translation $\longrightarrow$ energy conservation**
- **Invariance of $\mathcal{L}$ under spatial translation $\longrightarrow$ momentum conservation**
- **Invariance of $\mathcal{L}$ under rotational translation $\longrightarrow$ ang. mom. conservation**
- **Gauge Invariance of electric potential $\longrightarrow$ charge conservation**

Some of these symmetries are immediately evident from the Lagrangian:

- **If $\mathcal{L}$ does not explicitely depend on t then E is conserved**
- **If $\mathcal{L}$ does not explicitely depend on q_i then p_i is conserved**

Poisson Brackets I

DEFINITION: Let $A(\vec{q}, \vec{p})$ and $B(\vec{q}, \vec{p})$ be two functions of the generalized coordinates and their conjugate momenta, then the **Poisson bracket** of A and B is defined by

$$[A, B] = \sum_{i=1}^{n} \left[\frac{\partial A}{\partial q_i} \frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i} \frac{\partial B}{\partial q_i} \right]$$

Let $f = f(\vec{q}, \vec{p}, t)$ then

$$\mathrm{d}f = \frac{\partial f}{\partial q_i} \mathrm{d}q_i + \frac{\partial f}{\partial p_i} \mathrm{d}p_i + \frac{\partial f}{\partial t} \mathrm{d}t$$

where we have used the summation convention. This differential of f, combined with **Hamilton's equations**, allows us to write

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q_i} \frac{\partial \mathcal{H}}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial \mathcal{H}}{\partial q_i}$$

which reduces to

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + [f, \mathcal{H}]$$

This is often called **Poisson's equation of motion**. It shows that the time-evolution of any dynamical variable is governed by the Hamiltonian through the Poisson bracket of the variable with the Hamiltonian.

Poisson Brackets II

Using the Poisson brackets we can write

$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = \frac{\partial \mathcal{H}}{\partial t} + [\mathcal{H}, \mathcal{H}] = \frac{\partial \mathcal{H}}{\partial t} = \frac{\partial \mathcal{L}}{\partial t}$$

where the latter equality follows from $\mathcal{H} = \vec{p} \cdot \dot{\vec{q}} - \mathcal{L}$.

For an equilibrium system with a time-independent potential, $\partial\Phi/\partial t = 0$, we have that $\partial\mathcal{H}/\partial t = 0$ and thus also $\mathrm{d}\mathcal{H}/\mathrm{d}t = 0$. Since in this case the Hamiltonian is equal to the total energy, this simply reflects the energy conservation. Note that for any conservative system, $\mathcal{H}$ does not explicitely depend on time, and thus $\mathrm{d}\mathcal{H}/\mathrm{d}t = 0$

With the help of the Poisson brackets we can write Hamilton's equations in a more compact form

$$\dot{q}_i = [q_i, \mathcal{H}] \qquad\qquad \dot{p}_i = [p_i, \mathcal{H}]$$

Note that it is explicit that these equations of motion are valid in any system of generalized coordinated $(q_1, q_2, .., q_n)$ and their conjugate momenta $(p_1, p_2, .., p_n)$. As we will see next, in fact Hamilton's equations hold for any so-called canonical coordinate system.

Canonical Coordinate Systems

If we write $w_i = q_i$ and $w_{n+i} = p_i$ with $i = 1, .., n$ and we define the symplectic matrix c as

$$c_{\alpha\beta} \equiv [w_\alpha, w_\beta] = \begin{cases} \pm 1 & \text{if } \alpha = \beta \mp n \\ 0 & \text{otherwise} \end{cases}$$

with $\alpha, \beta \in [1, 2n]$, then

$$[A, B] = \sum_{\alpha,\beta=1}^{2n} c_{\alpha\beta} \frac{\partial A}{\partial w_\alpha} \frac{\partial B}{\partial w_\beta}$$

DEFINITION: Any set of $2n$ phase-space coordinates $\{w_\alpha, \alpha = 1, .., 2n\}$ is called canonical if $[w_\alpha, w_\beta] = c_{\alpha\beta}$.

Hamilton's equations can now be written in the extremely compact form:

$$\dot{w}_\alpha = [w_\alpha, \mathcal{H}]$$

which makes it explicit that they hold for any canonical coordinate system.

Note that the generalized coordinates and momenta $(\vec{q}, \vec{p})$ form a canonical coordinate system, since they obey the canonical commutation relations

$$[q_i, q_j] = [p_i, p_j] = 0 \qquad [p_i, q_j] = \delta_{ij}$$

Canonical Transformations I

Canonical Transformation: a transformation $(\vec{q}, \vec{p}) \longrightarrow (\vec{Q}, \vec{P})$ between two canonical coordinate systems that leaves the equations of motion invariant.

In order to reveal the form of these transformations, we first demonstrate the **non-uniqueness** of the **Lagrangian**.

