

The Distribution Function I

We have seen that the dynamics of our discrete system of N point masses is given by $6N$ equations of motion, which allow us to compute $6N$ unknowns (\vec{x}, \vec{v}) as function of time t .

The system is completely specified by $6N$ initial conditions (\vec{x}_0, \vec{v}_0)

We can specify these initial conditions by defining the **distribution function (DF)**, also called the **phase-space density**

$$f(\vec{x}, \vec{v}, t_0) = \sum_{i=1}^N \delta(\vec{x} - \vec{x}_{i,0}) \sum_{i=1}^N \delta(\vec{v} - \vec{v}_{i,0})$$

Once $f(\vec{x}, \vec{v}, t)$ is specified at any time t , we can infer $f(\vec{x}, \vec{v}, t')$ at any other time t'

The DF $f(\vec{x}, \vec{v}, t)$ completely specifies a collisionless system

In the case of our **smooth** density distribution we define the 6 dimensional **phase-space density** :

$$f(\vec{x}, \vec{v}, t) d^3 \vec{x} d^3 \vec{v}$$

NOTE: A necessary, physical condition is that $f \geq 0$

The Distribution Function II

The density $\rho(\vec{x})$ follows from $f(\vec{x}, \vec{v})$ by integrating over velocity space:

$$\rho(\vec{x}, t) = \int \int \int f(\vec{x}, \vec{v}, t) d^3\vec{v}$$

while the total mass follows from

$$M(t) = \int \int \int \rho(\vec{x}, t) d^3\vec{x} = \int d^3\vec{x} \int d^3\vec{v} f(\vec{x}, \vec{v}, t)$$

It is useful to think about the DF as a **probability function** (once normalized by M), which expresses the probability of finding a star in a phase-space volume $d^3\vec{x}d^3\vec{v}$. This means we can compute the **expectation value** for any quantity Q as follows:

$$\begin{aligned}\langle Q(\vec{x}, t) \rangle &= \frac{1}{\rho(\vec{x})} \int d^3\vec{v} Q(\vec{x}, \vec{v}) f(\vec{x}, \vec{v}, t) \\ \langle Q(t) \rangle &= \frac{1}{M} \int d^3\vec{x} \int d^3\vec{v} Q(\vec{x}, \vec{v}) f(\vec{x}, \vec{v}, t)\end{aligned}$$

EXAMPLES:

RMS velocity: $\langle v_i^2 \rangle = \frac{1}{M} \int d^3\vec{x} \int d^3\vec{v} v_i^2 f(\vec{x}, \vec{v}, t)$

Velocity Profile: $\mathcal{L}(x, y, v_z) = \int \int \int f(\vec{x}, \vec{v}, t) dz dv_x dv_y$

Surface Brightness: $\Sigma(x, y) = \int \int \int \int f(\vec{x}, \vec{v}, t) dz dv_x dv_y dv_z$

The Distribution Function III

Each particle (star) follows a **trajectory** in the 6D phase-space (\vec{x}, \vec{v}) , which is completely governed by Newtonian Dynamics (for a collisionless system).

This trajectory projected in the 3D space \vec{x} is called the **orbit** of the particle.

As we will see later, the **Lagrangian** time-derivative of the DF, i.e. the time-derivative of $f(\vec{x}, \vec{v}, t)$ as seen when travelling through phase-space **along the particle's trajectory**, is

$$\frac{df}{dt} = 0$$

This simple equation is the single most important equation for collisionless dynamics. It completely specifies the evolution of a collisionless system, and is called the **Collisionless Boltzmann Equation (C.B.E.)** or **Vlasov equation**.

The flow in phase-space is incompressible.

NOTE: Don't confuse this with $\frac{\partial f}{\partial t} = 0$!!!!

This is the **Eulerian** time-derivative as seen from a **fixed** phase-space location, which is only equal to zero for a system in **steady-state equilibrium**.

Collisionless Dynamics in a Nutshell

$$\begin{aligned}\rho(\vec{x}) &= \int f(\vec{x}, \vec{v}) d^3\vec{v} \\ \nabla^2 \Phi(\vec{x}) &= 4\pi G \rho(\vec{x}) \\ \frac{df}{dt} &= 0\end{aligned}$$

The **self-consistency problem** of finding the orbits that reproduce $\rho(\vec{x})$ is equivalent to finding the DF $f(\vec{x}, \vec{v})$ which yields $\rho(\vec{x})$.

