Kinematics I

We now focus shortly on how to extract kinematic information from spectra of galaxies.

The spectrum at a given location (x, y) of the projected galaxy is the sum of the spectra of all stars along the line-of-sight (los) Doppler shifted according to their velocity along the los

For a source with los-velocity v the observed and emitted frequencies are related according to

$$\boxed{\nu_{\rm obs} = (1-\beta)\gamma\nu_{\rm em}}$$

with $\beta \equiv v/c$ and $\gamma \equiv (1-\beta^2)^{-1/2}$. To lowest order in v/c we have that $\gamma = 1$ and thus

$$\nu_{\rm obs} = \left(1 - \frac{v}{c}\right)\nu_{\rm em}$$

Using that the wavelength $\lambda = c/\nu$ we obtain, again to lowest order in v/c that

$$\boxed{\lambda_{\rm obs} = \left(1 + \frac{v}{c}\right)\lambda_{\rm em}}$$

Kinematics II

If we now define $x \equiv \ln\lambda$ then

$$\Delta x = \ln\left(\frac{\lambda_{\rm obs}}{\lambda_{\rm em}}\right) = \ln\left(1 + \frac{\Delta v}{c}\right) \simeq \frac{\Delta v}{c}$$

where we have used the first term of the series expansion of $\ln(1+x)$

Let $S(x)$ correspond to an observed spectrum, rebinned linearly in x. Then we may write that

$$\boxed{S(x) = T(x) \otimes B(x)}$$

Here $T(x)$ is the template spectrum, which is luminosity weighted spectrum of all the various stars along the los, $B(x)$ is the broadening function, which gives the probability distribution of the los velocities of all these stars, and $\otimes$ denotes convolution.

Since convolution in real space corresponds to multiplication in Fourier space we have that

$$\boxed{\hat{S}(k) = \hat{T}(k) \cdot \hat{B}(k)}$$

where $\hat{F}$ indicates the Fourier Transform of F, and k is the wavenumber.

Kinematics III

This gives us immediately an unparameterized method to obtain the broadening function from the observed spectrum

$$B(x) = \widehat{\left[\frac{\hat{S}}{\hat{T}}\right]}$$

This method has the advantage that no assumption is made regarding the functional form of $B(x)$. However, the disadvantage is that one needs to adopt some noise filtering which introduces correlations between different points in the $B(x)$ estimate. This complicates comparison with models.

The alternative is to use a parameterized method, by assuming a functional form $\tilde{B}(x)$ for the broadening function which has n free parameters. The best-fit parameters are obtained by minimizing

$$\chi^2 = \int \left[S(x) - \tilde{B}(x) \otimes T(x)\right]^2 \mathrm{d}x$$

Rather than talking about the broadening function $B(x)$ one often prefers to talk about the velocity profile $\mathcal{L}(v) \equiv B(v/c)$, also called the LOSVD

Kinematics IV

A typical functional form to assume for $\mathcal{L}(v)$ is a simple Gaussian

$$\mathcal{L}(v) = \frac{1}{\sqrt{2\pi}\sigma} \mathrm{e}^{-\frac{1}{2}w^2}$$

where $w \equiv (v - V)/\sigma$. Note that this parameterization has two free parameters: V and σ.

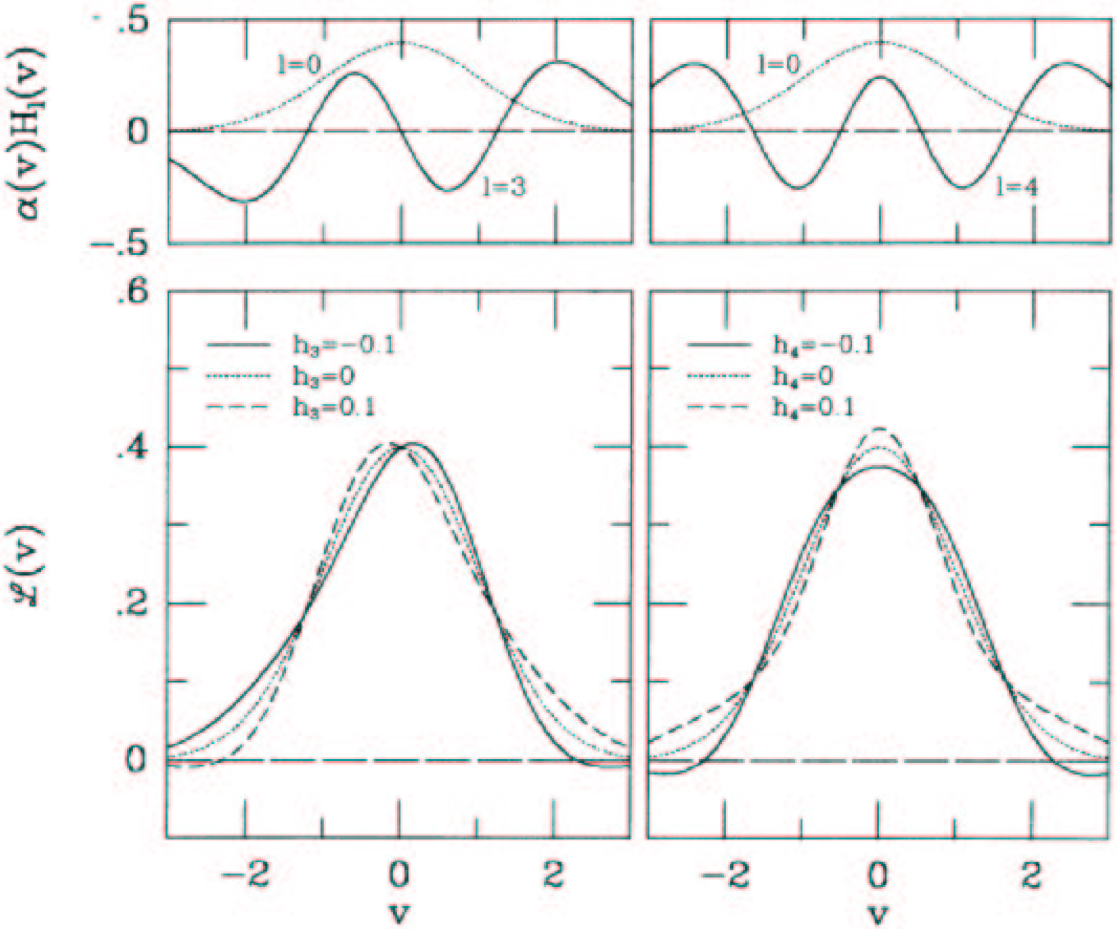
However, $\mathcal{L}(v)$ is generally not Gaussian, and a more general parameterization is required. Van der Marel & Franx (1993) and Gerhard (1993) have suggested using the Gauss-Hermite series