Consider a transformation $\mathcal{L} \longrightarrow \mathcal{L}' = \mathcal{L} + \frac{\mathrm{d}F}{\mathrm{d}t}$ where $F = F(\vec{q}, t)$

Under this transformation the **action integral** becomes

$$I' = \int_{t_0}^{t_1} \mathcal{L}' \mathrm{d}t = \int_{t_0}^{t_1} \mathcal{L} \mathrm{d}t + \int_{t_0}^{t_1} \frac{\mathrm{d}F}{\mathrm{d}t} \mathrm{d}t = I + F(t_1) - F(t_0)$$

Recall that the **equations of motion** correspond to $\delta I = 0$ (i.e., the action is stationary). Since the addition of $\mathrm{d}F/\mathrm{d}t$ only adds a **constant**, namely $F(t_1) - F(t_0)$ to the action, it leaves the equations of motion invariant.

Canonical Transformations II

Now consider our transformation $(\vec{q}, \vec{p}) \longrightarrow (\vec{Q}, \vec{P})$ with corresponding

Lagrangians $\mathcal{L}(\vec{q}, \dot{\vec{q}}, t)$ and $\mathcal{L}'(\vec{Q}, \dot{\vec{Q}}, t)$.

We start by writing the Lagrangians in terms of the corresponding Hamiltonians:

$$\begin{aligned} \mathcal{L}(\vec{q}, \vec{p}, t) &= \vec{p} \cdot \dot{\vec{q}} - \mathcal{H}(\vec{q}, \vec{p}, t) \\ \mathcal{L}'(\vec{Q}, \vec{P}, t) &= \vec{P} \cdot \dot{\vec{Q}} - \mathcal{H}'(\vec{Q}, \vec{P}, t) \end{aligned}$$

In order for the equations of motion to be invariant, we have the requirement that

$$\begin{aligned} & \mathcal{L}(\vec{q}, \vec{p}, t) = \mathcal{L}'(\vec{Q}, \vec{P}, t) + \frac{\mathrm{d}F}{\mathrm{d}t} \\ \Leftrightarrow \quad & \frac{\mathrm{d}F}{\mathrm{d}t} = \vec{p} \cdot \dot{\vec{q}} - \mathcal{H}(\vec{q}, \vec{p}, t) - \left[\vec{P} \cdot \dot{\vec{Q}} - \mathcal{H}'(\vec{Q}, \vec{P}, t)\right] \\ \Leftrightarrow \quad & \mathrm{d}F = p_i \mathrm{d}q_i - P_i \mathrm{d}Q_i + (\mathcal{H}' - \mathcal{H})\mathrm{d}t \end{aligned}$$

If we take $F = F(\vec{q}, \vec{Q}, t)$ then we also have that

$$\mathrm{d}F = \frac{\partial F}{\partial q_i}\mathrm{d}q_i + \frac{\partial F}{\partial Q_i}\mathrm{d}Q_i + \frac{\partial F}{\partial t}\mathrm{d}t$$

Canonical Transformations III

Equating the two expressions for the differential $\mathrm{d}F$ yields the transformation rules

$$p_i = \frac{\partial F}{\partial q_i} \qquad P_i = -\frac{\partial F}{\partial Q_i} \qquad \mathcal{H}' = \mathcal{H} + \frac{\partial F}{\partial t}$$

The function $F(\vec{q}, \vec{Q}, t)$ is called the generating function of the canonical transformation $(\vec{q}, \vec{p}) \longrightarrow (\vec{Q}, \vec{P})$

In order to transform $(\vec{q}, \vec{p}) \longrightarrow (\vec{Q}, \vec{P})$ one proceeds as follows:

- Find a function $F(\vec{q}, \vec{Q})$ so that $p_i = \partial F / \partial q_i$. This yields $Q_i(q_j, p_j)$
- Substitute $Q_i(q_j, p_j)$ in $P_i = \partial F / \partial Q_i$ to obtain $P_i(q_j, p_j)$

As an example consider the generating function $F(\vec{q}, \vec{Q}) = q_i O_i$.
According to the transformation rules we have that

$$p_i = \frac{\partial F}{\partial q_i} = Q_i \qquad P_i = -\frac{\partial F}{\partial Q_i} = -q_i$$

We thus have that $Q_i = p_i$ and $P_i = -q_i$: the canonical transformation has changed the roles of coordinates and momenta, eventhough the equations of motion have remained invariant! This shows that there is no special status to either generalized coordinates or their conjugate momenta

Canonical Transformations IV

For reasons that will become clear later, in practice it is more useful to consider a generating function of the form $S = S(\vec{q}, \vec{P}, t)$, i.e., one that depends on the old coordinates and the new momenta.