Problem: For most systems we only have constraints on a 3D projection of the 6D distribution function.

Recall: $\mathcal{L}(x, y, v_z) = \int \int \int f(\vec{x}, \vec{v}, t) dz dv_x dv_y$

Circular & Escape Velocities I

Consider a spherical density distribution $\rho(r)$ for which the **Poisson Equation** reads

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \Phi}{\partial r} \right) = 4\pi G \rho(r)$$

from which we obtain that

$$r^2 \frac{\partial \Phi}{\partial r} = 4\pi G \int_0^r \rho(r) r^2 dr = GM(r)$$

with $M(r)$ the enclosed mass. This allows us to write

$$\vec{F}_{\text{grav}}(\vec{r}) = -\vec{\nabla} \Phi(\vec{r}) = -\frac{d\Phi}{dr} \vec{e}_r = -\frac{GM(r)}{r^2} \vec{e}_r$$

Because gravity is a **central, conservative** force, both the energy and angular momentum are conserved, and a particle's orbit is confined to a plane.

Introducing the **polar coordinates** (r, θ) we write

$$\begin{aligned} E &= \frac{1}{2} (\dot{r}^2 + r^2 \dot{\theta}^2) + \Phi(r) \\ J &= r^2 \dot{\theta} \end{aligned}$$

Eliminating $\dot{\theta}$ we obtain the **Radial Energy Equation**:

$$\frac{1}{2} \dot{r}^2 + \frac{J^2}{2r^2} + \Phi(r) = E$$

Circular & Escape Velocities II

In the **co-rotating frame**, the equation of motion reduces to a one-dimensional radial motion under influence of the **effective potential** $U(r) = \frac{J^2}{2r^2} + \Phi(r)$. The 'extra' term arises due to the non-inertial nature of the reference frame, and corresponds to the **centrifugal force**

$$\vec{F}_{\text{cen}} = -\frac{d}{dr} \left(\frac{J^2}{2r^2} \right) \vec{e}_r = \frac{J^2}{r^3} \vec{e}_r = \frac{v_\theta^2}{r} \vec{e}_r$$

For a **circular orbit** we have that $\vec{F}_{\text{cen}} = -\vec{F}_{\text{grav}}$, so that we obtain the **circular speed**.

$$v_c(r) = \sqrt{r \frac{d\Phi}{dr}} = \sqrt{\frac{GM(r)}{r}}$$

Thus, $rv_c^2(r)$ measures the mass enclosed within radius r (in spherical symmetry). Note that for a point mass $v_c(r) \propto r^{-1/2}$, which is called a **Keplerian rotation curve**

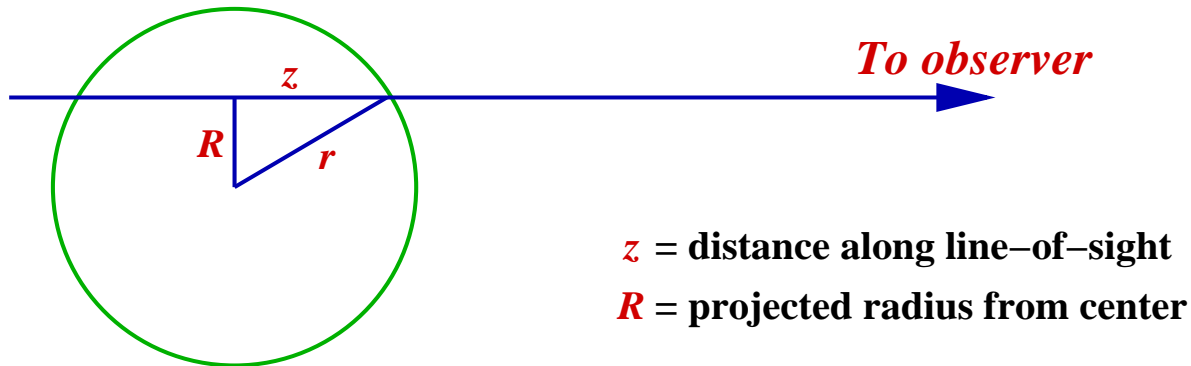
Escape Speed: The speed a particle needs in order to 'escape' to infinity

$$v_{\text{esc}}(r) = \sqrt{2|\Phi(r)|}$$

Recall: The energy per unit mass is $E = \frac{1}{2}v^2 + \Phi(r)$. In order to escape to infinity we need $E \geq 0$, which translates into $v^2 \geq 2|\Phi(r)|$