$$\mathcal{L}(v) = \frac{1}{\sqrt{2\pi}\sigma} \mathrm{e}^{-\frac{1}{2}w^2} \left[1 + \sum_{j=3}^{N} h_j \, H_j(w) \right]$$

where $H_l(x)$ are Hermite Polynomials of degree l, and h_j are additional free parameters that describe the deviation of $\mathcal{L}(v)$ from a pure Gaussian (i.e., $h_j = 0$ for a Gaussian LOSVD).

Typically one truncates the series expansion at $N = 4$ so that the LOSVD has four free parameters: V, σ, h_3, and h_4. Their best-fit values are determined by minimizing the χ^2 defined above.

Kinematics V



(from: van der Marel & Franx 1993)

Note that the odd Gauss-hermite coefficients (h_3, h_5, etc.) characterize anti-symmetric deviations of $\mathcal{L}(v)$ from a Gaussian, while the even coefficients (h_4, h_6, etc.) characterize the symmetric deviations.

The LOSVD shape contains useful information to break the degeneracy between mass and anisotropy

The Relaxation Puzzle

Relaxation: the process by which a physical system acquires equilibrium or returns to equilibium after a disturbance.

Often, but not always, relaxation erases the system's 'knowledge' of its initial conditions.

In the first lecture we defined the two-body relaxation time as the time required for a particle's kinetic energy to change by its own amount due to long-range collisions. We found that

$$t_{\rm relax} = \frac{N}{10 \ln N} t_{\rm cross}$$

For galaxies and DM haloes one always finds that $t_{\rm relax} \gg t_{\rm Hubble}$. As first pointed out by Zwicky (1939), how is it possible that galaxies appear relaxed?

In particular, if galaxies did not form near equilibrium, two-body relaxation certainly did not have the time to get them there. This would imply that galaxies all form in (virial) equilibrium, which seems very contrived.

The solution to this puzzle is that alternative relaxation mechanisms exist.