To derive the corresponding transformation rules, we start with the generating function $F = F(\vec{q}, \vec{Q}, t)$, and recall that

$$\mathrm{d}F = p_i \mathrm{d}q_i - P_i \mathrm{d}Q_i + (\mathcal{H}' - \mathcal{H})\mathrm{d}t$$

using that $P_i \mathrm{d}Q_i = \mathrm{d}(Q_i P_i) - Q_i \mathrm{d}P_i$, we obtain

$$\mathrm{d}(F + Q_i P_i) = p_i \mathrm{d}q_i + Q_i \mathrm{d}P_i + (\mathcal{H}' - \mathcal{H})\mathrm{d}t$$

Defining the new generator $S(\vec{q}, \vec{Q}, \vec{P}, t) \equiv F(\vec{q}, \vec{Q}, t) + \vec{Q} \cdot \vec{P}$, for which

$$\mathrm{d}S = \frac{\partial S}{\partial q_i}\mathrm{d}q_i + \frac{\partial S}{\partial Q_i}\mathrm{d}Q_i + \frac{\partial S}{\partial P_i}\mathrm{d}P_i + \frac{\partial S}{\partial t}\mathrm{d}t$$

Equating this to the above we find the transformation rules

$$p_i = \frac{\partial S}{\partial q_i} \qquad Q_i = \frac{\partial S}{\partial P_i} \qquad \frac{\partial S}{\partial Q_i} = 0 \qquad \mathcal{H}' = \mathcal{H} + \frac{\partial S}{\partial t}$$

Note that the third of these rules implies that $S = S(\vec{q}, \vec{P}, t)$ as intended.

Canonical Transformations V

The potential strength of canonical transformations becomes apparent from the following: Suppose one can find a canonical transformation $(\vec{q}, \vec{p}) \longrightarrow (\vec{Q}, \vec{P})$ such that $\mathcal{H}(\vec{q}, \vec{p}) \longrightarrow \mathcal{H}'(\vec{P})$, i.e., such that the new Hamiltonian does not explicitely depend on the new coordinates Q_i.

Hamilton's equation of motion then become

$$\frac{\partial \mathcal{H}'}{\partial Q_i} = -\dot{P}_i = 0 \qquad\qquad \frac{\partial \mathcal{H}'}{\partial P_i} = -\dot{Q}_i$$

Thus, we have that all the conjugate momenta P_i are constant, and this in turn implies that none of $\dot{Q}_i$ can depend on time either. The equations of motion in our new, canonical coordinate system are therefore extremely simple:

$$Q_i(t) = \Omega_i t + k_i \qquad\qquad P_i = \text{constant}$$

Here $\Omega_i = \partial \mathcal{H}'/\partial P_i$ are constants and k_i are integration constants. Any generalized coordinate whose conjugate momentum is a conserved quantity, is called a cyclic variable. The question that remains now is how to find the generator $S(q, P, t)$ of the canonical transformation $(\vec{q}, \vec{p}) \longrightarrow (\vec{Q}, \vec{P})$ which leads to only cyclic variables Q_i.

The Hamilton-Jacobi Equation I

Recall the transformation rules for the generator $S(\vec{q}, \vec{P}, t)$:

$$p_i = \frac{\partial S}{\partial q_i} \qquad Q_i = \frac{\partial S}{\partial P_i} \qquad \mathcal{H}' = \mathcal{H} + \frac{\partial S}{\partial t}$$

If for simplicity we consider a generator that does not explicitely depend on time, i.e., $\partial S/\partial t = 0$ then we have that $\mathcal{H}(\vec{q}, \vec{p}) = \mathcal{H}'(\vec{P}) = E$. If we now substitute $\partial S/\partial q_i$ for p_i in the original Hamiltonian we obtain

$$\mathcal{H}\left(\frac{\partial S}{\partial q_i}, q_i\right) = E$$

This is the Hamilton-Jacobi equation, which is a partial differential equation. If it can be solved for $S(\vec{q}, \vec{P})$ than, as we have seen above, basically the entire dynamics are solved.

Thus, for a dynamical system with n degrees of freedom, one can solve the dynamics in one of the three following ways:

- Solve n second-order differential equations (Lagrangian formalism)
- Solve $2n$ first-order differential equations (Hamiltonian formalism)
- Solve a single partial differential equation (Hamilton-Jacobi equation)

The Hamilton-Jacobi Equation II

Although it may seem an attractive option to try and solve the Hamilton-Jacobi equation, solving partial differential equations is in general much more difficult than solving ordinary differential equations, and the Hamilton-Jacobi equation is no exception.

However, in the specific case where the generator S is separable, i.e., if

$$S(\vec{q}, \vec{P}) = \sum_{i=1}^{n} f_i(q_i)$$

with f_i a set of n independent functions, then the Hamilton-Jacobi equation splits in a set of n ordinary differential equations which are easily solved by quadrature. The integration constants are related to the (constant) conjugate momenta P_i.

A Hamiltonian is called 'integrable' if the Hamilton-Jacobi equation is separable

Integrable Hamiltonians are extremely rare. Mathematically speaking they form a set of measure zero in the space of all Hamiltonians. In what follows, we establish the link between so-called isolating integrals of motion and whether or not a Hamiltonian is integrable.