Projected Surface Density

Consider a spherical system with intrinsic, 3D **luminosity** distribution $\nu(r)$. An observer, at large distance, observes the projected, 2D **surface brightness distribution** $\Sigma(R)$



$$\Sigma(R) = 2 \int_0^{\infty} \nu(r) dz = 2 \int_R^{\infty} \nu(r) \frac{r dr}{\sqrt{r^2 - R^2}}$$

This is a so-called **Abel Integral**, for which the inverse is:

$$\nu(r) = -\frac{1}{\pi} \int_r^{\infty} \frac{d\Sigma}{dR} \frac{dR}{\sqrt{R^2 - r^2}}$$

Thus, an observed surface brightness distribution $\Sigma(R)$ of a **spherical system** can be **deprojected** to obtain the 3D light distribution $\nu(r)$. However, because it requires the determination of a derivative, it can be fairly noisy.

Spherical Potential-Density Pairs

To compute the potential of a spherical density distribution $\rho(r)$ we can make use of **Newton's Theorems**

First Theorem A body inside a spherical shell of matter experiences no net gravitational force from that shell.

Second Theorem The gravitational force on a body that lies outside a closed spherical shell of mass M is the same as that of a point mass M at the center of the shell.

Based on these two Theorems, we can compute $\Phi(r)$ by splitting $\rho(r)$ in spherical shells, and adding the potentials of all these shells:

$$\Phi(r) = -4\pi G \left[\frac{1}{r} \int_0^r \rho(r') r'^2 dr' + \int_r^\infty \rho(r') r' dr' \right]$$

Using the definition of the **enclosed mass** $M(r) = 4\pi \int_0^r \rho(r') r'^2 dr'$ this can be rewritten as

$$\Phi(r) = -\frac{GM(r)}{r} - 4\pi G \int_r^\infty \rho(r') r' dr'$$

Power-law Density Profiles I

Consider a spherical system with a simple power-law density distribution

$$\rho(r) = \rho_0 \left(\frac{r}{r_0} \right)^{-\alpha}$$

$$\Sigma(R) = 2 \int_R^{\infty} \rho(r) \frac{r \, dr}{\sqrt{r^2 - R^2}} = \rho_0 r_0^\alpha B\left(\frac{\alpha}{2} - \frac{1}{2}, \frac{1}{2}\right) R^{1-\alpha}$$

$$M(< r) = 4\pi \int_0^r \rho(r') r'^2 \, dr' = \frac{4\pi\rho_0 r_0^\alpha}{3-\alpha} r^{3-\alpha} \quad (\alpha < 3)$$

$$M(> r) = 4\pi \int_r^{\infty} \rho(r') r'^2 \, dr' = \frac{4\pi\rho_0 r_0^\alpha}{\alpha-3} r^{3-\alpha} \quad (\alpha > 3)$$

NOTE: For $\alpha \geq 3$ the enclosed mass is infinite, while for $\alpha \leq 3$ the total mass ($r \rightarrow \infty$) is infinite: **A pure power-law system can not exist in nature!**

A more realistic density distribution consists of a **double power-law**:

At small radii: $\rho \propto r^{-\alpha}$ with $\alpha < 3$

At large radii: $\rho \propto r^{-\beta}$ with $\beta > 3$

$B(x, y)$ is the so-called Beta-Function, which is related to the Gamma Function $\Gamma(x)$

$$B(x, y) = \frac{\Gamma(x) \Gamma(y)}{\Gamma(x+y)} = B(y, x)$$

Power-law Density Profiles II

The potential of a power-law density distribution is:

$$\Phi(r) = \begin{cases} \frac{4\pi G \rho_0 r_0^\alpha}{(\alpha-3)(\alpha-2)} r^{2-\alpha} & \text{if } 2 < \alpha < 3 \\ \infty & \text{otherwise} \end{cases}$$

