Dynamics of Collisionless Systems

Summer Semester 2005, ETH Zürich

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Useful Information

TEXTBOOK: **Galactic Dynamics, Binney & Tremaine**
Princeton University Press *Highly Recommended*

LECTURES: **Wed, 14.45-16.30, HPP H2. Lectures will be in English**

EXERSIZE CLASSES: **to be determined**

HOMEWORK ASSIGNMENTS: $\pm$ **every other week**

EXAM: **Verbal (German possible), July/August 2005**

GRADING: **exam ($2/3$) plus homework assignments ($1/3$)**

TEACHER: **Frank van den Bosch (vdbosch@phys.ethz.ch), HPT G6**

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Outline

Summary of Vector Calculus I

$$\vec{A}\cdot\vec{B} = \text{scalar} = |\vec{A}|\,|\vec{B}|\cos\psi = A_i B_i$$ **(summation convention)**

$$\vec{A}\times\vec{B} = \text{vector} = \epsilon_{ijk}\vec{e}_i A_j B_k$$ **(with ϵ_{ijk} the Levi-Civita Tensor)**

Useful to Remember

$$\vec{A}\times\vec{A} = 0$$
$$\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$$
$$\vec{A}\cdot(\vec{A}\times\vec{B}) = 0$$
$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(\vec{A}\times\vec{B})$$
$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$$
$$(\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{D}) = (\vec{A}\cdot\vec{C})(\vec{B}\cdot\vec{D}) - (\vec{A}\cdot\vec{D})(\vec{B}\cdot\vec{C})$$

$\vec{\nabla}$ =**vector operator**= $(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$

$$\boxed{\begin{array}{lclcl} \vec{\nabla}S & = & \text{grad}S & = & \text{vector} \\ \vec{\nabla}\cdot\vec{A} & = & \text{div}\vec{A} & = & \text{scalar} \\ \vec{\nabla}\times\vec{A} & = & \text{curl}\vec{A} & = & \text{vector} \end{array}}$$

Summary of Vector Calculus II

Laplacian: $\nabla^2 = \vec{\nabla} \cdot \vec{\nabla} =$ scalar operator $= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

$$\boxed{\begin{array}{lcccl} \nabla^2 S & = & \vec{\nabla} \cdot (\vec{\nabla} S) & = & \text{scalar} \\ \nabla^2 \vec{A} & = & (\vec{\nabla} \cdot \vec{\nabla}) \vec{A} & = & \text{vector} \\ \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) & \neq & \nabla^2 \vec{A} & = & \text{vector} \end{array}}$$

$$\boxed{\begin{array}{lcll} \vec{\nabla} \times (\vec{\nabla} S) & = & 0 & \text{curl}(\text{grad} S) = 0 \\ \vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) & = & 0 & \text{div}(\text{curl}\, \vec{A}) = 0 \\ \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) & = & \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A} & \end{array}}$$

$$\boxed{\begin{array}{lcl} \vec{\nabla}(ST) & = & S\vec{\nabla} T + T\vec{\nabla} S \\ \vec{\nabla} \cdot (S\vec{A}) & = & S(\vec{\nabla} \cdot \vec{A}) + \vec{A} \cdot \vec{\nabla} S \\ \vec{\nabla} \times (S\vec{A}) & = & S(\vec{\nabla} \times \vec{A}) - \vec{A} \times \vec{\nabla} S \\ \vec{\nabla} \cdot (\vec{A} \times \vec{B}) & = & \vec{B} \cdot (\vec{\nabla} \times \vec{A}) - \vec{A} \cdot (\vec{\nabla} \times \vec{B}) \end{array}}$$

Integral Theorems I

Gradient Theorem: **Let γ be a curve running from $\vec{x}_0$ to $\vec{x}_1$, $\mathrm{d}\vec{l}$ is the directed element of length along γ, and $\phi(\vec{x})$ is a scalar field then:**

$$\boxed{\int_{\vec{x}_0}^{\vec{x}_1} \vec{\nabla}\phi \cdot \mathrm{d}\vec{l} = \int_{\vec{x}_0}^{\vec{x}_1} \mathrm{d}\phi = \phi(\vec{x}_1) - \phi(\vec{x}_0)}$$

It follows that

$$\boxed{\oint \vec{\nabla}\phi \cdot \mathrm{d}\vec{l} = 0}$$

Divergence Theorem (Gauss' Theorem): **Let V be a 3D volume bounded by a 2D surface S, and let $\vec{A}(\vec{x})$ be a vector field, then:**

$$\boxed{\int_V \vec{\nabla} \cdot \vec{A}\, \mathrm{d}^3\vec{x} = \int_S \vec{A} \cdot \mathrm{d}^2\vec{S}}$$

Curl Theorem (Stokes' Theorem): **Let S be a 2D surface bounded by a 1D curve γ, and let $\vec{A}(\vec{x})$ be a vector field, then:**

$$\boxed{\int_S (\vec{\nabla} \times \vec{A})\, \mathrm{d}^2\vec{S} = \oint_\gamma \vec{A} \cdot \mathrm{d}\vec{l}}$$

Integral Theorems II

NOTE: Since a conservative force $\vec{F}$ can always be written as the gradient of a scalar field ϕ, we have from the gradient theorem that

$$\boxed{\oint \vec{F} \cdot \mathrm{d}\vec{l} = 0}$$

From the curl theorem we immediately see that

$$\boxed{\vec{\nabla} \times \vec{F} = 0}$$

We immediately infer that a conservative force is curl free, and that the amount of work done ($\mathrm{d}W = \vec{F} \cdot \mathrm{d}\vec{r}$) is independent of the path taken.

From the divergence theorem we infer that

$$\boxed{\int_V \phi \vec{\nabla} \cdot \vec{A}\, \mathrm{d}^3\vec{x} = \int_S \phi \vec{A} \cdot \mathrm{d}^2\vec{S} - \int_V \vec{A} \cdot \vec{\nabla}\phi\, \mathrm{d}^3\vec{x}}$$

which is the three-dimensional analog of integration by parts

$$\boxed{\int u \frac{\mathrm{d}v}{\mathrm{d}x} \mathrm{d}x = \int \mathrm{d}(uv) - \int v \frac{\mathrm{d}u}{\mathrm{d}x} \mathrm{d}x}$$

Curvi-Linear Coordinate Systems I

In addition to the Cartesian coordinate system (x, y, z), we will often work with cylindrical (R, ϕ, z) or spherical (r, θ, ϕ) coordinate systems

Let (q_1, q_2, q_3) denote the coordinates of a point in an arbitrary coordinate system, defined by the metric tensor h_{ij}. The distance between (q_1, q_2, q_3) and $(q_1 + \mathrm{d}q_1, q_2 + \mathrm{d}q_2, q_3 + \mathrm{d}q_3)$ is

$$\mathrm{d}s^2 = h_{ij}\, \mathrm{d}q_i\, \mathrm{d}q_j \qquad \text{(summation convention)}$$

We will only consider orthogonal systems for which $h_{ij} = 0$ if $i \neq j$, so that $\mathrm{d}s^2 = h_i^2\, \mathrm{d}q_i^2$ with

$$h_i \equiv h_{ii} = \left|\frac{\partial \vec{x}}{\partial q_i}\right|$$

The differential vector is

$$\mathrm{d}\vec{x} = \frac{\partial \vec{x}}{\partial q_1}\mathrm{d}q_1 + \frac{\partial \vec{x}}{\partial q_2}\mathrm{d}q_2 + \frac{\partial \vec{x}}{\partial q_3}\mathrm{d}q_3$$

The unit directional vectors are

$$\vec{e}_i = \frac{\partial \vec{x}}{\partial q_i} / \left|\frac{\partial \vec{x}}{\partial q_i}\right| = \frac{1}{h_i}\frac{\partial \vec{x}}{\partial q_i}$$

so that $\mathrm{d}\vec{x} = \sum_i h_i\, \mathrm{d}q_i\, \vec{e}_i$ and $\mathrm{d}^3\vec{x} = h_1 h_2 h_3 \mathrm{d}q_1 \mathrm{d}q_2 \mathrm{d}q_3$.