Relaxation Mechanisms

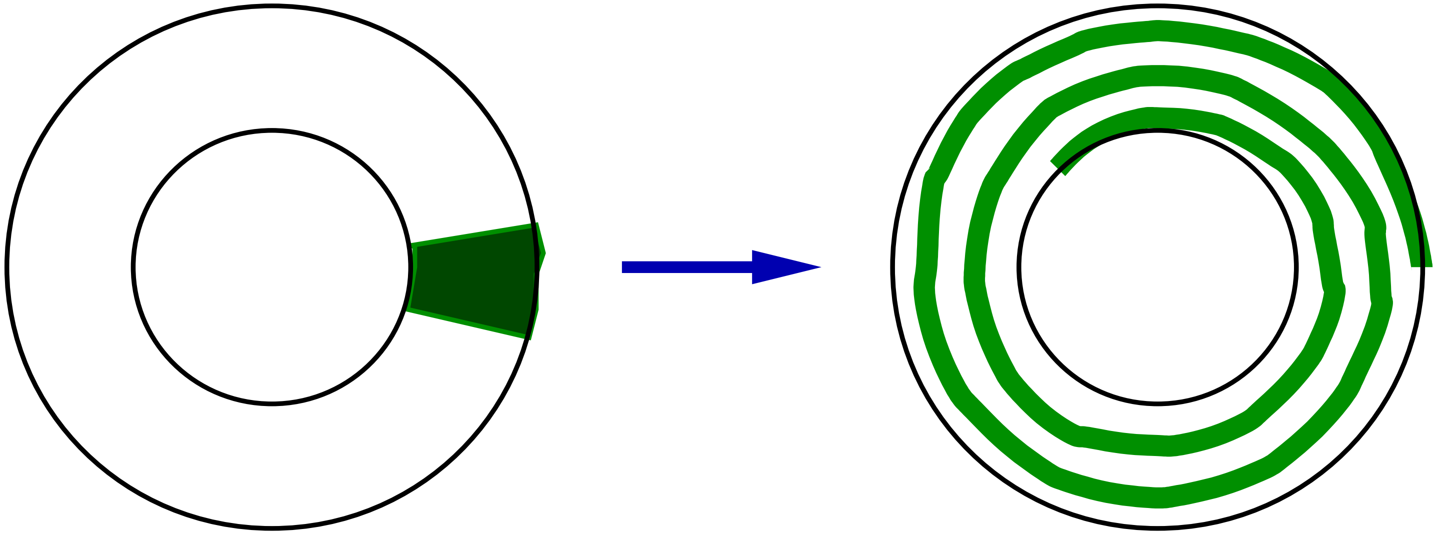
In gravitational N-body systems the four most important relaxation mechanisms are:

- **Phase Mixing** The spreading of neighboring points in phase-space due to the difference in frequencies between neighboring tori.
- **Chaotic Mixing** The spreading of neighboring points in phase-space due to the chaotic nature of their orbits.
- **Violent Relaxation** The change of energy of individual particles due to the change of the overall potential.
- **Landau Damping** The damping (and decay) of perturbations in the density and/or potential.

We will discuss each of these in turn.

As we will see later, Violent Relation and Landau Damping are basically specific examples of Phase Mixing!!.

Phase Mixing I



Consider circular motion in a disk with $V_c(R) = V_0 = \mathrm{constant}$. The frequency of a circular orbit at radius R is then

$$\omega = \frac{1}{T} = \frac{V_0}{2\pi R}$$

Thus, points in the disk that are initially close will separate according to

$$\Delta\phi(t) = \Delta\left(\frac{V_0}{R} t\right) = 2\pi\Delta\omega t$$

We thus see that the timescale on which the points are mixed over their accessible volume in phase-space is of the order of

$$\boxed{t_{\mathrm{mix}} \simeq \frac{1}{\Delta\omega}}$$

Phase Mixing II

In this example the phase-mixing occurs purely in 'real space'. In the more general case, however, the phase-mixing occurs in phase-space.

Phase mixing is the simplest mechanism that causes relaxation in gravitational N-body systems. Because the frequencies of regular motion on adjacent invariant tori are generally similar but different, two points on adjacent tori that are initially close together in phase-space, will seperate linearly with time. However, two points on the same torus do not phase-mix; their separation remains invariant.

Note that phase-mixing decreases the coarse-grained DF around a point, by mixing in 'vacuum' (i.e., unpopulated regions of phase-space). Nevertheless, as assured by the CBE, the flow in phase-space of a collisionless system is perfectly incompressible: unlike the coarse-grained DF, the fine-grained DF is not influenced by phase-mixing and is perfectly conserved.

Although phase-mixing is a relaxation mechanism, in that it drives the system towards a state in which the phase-space density is more and more uniform, it does not cause any loss of information: the system preserves all knowledge of the initial conditions.

In other words, in an integrable, Hamiltonian system phase mixing is a time-reversible relaxation mechanism!

The Lyapunov Exponent I

The Lyapunov exponent of a dynamical system is a measure that determines, for a given point in phase space, how quickly trajectories that begin in this point diverge over time.

Actually, for each point in a $2N$-dimensional phase space (N is the number of degrees of freedom) there are $2N$ Lyaponov exponents λ_i, but it is common to just refer to the largest one.

Consider a small $2N$-dimensional sphere with radius r centered on a phase-space point $\vec{x}$. Different points on the sphere's surface evolve differently with time, causing the sphere to deform into a $2N$-dimensional ellipsoid with principal axes $L_i(t)$.

The Lyapunov exponents for $\vec{x}$ are defined as

$$\boxed{\lambda_i = \lim_{t\to\infty} \frac{1}{t} \ln\left(\frac{\mathrm{d}L_i(t)}{\mathrm{d}r}\right)}$$

In a collisionless system

$$\sum_{i=1}^{2N} \lambda_i = 0$$

which expresses the incompressibility of the flow (conservation of volume).

The Lyapunov Exponent II

If the trajectory through $\vec{x}$ is regular then $\lambda_i = 0$ for $i = 1, ..., 2N$.

On the other hand, if the trajectory is stochastic then $\lambda \equiv \max \lambda_i > 0$.

