The 30th Jerusalem Winter School in Theoretical Physics

Lecture 2
The EPS Formalism

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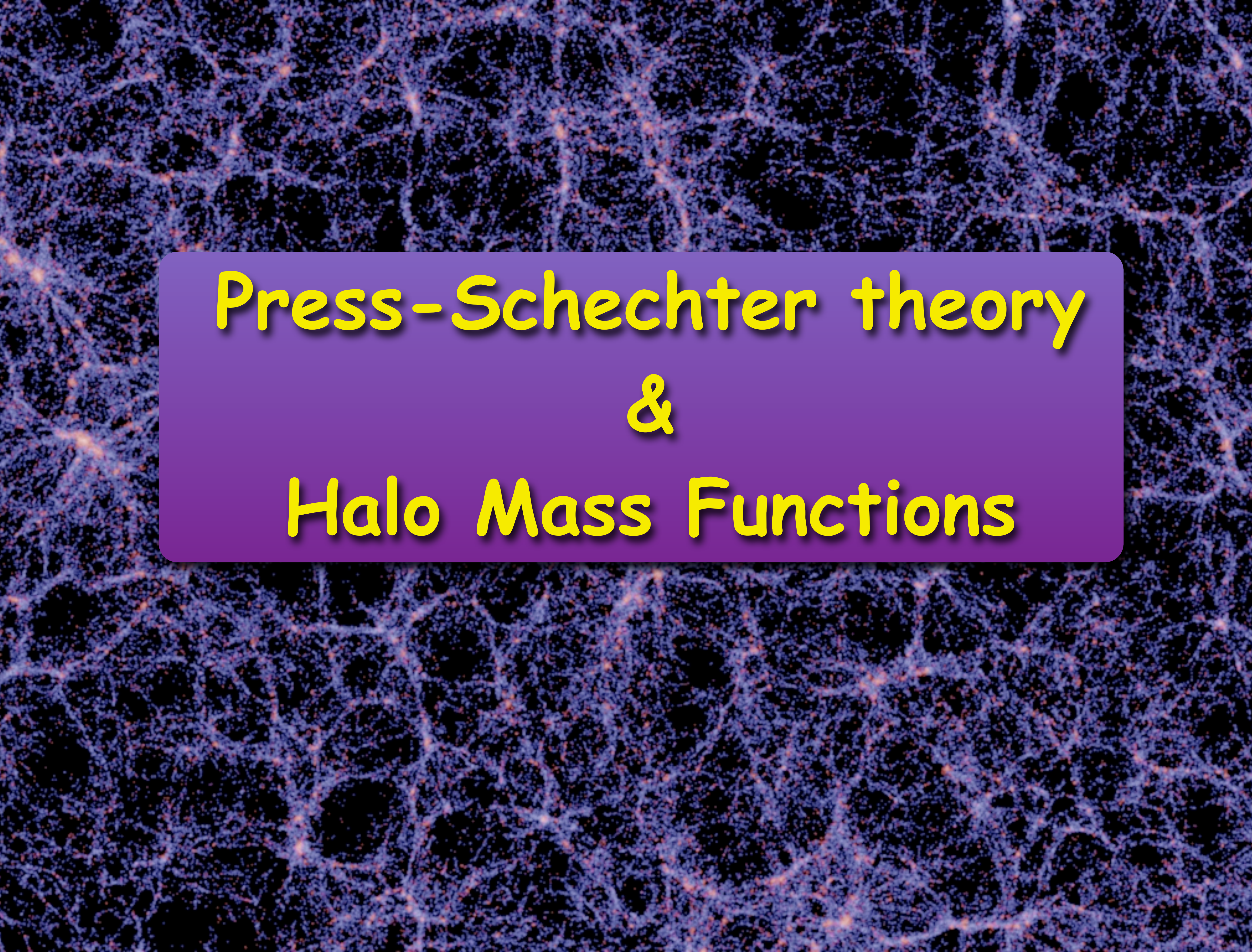
Yale University, Jan 2013