The circular and escape velocities of a power-law density distribution are:

$$v_c^2(r) = r \frac{d\Phi}{dr} = \frac{G M(r)}{r} = \frac{4\pi G \rho_0 r_0^\alpha}{3-\alpha} r^{2-\alpha}$$

$$v_{\text{esc}}^2(r) = \frac{2}{\alpha-2} v_c^2(r)$$

$\alpha = 2$: Singular Isothermal Sphere $v_c = \text{constant}$ (flat rotation curve)

$\alpha = 0$: Homogeneous Sphere $v_c \propto r$ (solid body rotation)

NOTE: For $\alpha > 3$ you find that $v_c(r)$ falls off more rapidly than **Keplerian**.

How can this be? After all, a Keplerian RC corresponds to a delta-function density distribution (point mass), which is the most concentrated mass distribution possible....

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answer: the circular velocity is defined via the gradient of the potential. As shown above, Φ is only defined for $2 < \alpha < 3$, and therefore so does v_c

Power-law Density Profiles: Summary

It is very useful to remember the following scaling relations:

$\rho(r)$	\propto	$r^{-\alpha}$	
$\Sigma(R)$	\propto	$R^{1-\alpha}$	
$\Phi(r)$	\propto	$r^{2-\alpha}$	$(2 < \alpha < 3)$
$v_c^2(r)$	\propto	$r^{2-\alpha}$	$(2 < \alpha < 3)$
$M(< r)$	\propto	$r^{3-\alpha}$	$(\alpha < 3)$
$M(> r)$	\propto	$r^{3-\alpha}$	$(\alpha > 3)$

Double Power-law Density Profiles

As we have seen, no realistic system can have a density distribution that is described by a **single** power-law. However, many often used density distributions have a **double** power-law.

$$\rho(r) = \frac{C}{r^\gamma (1+r^{1/\alpha})^{(\beta-\gamma)\alpha}}$$

At small radii, $\rho \propto r^{-\gamma}$, while at large radii $\rho \propto r^{-\beta}$. The parameter α determines the ‘sharpness’ of the break.

NOTE: In order for the mass to be finite, $\gamma < 3$ and $\beta > 3$

(α, β, γ)	Name	Reference
(1, 3, 1)	NFW Profile	Navarro, Frenk & White, 1997, ApJ, 490, 493
(1, 4, 1)	Hernquist Profile	Hernquist, 1990, ApJ, 356, 359
(1, 4, 2)	Jaffe Profile	Jaffe, 1983, MNRAS, 202, 995
(1, 4, $\frac{3}{2}$)	Moore Profile	Moore et al., 1999, MNRAS, 310, 1147
($\frac{1}{2}$, 2, 0)	Modified Isothermal Sphere	Sacket & Sparke, 1990, ApJ, 361, 409
($\frac{1}{2}$, 3, 0)	Modified Hubble Profile	Binney & Tremaine, p. 39
($\frac{1}{2}$, 4, 0)	Perfect Sphere	de Zeeuw, 1985, MNRAS, 216, 273
($\frac{1}{2}$, 5, 0)	Plummer Model	Plummer, 1911, MNRAS, 71, 460

Ellipsoids I

Thus far we have only considered **spherical systems**. However, only very few systems in nature are truly spherical. A more general, though still not fully general, form to consider is the **ellipsoid**.

Without losing generality, we will use the following definition of the **ellipsoidal radius**

$$m^2 = a_1^2 \sum_{i=1}^3 \frac{x_i^2}{a_i^2} \quad a_1 \geq a_2 \geq a_3$$

Note that we have taken the three **principal axes** to be aligned with our Cartesian coordinate system (x, y, z) . If $a_1 > a_2 > a_3$ then the ellipsoid is said to be **triaxial**

A body whose isodensity surfaces are **concentric ellipsoids** is called an **ellipsoidal body**.