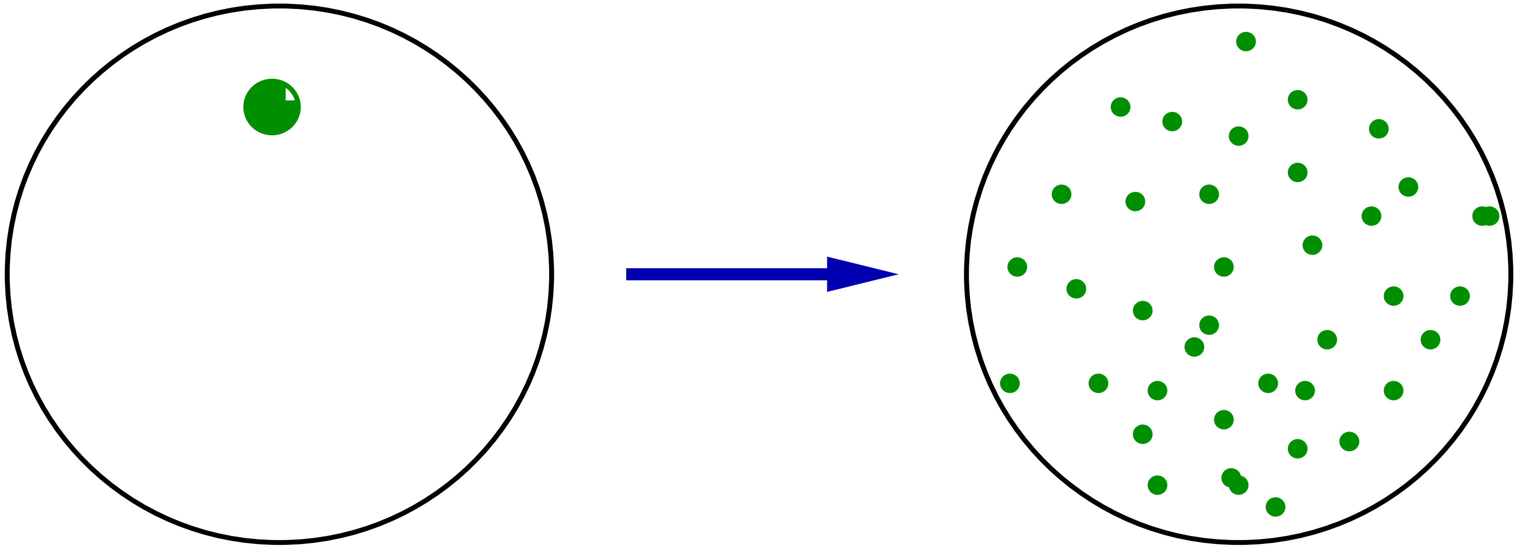
Note that a positive Lyapunov exponent implies that the neighboring trajectories diverge exponentially:

$$\delta\Gamma(t) \propto \mathrm{e}^{\lambda t}$$

The inverse of the largest Lyapunov exponent is called the Lyapunov time, and defines the characteristic e-folding time.

For a chaotic (stochastic) orbit the Lyapunov time is finite, while it is infinite for regular orbits.

Chaotic Mixing I



In the parts of phase-space that are not filled with regular, but with stochastic orbits, mixing occurs naturally due to the chaotic behavior of the obits.

Chaos implies a sensitivity to initial conditions: two stars intially close together separate exponentially with time.

After a sufficiently long time, the group of stars that were initially close together will have spread over the entire accessible phase-space (ie., the Arnold web). As for phase-mixing, chaotic mixing thus smooths out (i.e., relaxes) the coarse-grained DF, but leaves the fine-grained DF invariant.

Chaotic Mixing II

Unlike for phase-mixing, chaotic mixing is irreversible, in the sense that an infinitely precise fine-tuning of the phase-space coordinates is required to undo its effects.

Chaotic mixing occurs on the Lyapunov timescale.

However, the mixing rate of chaotic ensembles typically falls below the Lyapunov rate once the trajectories separate, because stochastic orbits are often confined over long periods of time to restricted parts of phase-space.

The time scale on which the orbits are uniformly spread over their accessible phase-space then becomes dependent on the efficiency of Arnold diffusion.

All in all, the effective rates of phase mixing and chaotic mixing might therefore be comparable in real galaxies.

For a nice review, see Merritt (1999)

Violent Relaxation I

Since $E = \frac{1}{2}v^2 + \Phi$ and $\Phi = \Phi(\vec{x}, t)$ we can write that