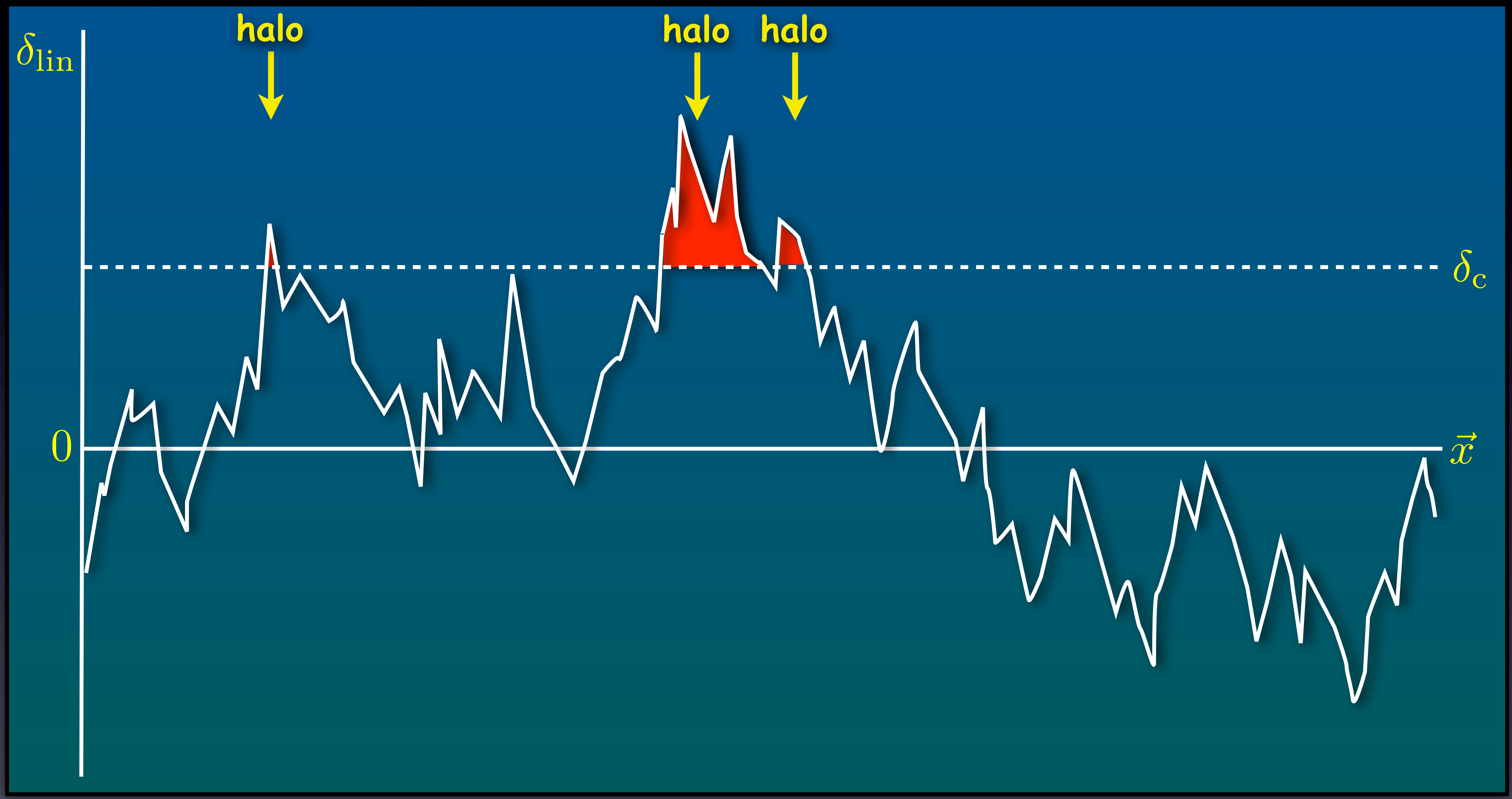
Press-Schechter theory
&
Halo Mass Functions