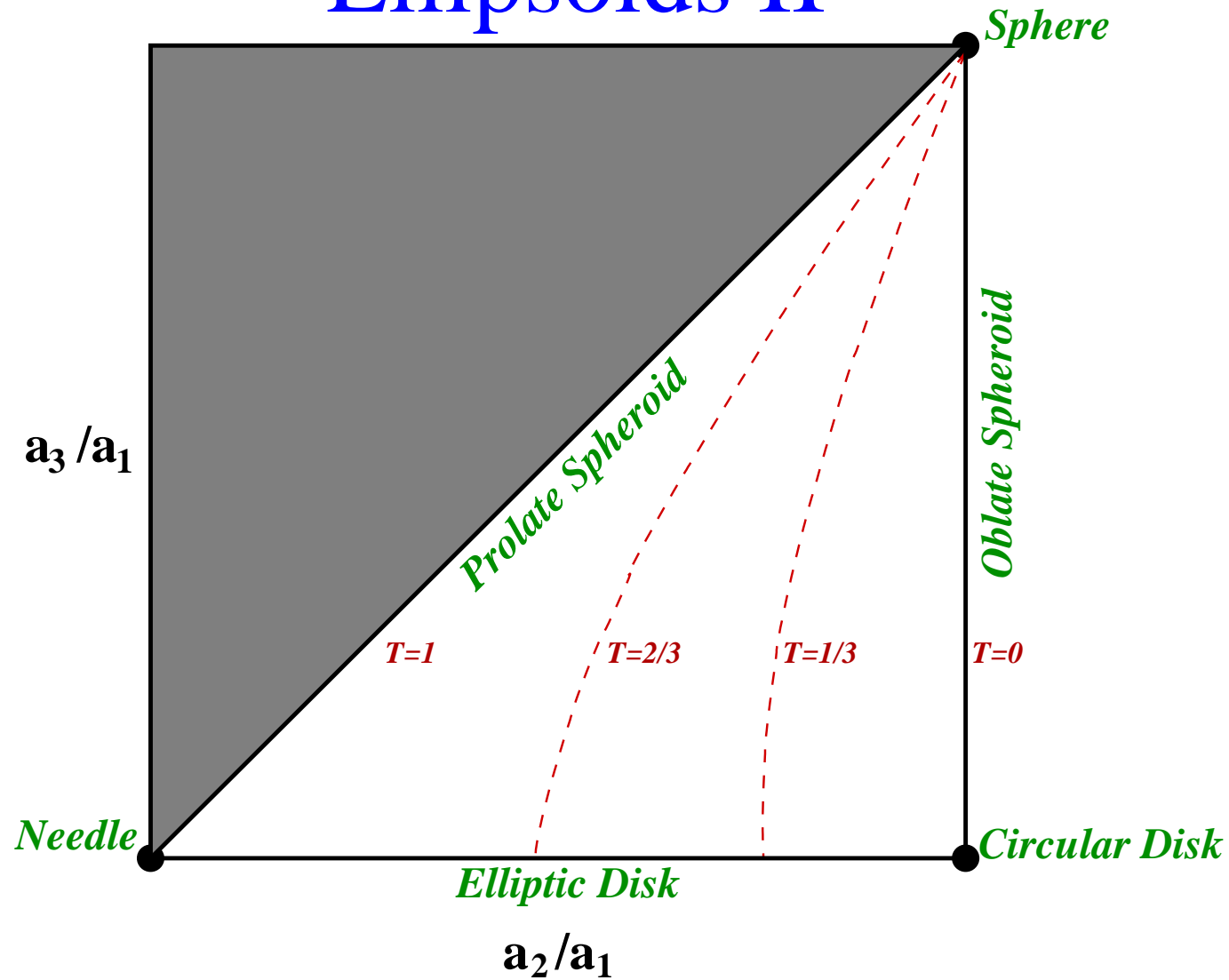
Triaxiality Parameter:

$$T \equiv \frac{1 - (a_2/a_1)^2}{1 - (a_3/a_1)^2}$$

A **spheroid** is an **axisymmetric ellipsoid** with two equal principal axes:

- **Oblate Spheroid:** $a_1 = a_2 > a_3$ ($T = 0$) (i.e. Earth)
- **Prolate Spheroid:** $a_1 > a_2 = a_3$ ($T = 1$) (i.e. Cigar)

Ellipsoids II



For an oblate spheroid with axis ratio $q = a_3/a_1$, we define:

Ellipticity: $\varepsilon = 1 - q$

Eccentricity: $e = \sqrt{1 - q^2}$

Ellipsoids III

A shell of **similar, concentric** ellipsoids is called a **homoeoid**. Note that the perpendicular distance d between the two ellipsoids is a function of the angular position.