Curvi-Linear Coordinate Systems II

The gradient:

$$\boxed{\vec{\nabla}\psi = \frac{1}{h_i}\frac{\partial\psi}{\partial q_i}\vec{e}_i}$$

The divergence:

$$\boxed{\vec{\nabla}\cdot\vec{A} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}(h_2h_3A_1) + \frac{\partial}{\partial q_2}(h_3h_1A_2) + \frac{\partial}{\partial q_3}(h_1h_2A_3)\right]}$$

The curl (only one component shown)**:**

$$\boxed{(\vec{\nabla}\times\vec{A})_3 = \frac{1}{h_1h_2}\left[\frac{\partial}{\partial q_1}(h_2A_2) - \frac{\partial}{\partial q_2}(h_1A_1)\right]}$$

The Laplacian:

$$\boxed{\nabla^2\psi = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\psi}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{h_3h_1}{h_2}\frac{\partial\psi}{\partial q_2}\right) + \frac{\partial}{\partial q_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\psi}{\partial q_3}\right)\right]}$$

Cylindrical Coordinates

For cylindrical coordinates (R, ϕ, z) we have that

$$x = R\cos\phi \qquad y = R\sin\phi \qquad z = z$$

The scale factors of the metric are:

$$\boxed{h_R = 1 \qquad h_\phi = R \qquad h_z = 1}$$

and the position vector is $\vec{x} = R\vec{e}_R + z\vec{e}_z$

Let $\vec{A} = A_R\vec{e}_R + A_\phi\vec{e}_\phi + A_z\vec{e}_z$ an arbitrary vector, then

$$\boxed{\begin{aligned} A_R &= A_x\cos\phi - A_y\sin\phi \\ A_\phi &= -A_x\sin\phi + A_y\cos\phi \\ A_z &= A_z \end{aligned}}$$

Velocity: $\vec{v} = \dot{R}\vec{e}_R + R\dot{\vec{e}}_R + \dot{z}\vec{e}_z = \dot{R}\vec{e}_R + R\dot{\phi}\vec{e}_\phi + \dot{z}\vec{e}_z$

Gradient & Laplacian:

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{R}\frac{\partial}{\partial R}(RA_R) + \frac{1}{R}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z}$$

$$\nabla^2\psi = \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial\psi}{\partial R}\right) + \frac{1}{R^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}$$

Spherical Coordinates

For spherical coordinates (r, θ, ϕ) we have that

$$x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi \qquad z = r\cos\theta$$

The scale factors of the metric are:

$$\boxed{h_r = 1 \qquad h_\theta = r \qquad h_\phi = r\sin\theta}$$

and the position vector is $\vec{x} = r\vec{e}_r$

Let $\vec{A} = A_r\vec{e}_r + A_\theta\vec{e}_\theta + A_\phi\vec{e}_\phi$ an arbitrary vector, then

$$\boxed{\begin{array}{lcl} A_r & = & A_x\sin\theta\cos\phi + A_y\sin\theta\sin\phi + A_z\cos\theta \\ A_\theta & = & A_x\cos\theta\cos\phi + A_y\cos\theta\sin\phi - A_z\sin\theta \\ A_\phi & = & -A_x\sin\phi + A_y\cos\phi \end{array}}$$

Velocity: $\vec{v} = \dot{r}\vec{e}_r + r\dot{\vec{e}}_r = \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta + r\sin\theta\dot{\phi}\vec{e}_\phi$

Gradient & Laplacian:

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\psi^2}$$

Introduction

COLLISIONLESS DYNAMICS: The study of the motion of large numbers of point particles orbiting under the influence of their mutual self-gravity

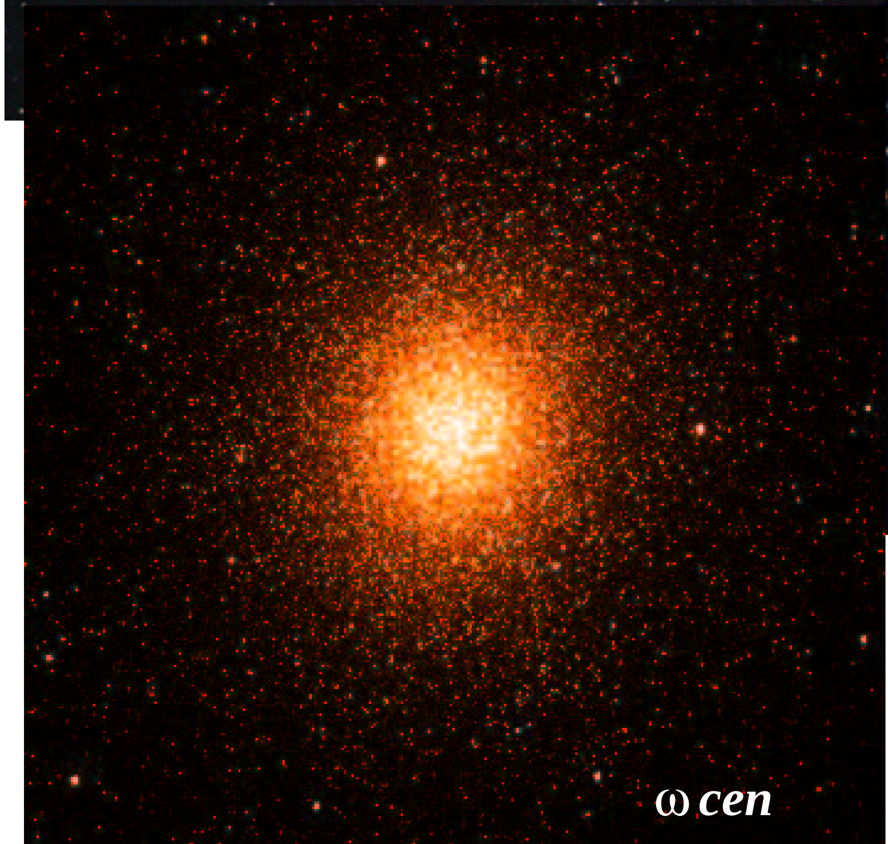
EAMPLES OF COLLISIONLESS SYSTEMS

- Galaxies (ellipticals & disk galaxies) $N \sim 10^6 - 10^{11}$
- Globular clusters $N \sim 10^4 - 10^6$
- Galaxy clusters $N \sim 10^2 - 10^3$
- Cold Dark Matter haloes $N \gg 10^{50}$