The Linear Cosmological Density Field

According to linear theory, the density field evolves as $\delta(\vec{x},t) = D(t)\,\delta_0(\vec{x})$

Here $\delta_0(\vec{x})$ is the density field linearly extrapolated to $t = t_0$, and $D(t)$ is the linear growth rate normalized to unity at $t = t_0$



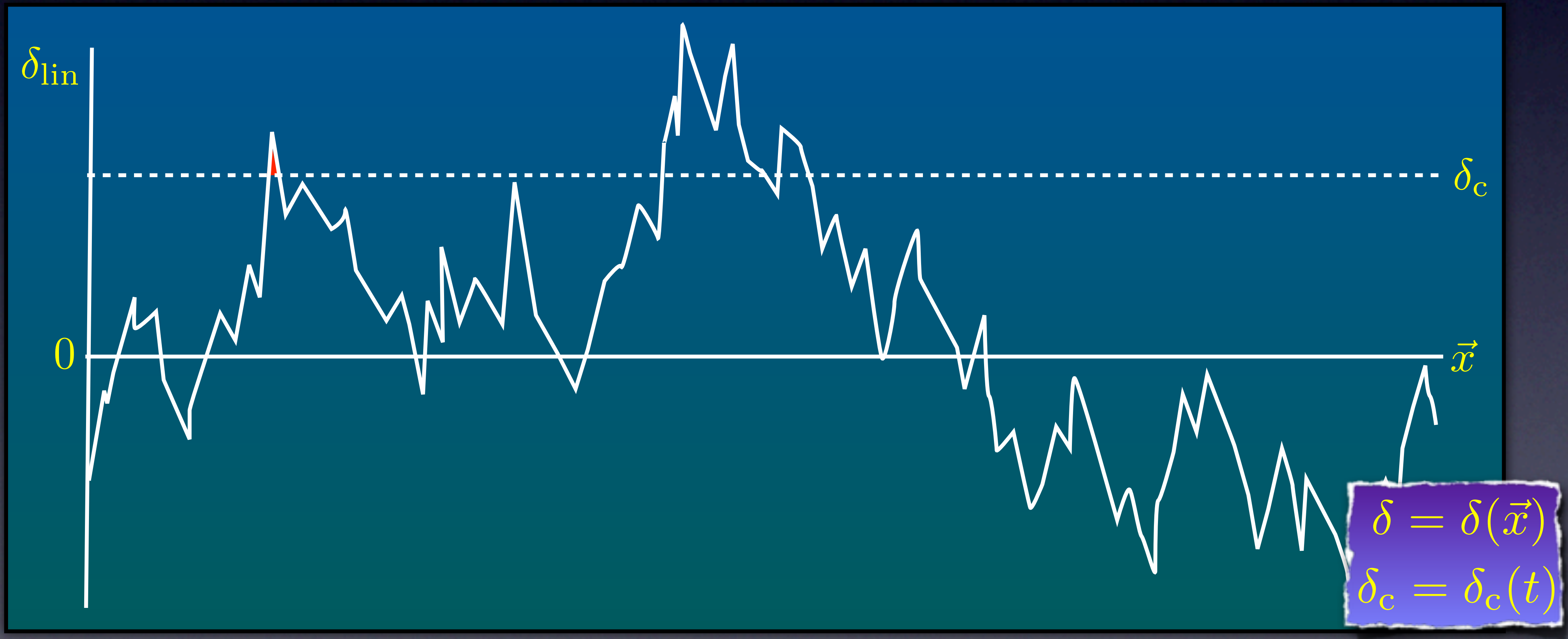
According to the spherical collapse model, regions with $\delta(\vec{x},t) > \delta_{\rm c} \simeq 1.686$ will have collapsed to produce dark matter haloes by time t . In this lecture we examine how to assign a halo mass to this structure. But first, we need to introduce some concepts...