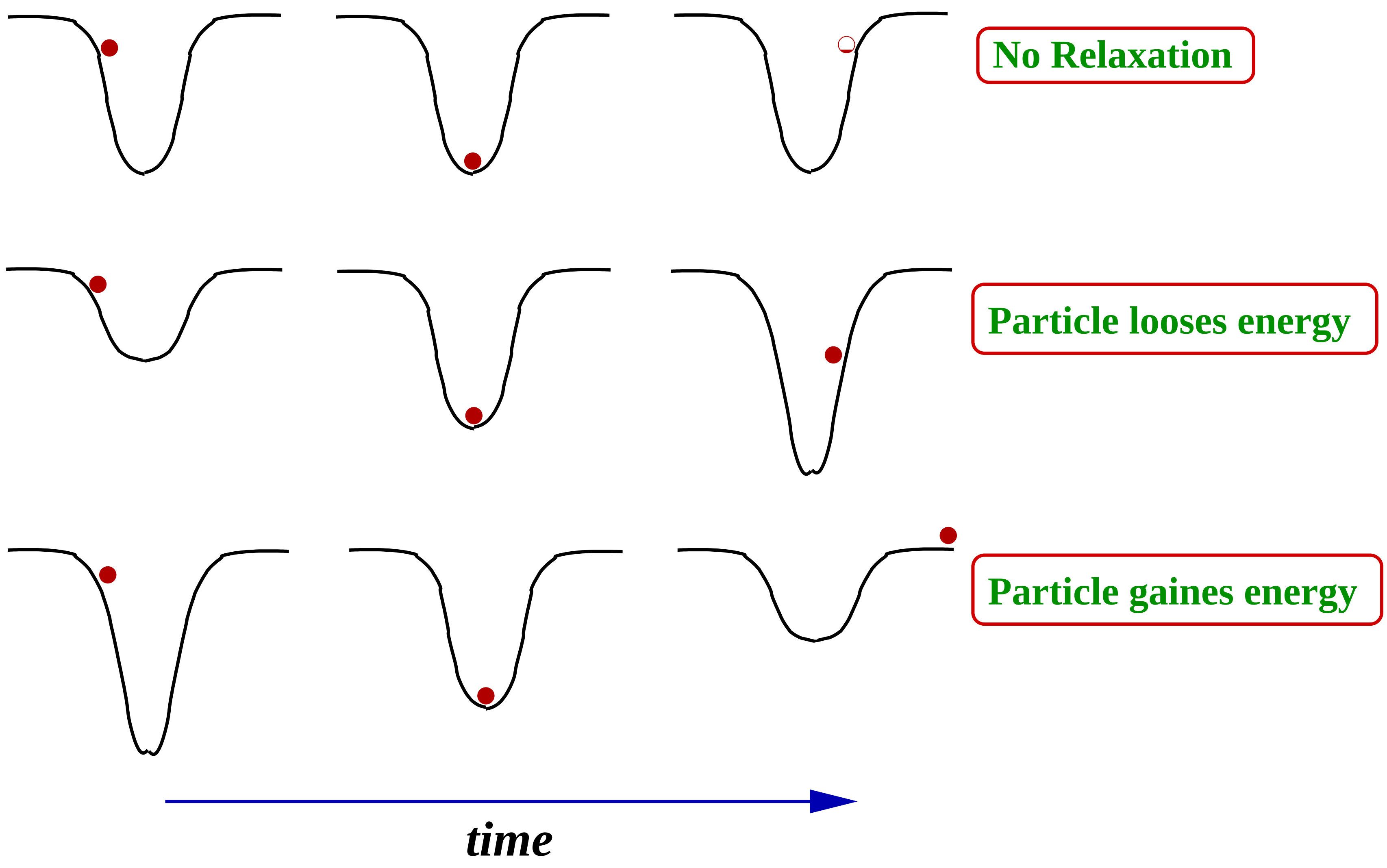
$$
\begin{aligned}
\frac{\mathrm{d}E}{\mathrm{d}t} &= \frac{\partial E}{\partial \vec{v}} \cdot \frac{\mathrm{d}\vec{v}}{\mathrm{d}t} + \frac{\partial E}{\partial \Phi}\frac{\mathrm{d}\Phi}{\mathrm{d}t} \\
&= -\vec{v} \cdot \vec{\nabla}\Phi + \frac{\mathrm{d}\Phi}{\mathrm{d}t} \\
&= -\vec{v} \cdot \vec{\nabla}\Phi + \frac{\partial \Phi}{\partial t} + \frac{\partial \Phi}{\partial \vec{x}} \cdot \frac{\mathrm{d}\vec{x}}{\mathrm{d}t} \\
&= -\vec{v} \cdot \vec{\nabla}\Phi + \frac{\partial \Phi}{\partial t} + \vec{v} \cdot \vec{\nabla}\Phi \\
&= \frac{\partial \Phi}{\partial t}
\end{aligned}
$$

Thus we see that the only way in which a particle's energy can change in a collisionless system, is by having a time-dependent potential.

An interesting case to consider is the collapse of a dark matter halo, or that of a galaxy. In this case the potential will vary as function of time, and the particles thus change their energy

Exactly how a particle's energy changes depends in a complex way on the particle's initial position and energy: particles can both gain or loose energy (see fig. on next page). Overall, however, the effect is to widen the range of energies.

Violent Relaxation II



Violent Relaxation III

A few remarks regarding Violent Relaxation:

- During the collapse of a collisionless system the CBE is still valid, i.e., the fine-grained DF does not evolve ($\mathrm{d}f/\mathrm{d}t = 0$). However, unlike for a 'steady-state' system, $\partial f/\partial t \neq 0$.