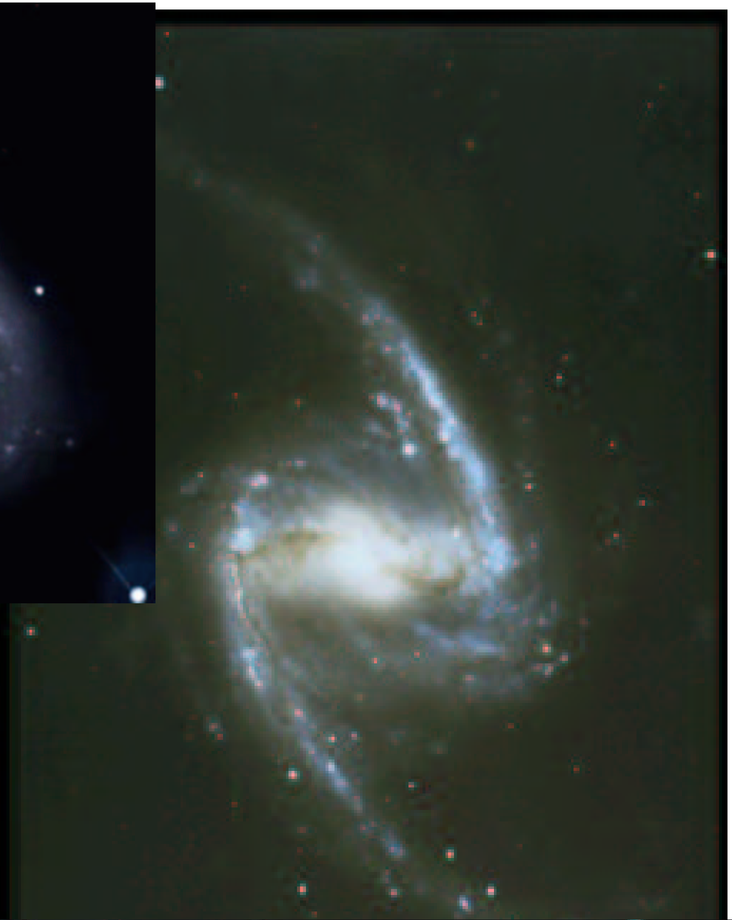
MAIN GOALS

- Infer mass distribution from observed kinematics. Comparison with light distribution $\Longrightarrow$ learn about dark matter and black holes
- Understand observed structure of galaxies:
  1. Galaxies formed this way $\Longrightarrow$ learn about Galaxy Formation
  2. Galaxies evolved this way $\Longrightarrow$ learn about Stability of galaxies