Notation & Convention

According to the spherical collapse model, regions with $\delta(\vec{x},t) > \delta_c \simeq 1.686$ will have collapsed to produce dark matter haloes by time t

Using that $\delta(\vec{x},t) = D(t)\,\delta_0(\vec{x})$ we can also phrase this differently: regions with $\delta_0(\vec{x}) > \delta_c/D(t)$ will have collapsed to produce dark matter haloes by time t

In this latter case, we consider the density field to be static (at the one linearly extrapolated to our reference time), while the `collapse barier' evolves with time.



In the Press-Schechter formalism, the latter will be our preferred `view'.

Smoothing

Given a density field $\delta(\vec{x})$, one can filter it using some window function (or "filter") $W(\vec{x};R)$ which is properly normalized such that $\int W(\vec{x};R)\,\mathrm{d}^3\vec{x} = 1$, to get a smoothed field

$$\delta(\vec{x};R) \equiv \int \delta(\vec{x}')\,W(\vec{x}+\vec{x}';R)\,\mathrm{d}^3\vec{x}'$$

For each filter, one can define a mass $M = \gamma_{\rm f}\,\bar{\rho}\,R^3$, where $\gamma_{\rm f}$ is some constant that depends on the shape of the filter. In what follows, we will characterize a filter intermittendly by its size R or its mass M.

The above equation for the smoothed density field is a convolution integral (density field is convolved with window function). Since convolution in real-space is equal to multiplication in Fourier space, we have that

$$\delta(\vec{k};R) = \int \delta(\vec{x};R)\,e^{-i\vec{k}\cdot\vec{x}}\,\mathrm{d}^3\vec{x} = \delta(\vec{k})\,\widetilde{W}(kR)$$

where $\widetilde{W}(kR) = \int W(\vec{x};R)\,e^{-i\vec{k}\cdot\vec{x}}\,\mathrm{d}^3\vec{x}$ is the Fourier Transform of the window function for which we have made it explicit that k and R only enter in the combination kR.

Window Functions

Throughout we will use either one of the following three window functions:

Top Hat Filter: $\gamma_{\rm f} = 4\pi/3$

$$W(\vec{x};R) = \begin{cases} \frac{3}{4\pi R^3} & r \leq R \\ 0 & r > R \end{cases} \qquad \widetilde{W}(kR) = \frac{3}{(kR)^3}\left[\sin(kR) - (kR)\,\cos(kR)\right]$$

Gaussian Filter: $\gamma_{\rm f} = (2\pi)^{3/2}$

$$W(\vec{x};R) = \frac{1}{(2\pi)^{3/2}\,R^3}\exp\left(-\frac{r^2}{2R^2}\right) \qquad \widetilde{W}(kR) = \exp\left(-\frac{(kR)^2}{2}\right)$$

Sharp k-space Filter: $\gamma_{\rm f} = 6\,\pi^2$

$$W(\vec{x};R) = \frac{1}{2\pi^2\,r^3}\left[\sin(r/R) - (r/R)\,\cos(r/R)\right] \qquad \widetilde{W}(kR) = \begin{cases} 1 & k \leq 1/R \\ 0 & k > 1/R \end{cases}$$

The Mass Variance

Similar to case without smoothing, we define the variance of the smoothed density field as

$$\sigma^2(R) = \langle \delta^2(\vec{x};R) \rangle = \frac{1}{2\pi^2} \int P(k)\, \widetilde{W}^2(kR)\, k^2\, \mathrm{d}k$$