- A time-varying potential does not guarantee violent relaxation. One can construct oscillating models that exhibit no tendency to relax: although the energies of the individual particles change as function of time, the relative distribution of energies is invariant (cf. Sridhar 1989).

Although fairly artificial, this demonstrates that violent relaxation requires both a time-varying potential and mixing to occur simultaneous.

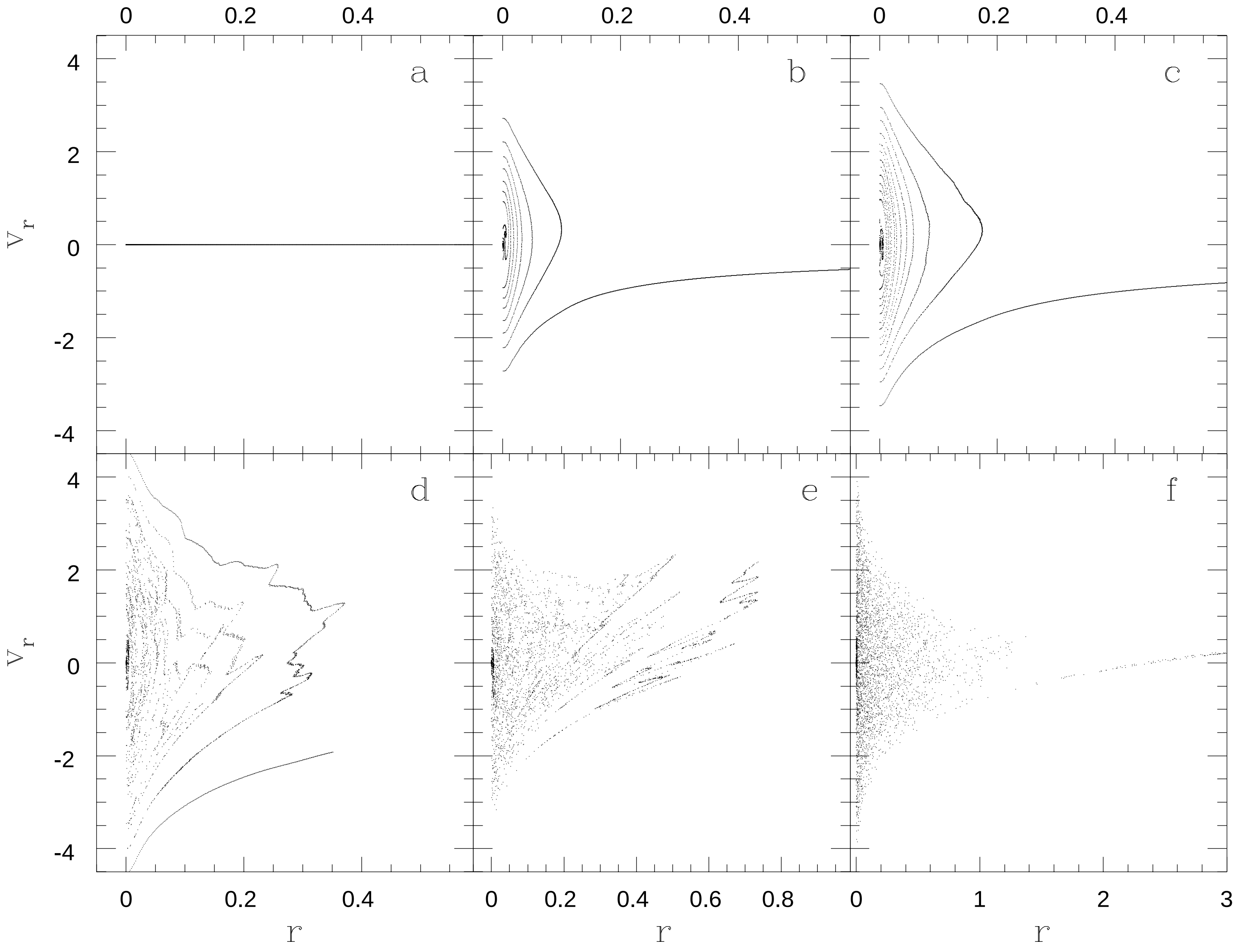
- The time-scale for violent relaxation is

$$t_{\rm vr} = \left\langle \frac{(\mathrm{d}E/\mathrm{d}t)^2}{E^2} \right\rangle^{-1/2} = \left\langle \frac{(\partial \Phi/\partial t)^2}{E^2} \right\rangle^{-1/2} = \frac{3}{4} \langle \dot{\Phi}^2/\Phi^2 \rangle^{-1/2}$$

where the last step follows from the time-dependent virial theorem (see Lynden-Bell 1967).