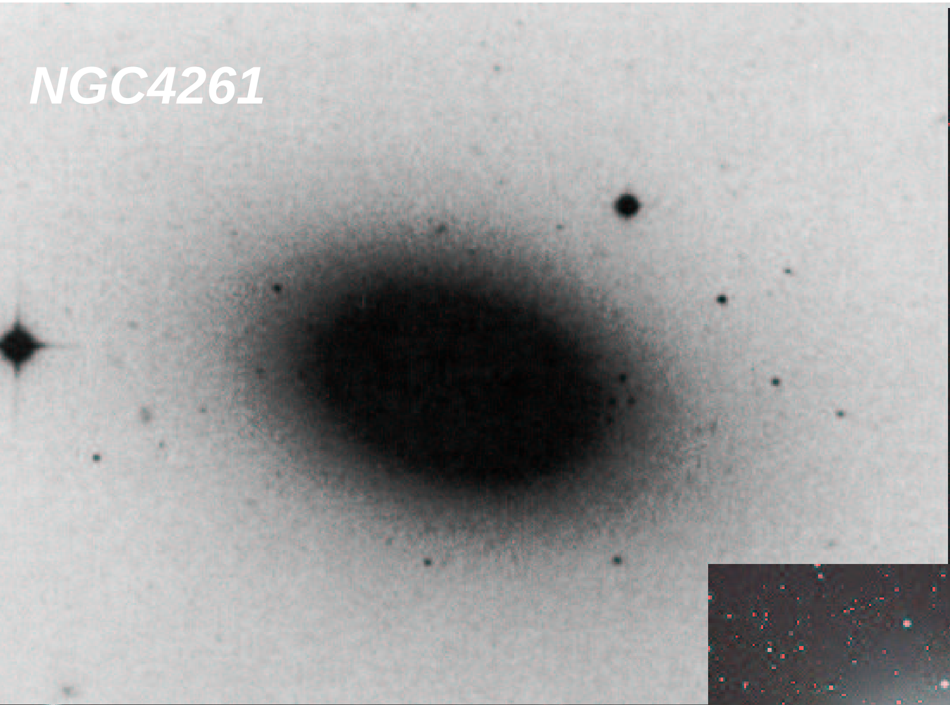
Globular Clusters

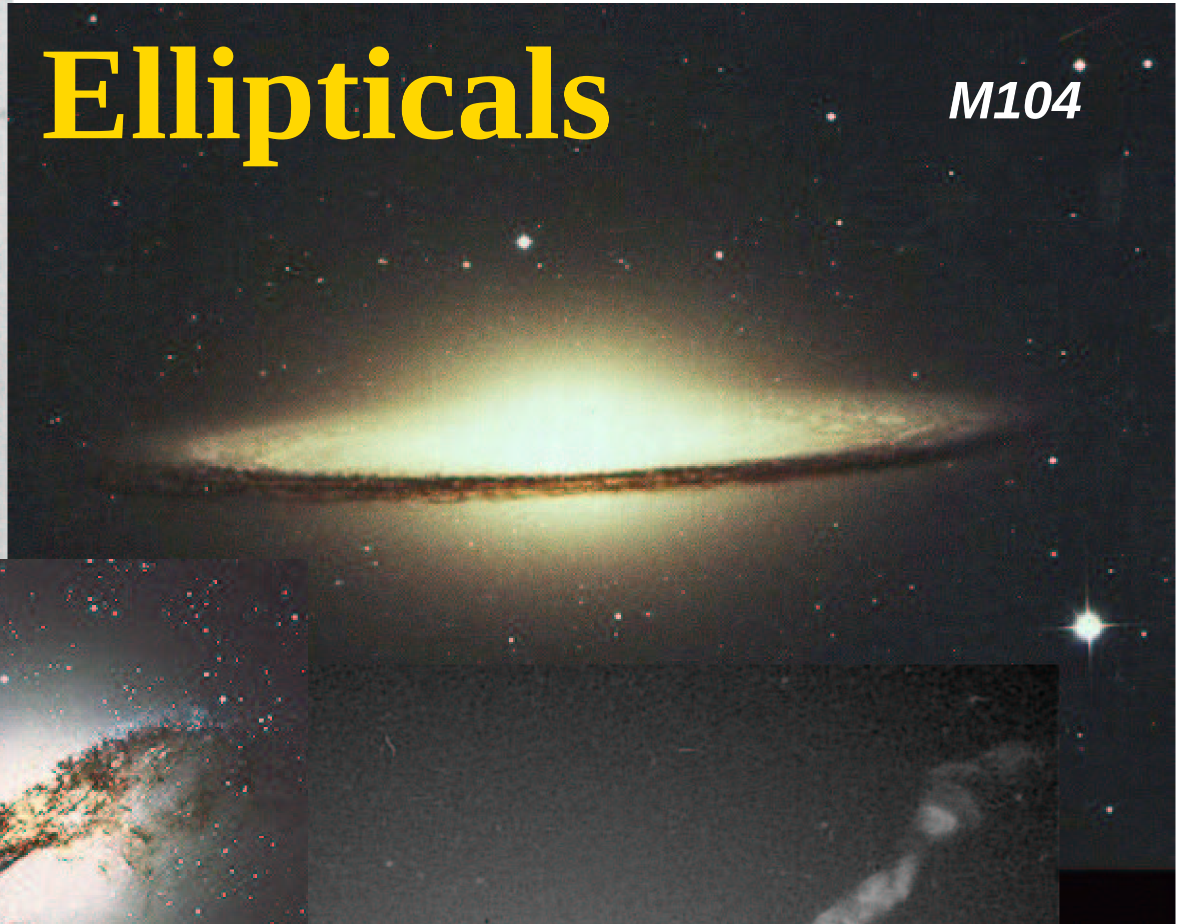


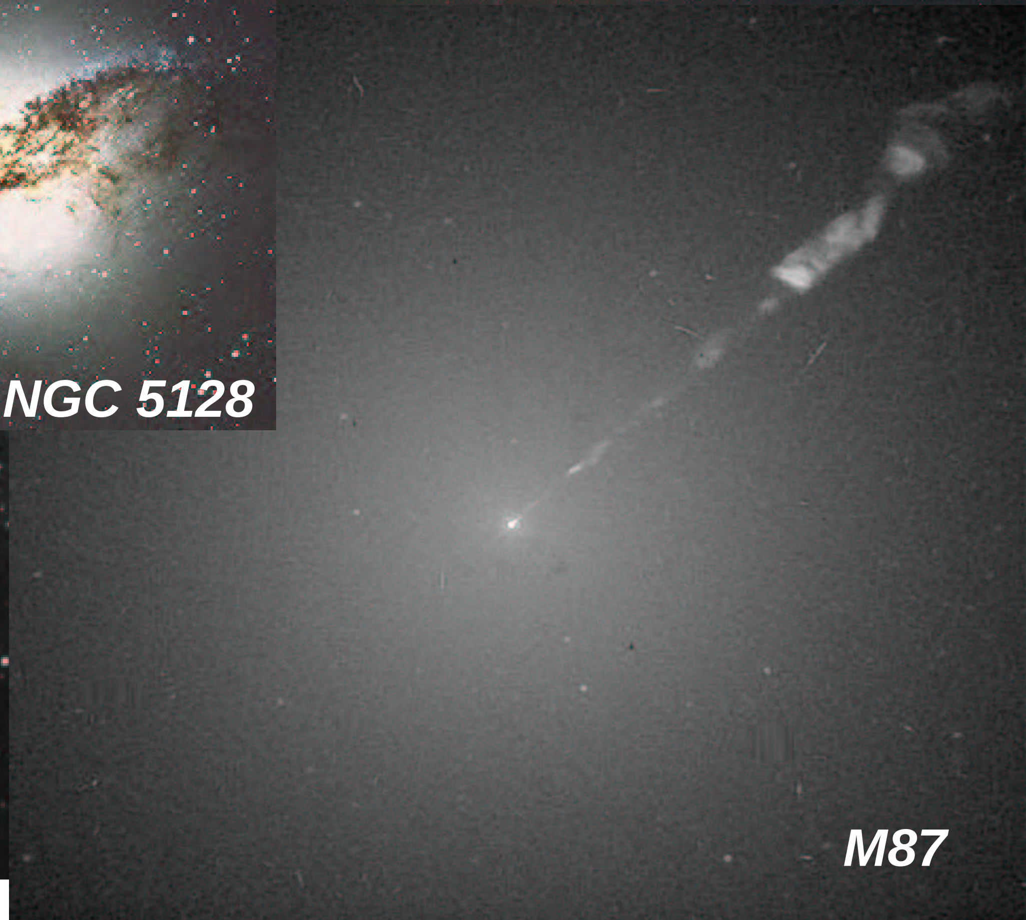
Disk Galaxies

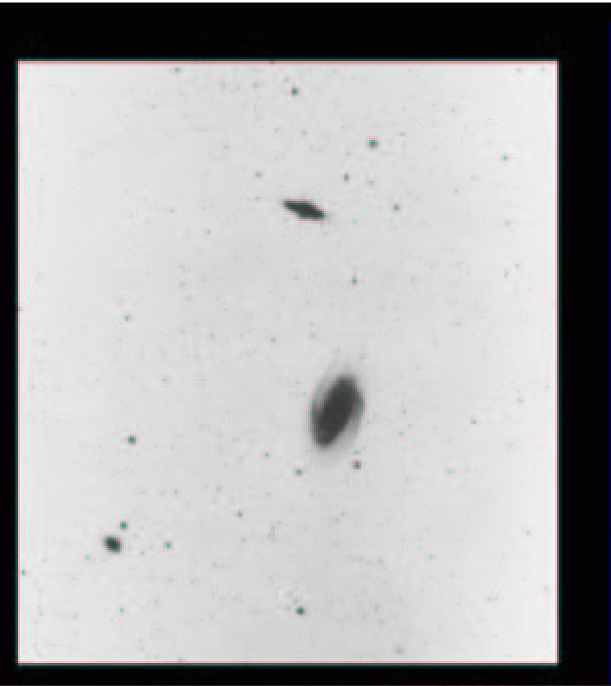


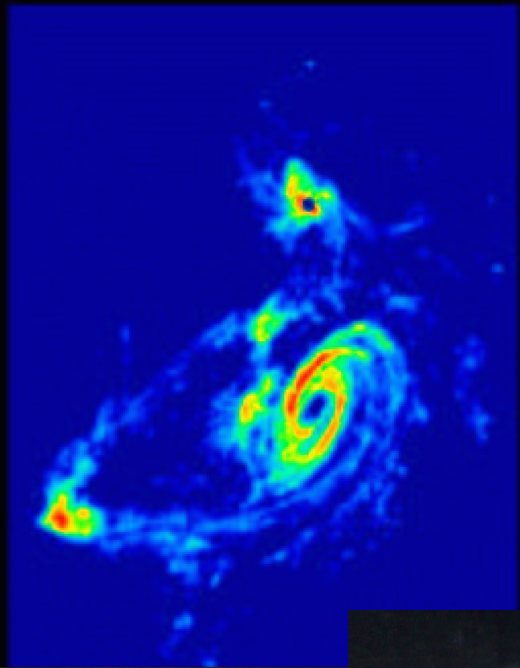
Ellipticals



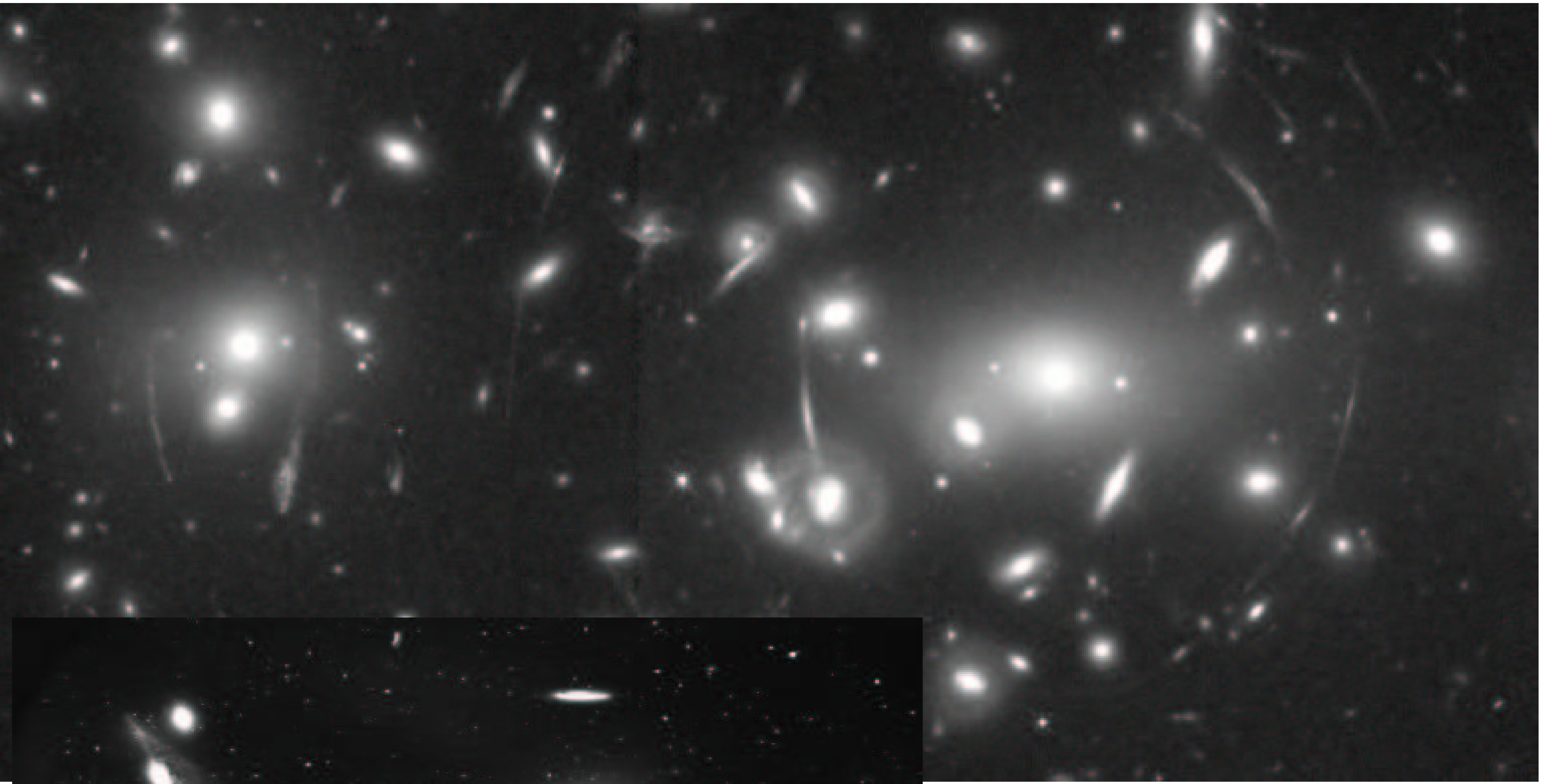








Weird Stuff...



Clusters & Groups

Newtonian Gravity

Newton's First Law: A body acted on by no forces moves with uniform velocity in a straight line

Newton's Second Law: $\vec{F} = m\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = \frac{\mathrm{d}\vec{p}}{\mathrm{d}t}$ (equation of motion)

Newton's Third Law: $\vec{F}_{ij} = -\vec{F}_{ji}$ (action = reaction)

Newton's Law of Gravity: $\vec{F}_{ij} = -G\frac{m_i m_j}{|\vec{x}_i - \vec{x}_j|^3}(\vec{x}_i - \vec{x}_j)$

$$\boxed{G = 6.67 \times 10^{-11} N m^2\,\mathrm{kg}^{-2} = 4.3 \times 10^{-9}(\,\mathrm{km\,s}^{-1})^2\,\mathrm{M}_{\odot}{}^{-1}\,\mathrm{Mpc}}$$

Gravity is a conservative Force. This implies that:

- $\exists$ scalar field $V(\vec{x})$ (potential energy), so that $\vec{F} = -\vec{\nabla} V(\vec{x})$
- The total energy $E = \frac{1}{2}mv^2 + V(\vec{x})$ is conserved
- Gravity is a curl-free field: $\vec{\nabla} \times \vec{F} = 0$

Gravity is a central Force. This implies that:

- The moment about the center vanishes: $\vec{r} \times \vec{F} = 0$
- Angular momentum $\vec{J} = m\vec{r} \times \vec{v}$ is conserved: $(\frac{\mathrm{d}\vec{J}}{\mathrm{d}t} = \vec{r} \times \vec{F} = 0)$

The Gravitational Potential

Potential Energy: $\vec{F}(\vec{x}) = -\vec{\nabla} V(\vec{x})$

Gravitational Potential: $\Phi(\vec{x}) = \frac{V(\vec{x})}{m}$

Gravitational Field: $\vec{g}(\vec{x}) = \frac{\vec{F}(\vec{x})}{m} = -\vec{\nabla}\Phi(\vec{x})$

From now on $\vec{F}$ is the force per unit mass so that $\vec{F}(\vec{x}) = -\vec{\nabla}\Phi(\vec{x})$

For a point mass M at $\vec{x}_0$: $\Phi(\vec{x}) = -\frac{GM}{|\vec{x}-\vec{x}_0|}$

For a density distribution $\rho(\vec{x})$: $\Phi(\vec{x}) = -G \int \frac{\rho(\vec{x}')}{|\vec{x}'-\vec{x}|} \mathrm{d}^3\vec{x}'$

The density distribution $\rho(\vec{x})$ and gravitational potential $\Phi(\vec{x})$ are related to each other by the Poisson Equation

$$\boxed{\nabla^2\Phi = 4\pi G\rho}$$

For $\rho = 0$ this reduces to the Laplace equation: $\nabla^2\Phi = 0$.

see B&T p.31 for derivation of Poisson Equation

Gauss's Theorem & Potential Theorem

If we integrate the Poisson Equation, we obtain

$$4\pi G \int \rho \, \mathrm{d}^3\vec{x} = 4\pi G M = \int_V \nabla^2 \Phi \, \mathrm{d}^3\vec{x} = \int_S \vec{\nabla}\Phi \mathrm{d}^2\vec{s}$$