In what follows we consider the family of ellipsoidal bodies whose density distribution is the sum of thin homoeoids.

Homoeoid Theorem: The exterior isopotential surface of a homoeoidal shell of negligible thickness are the spheroids that are **confocal** with the shell itself. Inside the shell the potential is constant.

This implies that:

- The equipotentials of a homoeoid become spherical at large radii.
- The equipotential of a thin homoeoid has the same shape as the homoeoid at the location of the homoeoid.

NOTE: the Homoeoid Theorem applies only to **thin** homoeoids. However, for **any** homoeoid of **any** thickness we have:

Newton's Third Theorem: A mass that is inside a homoeoid experiences no net gravitational force from the homoeoid. $\Phi_{\text{inside}} = \text{constant}$

Ellipsoids IV

Consider a spheroidal density distribution $\rho(R, z) = \rho(m^2)$ with $m^2 = R^2 + z^2/q^2$, then the potential is:

$$\Phi(R, z) = -2\pi Gq \frac{\arcsine}{e} \psi(\infty) + \pi Gq \int_0^\infty \frac{\psi(m) d\tau}{(\tau+1)\sqrt{\tau+q^2}}$$

Here $e = \sqrt{1 - q^2}$ is the **eccentricity**,

$$m^2 = \frac{R^2}{\tau+1} + \frac{z^2}{\tau+q^2}$$

and

$$\psi(m) \equiv \int_0^{m^2} \rho(m'^2) dm'^2$$

The corresponding **circular velocity** in the equatorial plane $z = 0$ is

$$v_c^2(R) = R \frac{\partial \Phi}{\partial R} = 4\pi Gq \int_0^R \frac{\rho(m^2) m^2 dm}{\sqrt{R^2 - m^2 e^2}}$$

Ellipsoids V

- In general one finds that $v_c(R)$ increases with larger flattening q : Flatter systems with the same spheroidal, enclosed mass have larger circular speeds at given R .
- Let $\varepsilon_\rho = 1 - q$ the **ellipticity** of the density distribution. One always has that $\varepsilon_\Phi \leq \varepsilon_\rho$. At a few characteristic radii, a reasonable rule of thumb is that $\varepsilon_\Phi \sim \frac{1}{3}\varepsilon_\rho$

We can generalize the equations on the previous page for a **triaxial, ellipsoidal** density distribution $\rho(\vec{x}) = \rho(m^2)$ with

$$m^2 = a_1^2 \sum_{i=1}^3 \frac{x_i^2}{a_i^2}$$

The corresponding potential is

$$\Phi(\vec{x}) = -\pi G \left(\frac{a_2 a_3}{a_1} \right) \int_0^\infty \frac{[\psi(\infty) - \psi(m)] d\tau}{\sqrt{(\tau + a_1^2)(\tau + a_2^2)(\tau + a_3^2)}}$$

with

$$\frac{m^2}{a_1^2} = \sum_{i=1}^3 \frac{x_i^2}{\tau + a_i^2}$$

Multipole Expansion I

In order to calculate the potential of an **arbitrary** density distribution, it is useful to consider a **Multipole expansion**.

Using **separation of variables**, $\Phi(r, \theta, \phi) = R(r) P(\theta) Q(\phi)$, one can write

$$\Phi(r, \theta, \phi) = -4\pi G \sum_{l=0}^{\infty} \sum_{m=-l}^l \frac{Y_l^m(\theta, \phi)}{2l+1} \left[\frac{1}{r^{(l+1)}} \int_0^r \rho_{lm}(r') r'^{(l+2)} dr' + r^l \int_r^{\infty} \rho_{lm}(r') \frac{dr'}{r'^{(l-1)}} \right]$$

Here

$$\rho_{lm}(r) = \int_0^{\pi} \sin \theta d\theta \int_0^{2\pi} \hat{Y}_l^m(\theta, \phi) \rho(r, \theta, \phi)$$

and

$$Y_l^m(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi} \frac{(l-|m|)!}{(l+|m|)!}} P_l^{|m|}(\cos \theta) e^{im\phi}$$

with $P_l(x)$ the **associated Legendre functions**, and $\hat{Y}_l^m(\theta, \phi)$ the complex conjugate of $Y_l^m(\theta, \phi)$