Note that $t_{\rm vr}$ is thus of the order of the time-scale on which the potential changes by its own amount, which is basically the collapse time. $\triangleright$ the relaxation mechanism is very fast. Hence the name 'violent relaxation'

Violent Relaxation IV



(from: Henriksen & Widrow 1997)

Collapse of a spherical system with $\rho_{\mathrm{init}} \propto r^{-3/2}$

Violent Relaxation V

In the collapse simulation of Henriksen & Widrow (1997), phase-mixing is the dominant relaxation mechanism during the initial phases of the collapse. After some time, there is a transition to a more 'chaotic' flow: due to the time-varying potential particles on neighboring phase-space streams start to mix rapidly (i.e., violent relaxation kicks in).

Violent relaxation leads to efficient fine-grained mixing of the DF, and erases the system's memory of its initial conditions.

For comparison: phase-mixing only leads to a relaxation of the coarse-grained DF and is reversible.

Another important aspect of violent relaxation is the fact that it changes a particle's energy in a way that is independent of the particle's mass. Thus violent relaxation drives the system to a relaxed state that is very different from the one promoted by collional relaxation, where momentum conservation in collisions causes equipartition of energy (ie. mass segregation).

Landau Damping I

In 1946 Landau showed that waves in a collsionless plasma can be damped, despite the fact that there is no dissipation.

This damping mechanism, called Landau Damping, also operates in gravitational, collisionless systems, and is thus another relaxation mechanism.

The physical reason for this collisionless damping arises from the detailed interaction of the wave with the orbits of the background particles which are not part of the wave (i.e., particle-wave interactions).

To gain insight, it is useful to start by considering a fluid. Perturbation analysis of the fluid shows that if the perturbation has a wavelength $\lambda < \lambda_J$, with λ_J the Jeans length, then the perturbation is stable, and the wave propagates with a phase velocity

$$v_p = c\sqrt{1 - \lambda^2/\lambda_J^2}$$

with c the sound speed. Note that larger waves move slower, which owes to the fact that self-gravity becomes more and more important.

When $\lambda = \lambda_J$ the wave no longer propagates. Rather, the perturbation is unstable: self-gravity overpowers the pressure, causing the perturbation to grow.

Landau Damping II

One can apply a similar perturbation analysis to collisionless, gravitational systems (see B&T, Section 5.1). This yields a similar Jeans criterion, but with the velocity dispersion of the stars, σ playing the role of the sound speed.

Once again, perturbations with $\lambda > \lambda_J$ are unstable and cause the perturbation to grow.

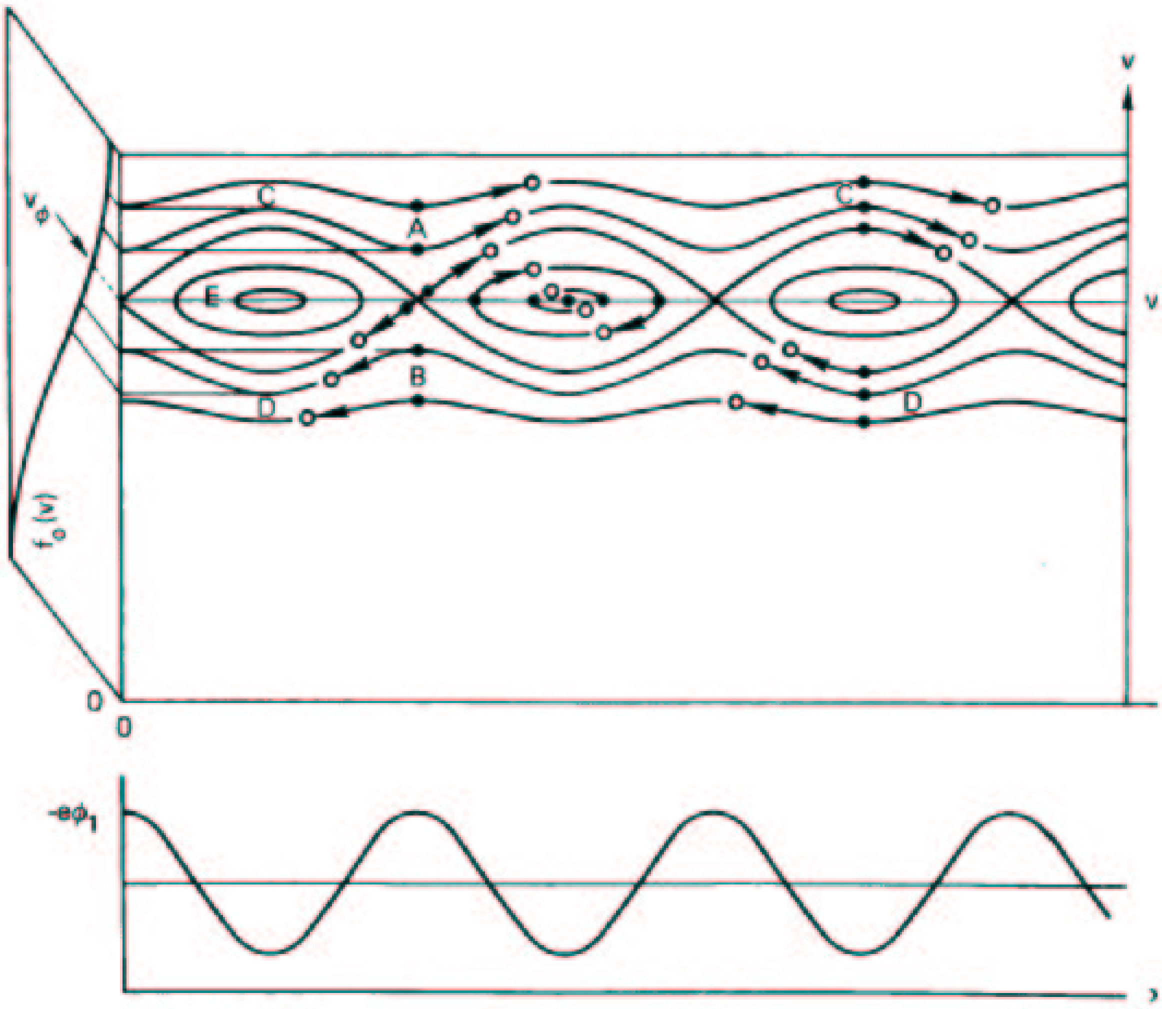
For $\lambda < \lambda_J$, however, the situation is somewhat different. While the fluid supports gravity-modified sound waves that are stable, the equivalent waves in gravitational systems experience Landau Damping.