Gauss's Theorem: $\boxed{\int_S \vec{\nabla}\Phi \, \mathrm{d}^2\vec{s} = 4\pi G M}$

Gauss's Theorem states that the integral of the normal component of the gravitational field [$\vec{g}(\vec{r}) = \vec{\nabla}\Phi$] over any closed surface S is equal to $4\pi G$ times the total mass enclosed by S.

cf. Electrostatics: $\boxed{\int_S \vec{E} \cdot \vec{n} \, \mathrm{d}^2\vec{s} = \frac{Q_{\mathrm{int}}}{\epsilon_0}}$

For a continuous density distribution $\rho(\vec{x})$ the total potential energy is:

$$\boxed{W = \frac{1}{2} \int \rho(\vec{x}) \, \Phi(\vec{x}) \, \mathrm{d}^3\vec{x}}$$

(see B&T p.33 for derivation)

NOTE: Here we follow B&T and use the symbol W instead of V.

The Discrete N-body Problem

The gravitational force on particle i due to particle j is:

$$\boxed{\vec{F}_{i,j} = \frac{G m_i m_j}{|\vec{x}_i - \vec{x}_j|^3}(\vec{x}_i - \vec{x}_j)}$$

(Newton's Inverse Square Law)

Equations of Motion: $\vec{F} = m\frac{\mathrm{d}\vec{v}}{\mathrm{d}t}$

For particle i, the equations of motion are:

$$\boxed{\begin{array}{rcll} \frac{\mathrm{d}v_{k,i}}{\mathrm{d}t} & = & G \sum\limits_{j=1, j\neq i}^{N} \frac{m_j}{(x_{k,i} - x_{k,j})^2} & \\ \frac{\mathrm{d}x_{k,i}}{\mathrm{d}t} & = & v_{k,i} & (k = 1, 3) \end{array}}$$

This corresponds to a closed set of $6N$ equations, for a total of $6N$ unknowns (x, y, z, v_x, v_y, v_z)

Since N is typically very, very large, we can't make progress studying the dynamics of these systems by solving the $6N$ equations of motion.

Even with the most powerful computers to date, we can only run N-body simulations with $N \lesssim 10^6$

From Discrete to Smooth

The density distribution and gravitational potential of N-body system are:

$$\boxed{\rho_N(\vec{x}) = \sum_{i=1}^{N} m_i\, \delta(\vec{x} - \vec{x}_i)}$$

with $\delta(\vec{x})$ the Dirac delta function (B&T p.652), and

$$\boxed{\Phi_N(\vec{x}) = -\sum_{i=1}^{N} \frac{G m_i}{|\vec{x} - \vec{x}_i|}}$$

$$\begin{aligned}
\vec{F}_i &= G \sum_{j=1, j\neq i}^{N} \frac{m_j}{|\vec{x}_j - \vec{x}_i|^3}(\vec{x}_j - \vec{x}_i) \\
&= G \sum_{j=1, j\neq i}^{N} \int \frac{(\vec{x}_j - \vec{x}_i)}{|\vec{x}_j - \vec{x}_i|^3} m_j \delta(\vec{x}_j - \vec{x})\mathrm{d}^3\vec{x} \\
&= G \int \frac{(\vec{x}_j - \vec{x}_i)}{|\vec{x}_j - \vec{x}_i|^3} \rho_N(\vec{x})\mathrm{d}^3\vec{x}
\end{aligned}$$

We will replace $\rho_N(\vec{x})$ and $\Phi_N(\vec{x})$ with smooth and continuous functions $\rho(\vec{x})$ and $\Phi(\vec{x})$

From Discrete to Smooth

For systems with large N, it is useful to try to use statistical descriptions of the system (cf. Thermodynamics)

Replacing a discrete density distribution by a continuous density distribution is familiar to us from fluid dynamics and plasma physics

However, there is one important difference:

$$\begin{array}{lcl} \text{Plasma \& Fluid} & \Longleftrightarrow & \text{short range forces} \\ \text{Gravitational System} & \Longleftrightarrow & \text{long range forces} \end{array}$$

Plasma: electrostatic forces are long-range forces, but because of Debye schielding the total charge $\longrightarrow 0$ at large r: short-range forces dominate. Plasma may be collisionless.

Fluid: collisional system dominated by short-range van der Waals forces between dipoles of molecules. Always attractive, but for large r dipoles vanish. For very small r force becomes strongly repulsive.

For both plasma and fluid energy is an extensive variable: total energy is sum of energies of subsystems.

For gravitational systems, energy is a non-extensive variable: sub-systems influence each other by long-range gravitational interaction.

From Discrete to Smooth

FLUID

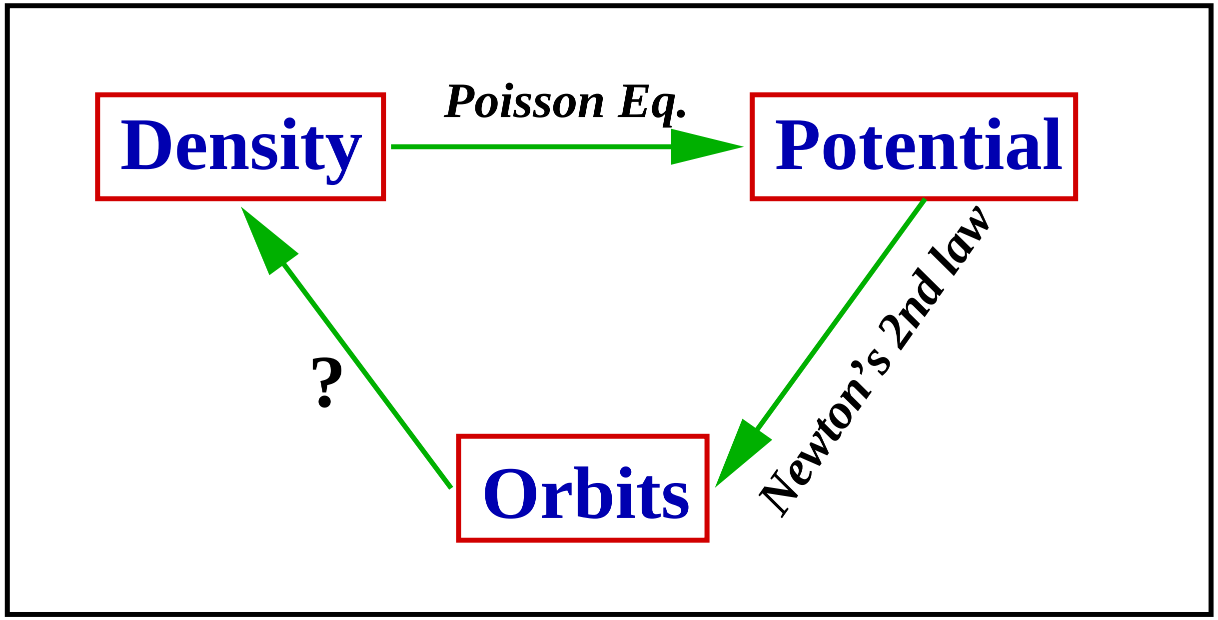
- mean-free path of molecules $\ll$ size of system
- molecules collide frequently, giving rise to a well defined collisional pressure. This pressure balances gravity in hydrostatic equilibrium.
- Pressure related to density by equation of state. Ie, the EOS determines the (hydrostatic) equilibrium.