Multipole Expansion II

Monopole	$l = 0$	1 term
Dipole	$l = 1$	3 terms
Quadrupole	$l = 2$	5 terms
Octopole	$l = 3$	7 terms
Hexadecapole	$l = 4$	9 terms

The **monopole term** describes the potential of a spherical system with $\rho(r, \theta, \phi) = \rho(r)$. Since $Y_0^0(\theta, \phi) = 1/\sqrt{4\pi}$ and $\rho_{00} = \sqrt{4\pi}\rho(r)$, the $(l = m = 0)$ -term of the multipole expansion is simply the equation for the potential of a spherical system:

$$\Phi(r) = -4\pi G \left[\frac{1}{r} \int_0^r \rho(r') r'^2 dr' + \int_r^\infty \rho(r') r' dr' \right]$$

In **electrostatics** you have both positive and negative charges. Consequently, the monopole term of the electrostatic potential often vanishes at large radii, while the dipole terms comes to dominate.

In **gravity** we have only positive charges (mass). Consequently, the monopole term always dominates at large radii, while the dipole term vanishes. The quadrupole term depends on the flattening of the density distribution.

Potentials of Disks

Since many galaxies have a dominant, thin disk component, it is useful to consider the potentials of **infinitesimally thin disks**.

There are three methods to compute the potential of an infinitesimally thin disk:

- Use the formalism for **ellipsoids**, and apply the limit $q \rightarrow 0$.
Cumbersome! involving complicated double integrals...this method is seldomly used.
- Use the general definition of the potential, which results in an expression in terms of **Elliptic Integrals**.
- Use the **Laplace equation** subject to appropriate boundary conditions on the disk and at infinity.

Disk Potentials via Elliptic Integrals

The potential of a thin disk with surface density $\Sigma(R)$ can be written as

$$\Phi(\vec{x}) = -G \int \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|} d^3\vec{x}' = -G \int_0^\infty \Sigma(R') R' dR' \int_0^{2\pi} \frac{d\phi'}{|\vec{x}-\vec{x}'|}$$

Expressing $|\vec{x} - \vec{x}'|$ in $(R, \phi = 0, z)$ and $(R', \phi', z' = 0)$ yields

$$\Phi(R, z) = -\frac{2G}{\sqrt{R}} \int_0^\infty K(k) k \Sigma(R') \sqrt{R'} dR'$$

with $k^2 \equiv 4 R R' / [(R + R')^2 + z^2]$. The corresponding **circular velocity** can be obtained from

$$R \frac{\partial \phi}{\partial R}(R, z) = \frac{G}{\sqrt{R}} \int_0^\infty dR' k \Sigma(R') \sqrt{R'} \times \left[K(k) - \frac{1}{4} \left(\frac{k^2}{1-k^2} \right) \left(\frac{R'}{R} - \frac{R}{R'} + \frac{z^2}{RR'} \right) E(k) \right]$$

with $K(k)$ and $E(k)$ so called **complete elliptic integrals**. In principle the evaluation at $z = 0$ is complicated (contains integrable singularity); in practice it often suffices to approximate the above at small z

Disk Potentials via Bessel Functions

The potential of a thin disk with surface density $\Sigma(R)$ can be written as

$$\Phi(R, z) = \int_0^{\infty} S(k) J_0(kR) e^{-k|z|} dk$$

with

$$S(k) = -2\pi G \int_0^{\infty} J_0(kR) \Sigma(R) R dR$$

Here $J_0(x)$ is the **cylindrical Bessel function** of order zero.

The corresponding **circular velocity** is given by

$$v_c^2(R) = R \left(\frac{\partial \Phi}{\partial R} \right)_{z=0} = -R \int_0^{\infty} S(k) J_1(kR) k dk$$

This method is simple, and most of the time well behaved. For an **exponential disk** with $\Sigma(R) = \Sigma_0 e^{-R/R_d}$ one finds

$$v_c^2(R) = 4\pi G \Sigma_0 R_d y^2 [I_0(y) K_0(y) - I_1(y) K_1(y)]$$

with $y = \frac{R}{2R_d}$ and $I_n(x)$ and $K_n(x)$ **modified Bessel functions** of the first and second kinds