Consider a density wave with $\lambda < \lambda_J$. While for a fluid the phase velocity $v_p < c$, in a gravitational system we have that $v_p < \sigma$.

Stars that move faster than the wave (i.e., with $v > v_p$) loose energy and tend to be captured by the wave: they tend to amplify the wave. Inversely, stars with $v < v_p$ gain energy and thus tend to damp the wave.

If, for simplicity, we assume a Gaussian distribution of velocities, centered on $v = 0$ and with a velocity dispersion of σ, we see immediately that there will be more stars with $v < v_p$ than with $v > v_p$. Consequently, the net effect will be to damp the wave.

Landau Damping III



Particles initially at A and D gain energy during the first quarter cycle of the wave (i.e., their net velocity increases), while those at B and C loose energy.

Since there are more particles at A than at C and more at D than at B (see distribution at left), the particles experience a net gain. Consequently, the wave has to experience a net loss.

The End-State of Relaxation I

The various relaxation mechanism discussed will drive the system to an equilibrium configuration. However, there are many different equilibrium configurations for a collisionless system of mass M and energy E. So to which of these configurations does a system settle?

This is a very complicated problem which is still largely unsolved.

One might expect the system to evolve to a most probable state.

This actually happens in a collisional gas, where the collisions cause a rapid exchange of energy between the particles. The most probable distribution is obtained by maximizing the entropy, which results in the Maxwell-Boltzmann velocity distribution.

If one applies the same logic to collisionless systems, and (somewhat naively) defines the systems entropy by

$$S = -\int f \ln f \, \mathrm{d}^3\vec{x} \, \mathrm{d}^3\vec{v}$$

then one finds (not surprisingly) that the functional form of f which maximizes S subject to given values of the system's mass and energy is that of a singular isothermal sphere, for which the DF again has a Maxwellian velocity distribution (see Lynden-Bell 1967).

The End-State of Relaxation II

However, a singular isothermal sphere has infinite mass, and is thus inconsistent with our constraint. Thus, no DF that is compatible with finite M and E maximizes S!

In fact, one can show that one can always increase the system's entropy (as defined above) by increasing the systems central concentration (see Tremaine et al. 1986).

Thus, violent relaxation will drive the system towards its equilibrium state by making the system more concentrated, but can never reach it. In other words, there is no end state. This seems unsatisfactory.

There are two possible 'solutions', both of which are probably correct:

(1) The relaxation is not complete.

(2) The expression for the entropy is not appropriate.

As an example of the latter, since gravitational systems do not obey extensive statistics, a more appropriate definition for the entropy may be

$$S_q = -\int f^q \ln_q f \, \mathrm{d}^3\vec{x} \, \mathrm{d}^3\vec{v}$$

where $\ln_q(x) = (x^{1-q} - 1)/(1-q)$ with $q \neq 1$ (see Hansen et al. 2004). Note that for $q = 1$ one recovers the normal Boltzmann-Gibbs entropy S.

The End-State of Relaxation III

Despite the uncertainties regarding the definition of entropy, it has become clear that relaxation is in general not complete. There are various reasons for this:

- **The time-scales for phase-mixing and chaotic mixing may not be small enough compared to the Hubble time.**
- **Violent relaxation is only efficient as long as the potential fluctuates. This requires global, coherent modes, which, however, as strongly damped by Landau Damping. This occurs roughly on the collapse time-scale, which is also the time-scale on which violent relaxation works. Thus it is not too suprising if it is not complete.**
- **This is strongly supported by numerical simulations, which show that the end-state of violent relaxation depends on the clumpiness of the initial conditions (van Albada 1982). This illustrates that the final state is not completely governed by statistical mechanics alone, but also by the details of the collapse.**

Much work is still required before we have a proper understanding of why dark matter haloes and galaxies have the (semi)-equilibrium states they have.