GRAVITATIONAL SYSTEM

- mean-free path of particles $\gg$ size of system
- No collisional pressure, although kinetic energy of particles act as a source of ‘pressure’, balancing the potential energy in virial equilibrium.
- No equivalent of equation of state. Pressure follows from kinetic energy, but kinetic energy follows from the actual orbits within gravitational potential, which in turn follows from the spatial distribution of the particles (Self-Consistency Problem)

The Self-Consistency Problem

Given a density distribution $\rho(\vec{x})$, the Poisson equation yields the gravitational potential $\Phi(\vec{x})$. In this potential I can integrate orbits using Newton's equations of motion. The self-consistency problem is the problem of finding that combination of orbits that reproduces $\rho(\vec{x})$.



Think of self-consistency problem as follows: Given $\Phi(\vec{x})$, integrate all possible orbits $\mathcal{O}_i(\vec{x})$, and find the orbital weights w_i such that $\rho(\vec{x}) = \sum w_i \mathcal{O}_i(\vec{x})$. Here $\mathcal{O}_i(\vec{x})$ is the density contributed to $\vec{x}$ by orbit i.

Timescales for Collisions

Following fluid dynamics and plasma physics, we replace our discrete $\rho_N(\vec{x})$ with a smooth, continuous $\rho(\vec{x})$. Orbits are then integrated in the corresponding smooth potential $\Phi(\vec{x})$.

In reality, the true orbits will differ from these orbits, because the true potential is not smooth.

In addition to direct collisions ('touching' particles), we also have long-range collisions, in which the long-range gravitational force of the granularity of the potential causes small deflections.

Over time, these deflections accumulate to make the description based on the smooth potential inadequate.

It is important to distinguish between long-range interactions, which only cause a small deflection per interaction, and short-range interactions, which cause a relatively large deflection per interaction.

Direct Collisions

Consider a system of size R consisting of N identical bodies of radius r

The cross section for a direct collision is $\sigma = 4\pi r^2$

The mean free path of a particle is $\lambda = \frac{1}{n\sigma}$, with $n = \frac{3N}{4\pi R^3}$ the number density of bodies

$$\boxed{\frac{\lambda}{R} = \frac{4\pi R^3}{3\,N\,4\pi r^2\,R} \simeq \left(\frac{R}{r}\right)^2 \frac{1}{N}}$$

It takes a crossing time $t_{\rm cross} \sim R/v$ to cross the system, so that the time scale for direct collisions is

$$\boxed{t_{\rm coll} = \left(\frac{R}{r}\right)^2 \frac{1}{N} t_{\rm cross}}$$

Example: A Milky Way like galaxy has $R = 10\ {\rm kpc} = 3.1 \times 10^{17}\ {\rm km}$, $v \simeq 200\ {\rm km\,s^{-1}}$, $N \simeq 10^{10}$, and r is roughly the radius of the Sun $(r = 6.9 \times 10^5\ {\rm km})$. This yields $\lambda = 2 \times 10^{13} R$. In other words, a direct collision occurs on average only once per 2000 billion crossings! The crossing time is $t_{\rm cross} = R/v = 5 \times 10^7\ {\rm yr}$, so that $t_{\rm coll} \simeq 10^{21}\ {\rm yr}$. This is about 10^{11} times the age of the Universe!!!

Relaxation Time I

Now that we have seen that direct collisions are completely negligble, let's focus on encounters

Consider once again a system of size R consisting of N identical particles of mass m. Consider one such particle crossing the system with velocity v. As we will see later, a typical value for the velocity is

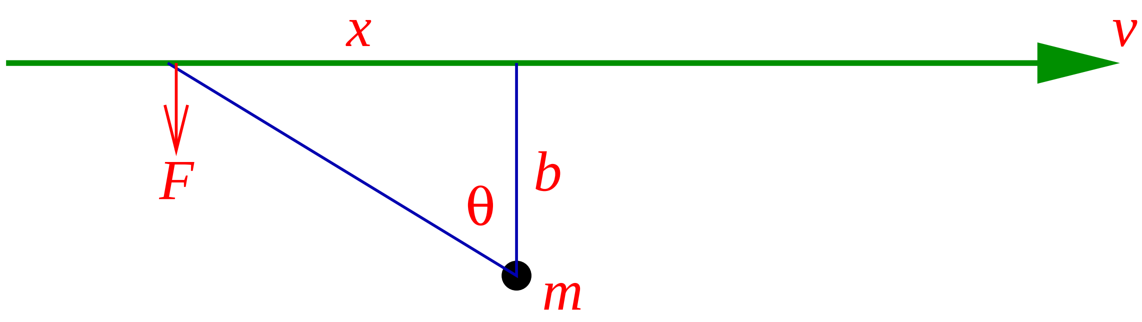
$$v = \sqrt{\frac{G\,M}{R}} = \sqrt{\frac{G\,N\,m}{R}}$$

We want to calculate how long it takes before the cumulative effect of many encounters has given our particle a kinetic energy $E_{\rm kin} \propto v^2$ in the direction perpendicular to its original motion of the order of its its initial kinetic energy.

Note that for a sufficiently close encounter, this may occur in a single encounter. We will treat this case seperately, and call such an encounter a close encounter.

Relaxation Time II

First consider a single encounter



Here b is the impact parameter, $x = v\,t$, with $t = 0$ at closest approach,

and $\cos\theta = \frac{b}{\sqrt{x^2+b^2}} = \left[1 + \left(\frac{vt}{b}\right)^2\right]^{-1/2}$

At any given time, the gravitational force in the direction perpendicular to the direction of the particle is

$$F_\perp = G\frac{m^2}{x^2+b^2}\cos\theta = \frac{Gm^2}{b^2}\left[1 + \left(\frac{vt}{b}\right)^2\right]^{-3/2}$$

This force $F_\perp$ causes an acceleration in the $\perp$-direction: $F_\perp = m\frac{\mathrm{d}v_\perp}{\mathrm{d}t}$

We now compute the total $\Delta v_\perp$ integrated over the entire encounter, where we make the simplifying assumption that the particle moves in a straight line. This assumption is OK as long as $\Delta v_\perp \ll v$

Relaxation Time III

$$\begin{aligned}
\Delta v_\perp &= 2\int_0^\infty \frac{Gm}{b^2}\left[1+\left(\frac{vt}{b}\right)^2\right]^{-3/2}\mathrm{d}t \\
&= \frac{2Gm}{b^2}\frac{b}{v}\int_0^\infty (1+s^2)^{-3/2}\mathrm{d}s \\
&= \frac{2Gm}{bv}
\end{aligned}$$

As discussed above, this is only valid as long as $\Delta v_\perp \ll v$. We define the minimum impact parameter $b_{\rm min}$, which borders long- and short-range interactions as: $\Delta v_\perp(b_{\rm min}) = v$

$$\boxed{b_{\rm min} = \frac{2Gm}{v^2} \simeq R/N}$$

For a MW-type galaxy, with $R = 10$ kpc and $N = 10^{10}$ we have that $b_{\rm min} \simeq 3\times 10^7\ {\rm km} \simeq 50\ {\rm R}_\odot$

In a single, close encounter $\Delta E_{\rm kin} \sim E_{\rm kin}$. The time scale for such a close encounter to occur can be obtained from the time scale for direct collisions, by simply replacing r by $b_{\rm min}$.

$$\boxed{t_{\rm short} = \left(\frac{R}{b_{\rm min}}\right)^2 \frac{t_{\rm cross}}{N} = N\, t_{\rm cross}}$$

Relaxation Time IV

Now we compute the number of long-range encounters per crossing. Here we use that $(\Delta v_\perp)^2$ adds linearly with the number of encounters. (Note: this is not the case for $\Delta v_\perp$ because of the random directions).

When the particle crosses the system once, it has $n(< b)$ encounters with an impact parameter less than b, where

$$n(< b) = N \frac{\pi b^2}{\pi R^2} = N \left(\frac{b}{R}\right)^2$$

Differentiating with respect to b yields

$$n(b)\mathrm{d}b = \frac{2Nb}{R^2}\mathrm{d}b$$

Thus the total $(\Delta v_\perp)^2$ per crossing due to encounters with impact parameter $b, b + \mathrm{d}b$ is

$$(\Delta v_\perp)^2(b)\mathrm{d}b = \left(\frac{2Gm}{bv}\right)^2 \frac{2Nb}{R^2}\mathrm{d}b = 8N \left(\frac{Gm}{Rv}\right)^2 \frac{\mathrm{d}b}{b}$$

Integrating over the impact parameter yields

$$(\Delta v_\perp)^2 = 8N \left(\frac{Gm}{Rv}\right)^2 \int\limits_{b_{\min}}^{R} \frac{\mathrm{d}b}{b} \equiv 8N \left(\frac{Gm}{Rv}\right)^2 \ln\Lambda$$

with $\ln\Lambda = \ln\left(\frac{R}{b_{\min}}\right) = \ln N$ the Coulomb logarithm

Relaxation Time V

We thus have that

$$(\Delta v_\perp)^2 = \left(\frac{GNm}{R}\right)^2 \frac{1}{v^2}\frac{8\ln N}{N}$$

Substituting the characteristic value for v then yields that

$$\boxed{\frac{(\Delta v_\perp)^2}{v^2} \simeq \frac{10\ln N}{N}}$$

Thus it takes of the order of $N/(10\ln N)$ crossings for $(\Delta v_\perp)^2$ to become comparable to v^2. This defines the relaxation time

$$\boxed{t_{\mathrm{relax}} = \frac{N}{10\ln N} t_{\mathrm{cross}}}$$

Summary of Time Scales

Let R be the size of the system, r the size of a particle (e.g., star), v the typical velocity of the particles, and N the number of particles in the system.

Hubble time: The age of the Universe. $t_H \simeq 1/H_0 \simeq 10^{10}\ \mathrm{yr}$

Formation time: The time it takes the system to form. $t_{\mathrm{form}} = \frac{\dot{M}}{M} \simeq t_H$

Crossing time: The typical time needed to cross the system. $t_{\mathrm{cross}} = R/v$

Collision time: The typical time between two direct collisions.

$$t_{\mathrm{coll}} = \left(\frac{R}{r}\right)^2 \frac{t_{\mathrm{cross}}}{N}$$

Relaxation time: The time over which the change in kinetic energy due to the long-range collisions has accumulated to a value that is comparable to the intrinsic kinetic energy of the particle.

$$t_{\mathrm{relax}} = \frac{N}{10 \ln N} t_{\mathrm{cross}}$$

Interaction time: The typical time between two short-range interactions that cause a change in kinetic energy comparable to the intrinsic kinetic energy of the particle.

$$t_{\mathrm{short}} = N t_{\mathrm{cross}}$$

For Trully Collisionless systems:

$$\boxed{t_{\mathrm{cross}} \ll t_H \simeq t_{\mathrm{form}} \ll t_{\mathrm{relax}} \ll t_{\mathrm{short}} \ll t_{\mathrm{coll}}}$$

Some other useful Time Scales

NOTE: Using that $v = \sqrt{\frac{GM}{R}}$ and $\bar{\rho} = \frac{3M}{4\pi R^3}$ we can write

$$\boxed{t_{\rm cross} = \sqrt{\frac{3}{4\pi G\bar{\rho}}}}$$

Dynamical time: the time required to travel halfway across the system.

$$t_{\rm dyn} = \sqrt{\frac{3\pi}{16G\rho}} = \frac{\pi}{2} t_{\rm cross}$$

Free-fall time: the time it takes a sphere with zero pressure to collapse to a point.

$$t_{\rm ff} = \sqrt{\frac{3\pi}{32G\rho}} = t_{\rm dyn}/\sqrt{2}$$

Orbital time: the time it takes to complete a (circular) orbit.

$$t_{\rm orb} = \sqrt{\frac{3\pi}{G\rho}} = 2\pi t_{\rm cross}$$

NOTE: All these timescales are the same as the crossing time, except for some pre-factors

$$\boxed{t_{\rm cross} \lesssim t_{\rm ff} \lesssim t_{\rm dyn} \lesssim t_{\rm orb}}$$

Example of Time Scales

System	Mass $\mathrm{M}_\odot$	Radius kpc	Velocity $\mathrm{km\,s^{-1}}$	N	$t_{\rm cross}$ yr	$t_{\rm relax}$ yr
Galaxy	10^{10}	10	100	10^{10}	10^{8}	$> 10^{15}$
DM Halo	10^{12}	200	200	$> 10^{50}$	10^{9}	$> 10^{60}$
Cluster	10^{14}	1000	1000	10^{3}	10^{9}	$\sim 10^{10}$
Globular	10^{4}	0.01	2	10^{4}	5×10^{6}	5×10^{8}

- **Dark Matter Haloes and Galaxies are collisionless**
- **Collisions may or may not be important in clusters of galaxies**
- **Relaxation is expected to have occured in (some) globular clusters**

NOTE: For a self-gravitating system, the typical velocities are $v \simeq \sqrt{\frac{GM}{R}}$

For the crossing time this implies: $t_{\rm cross} = \frac{R}{v} = \sqrt{\frac{R^3}{GM}} = \sqrt{\frac{3}{4\pi G\rho}}$

Useful to remember: $1\ \mathrm{km/\ s} \simeq 1\ \mathrm{kpc/\ Gyr}$

$1\ \mathrm{yr} \simeq \pi \times 10^{7}\ \mathrm{s}$

$1\ \mathrm{M}_\odot \simeq 2 \times 10^{30}\ \mathrm{kg}$

$1\ \mathrm{pc} \simeq 3.1 \times 10^{13}\ \mathrm{km}$