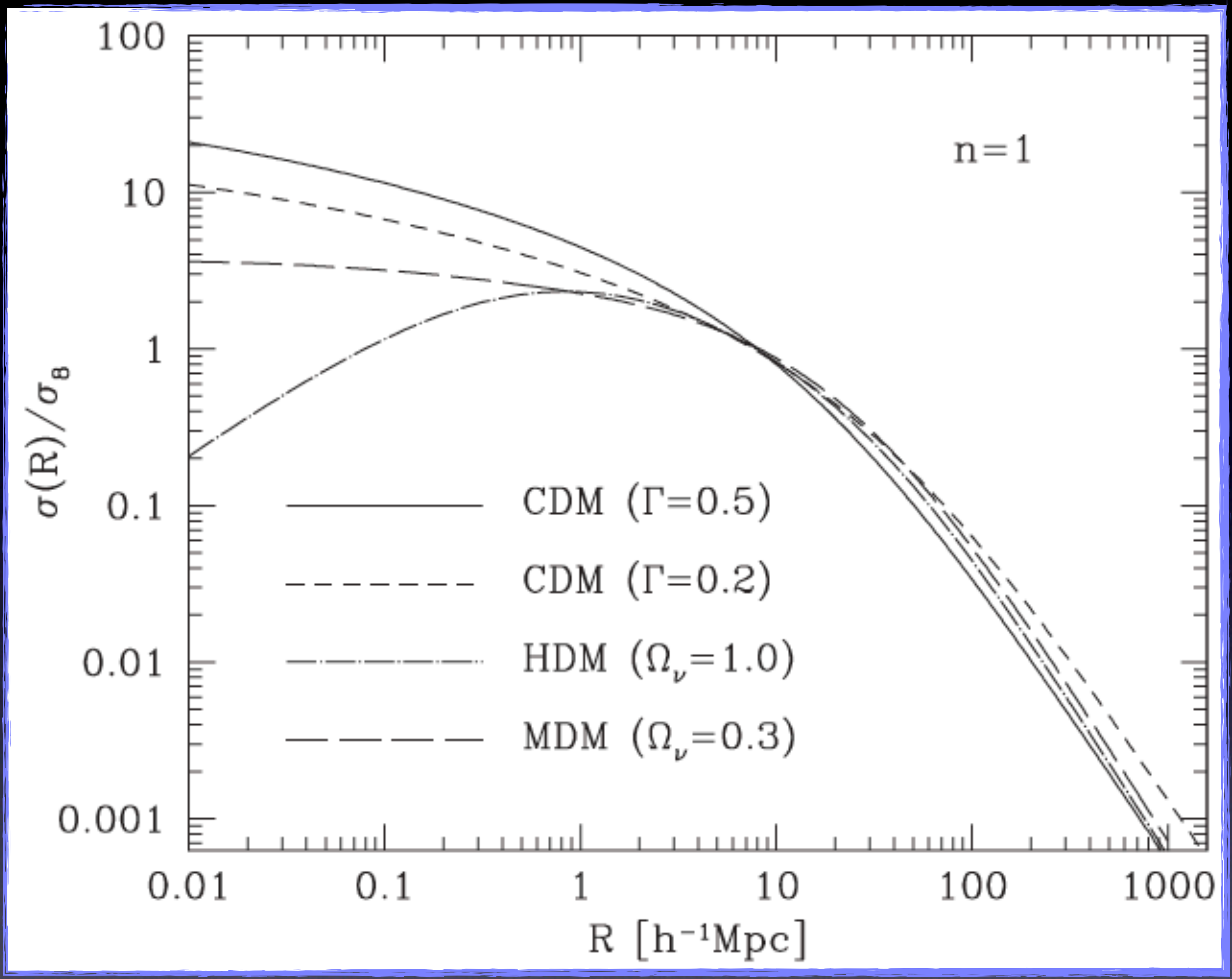
Since we can equally label a filter by its size R or its mass M, we can write $\sigma^2(R) = \sigma^2(M)$. The latter is called the mass variance, and plays an important role in what follows.

NOTE: If $\delta(\vec{x})$ is a Gaussian random field, then so is $\delta(\vec{x};R)$. In particular

$$\mathcal{P}(\delta_M)\, \mathrm{d}\delta_M = \frac{1}{\sqrt{2\pi}\,\sigma_M} \exp\left[-\frac{\delta_M^2}{2\sigma_M^2}\right] \mathrm{d}\delta_M$$

where we have used the shorthand notation $\delta_M = \delta(\vec{x};M)$ and $\sigma_M = \sigma(M)$.

Mass Variance



The variance of the smoothed, linear density field as a function of the size R of the top-hat filter. Results are shown for four different cosmogonies. The variance is normalized such that $\sigma_8 = 1$.

(see MWB §6.1.3)

Note: the shape parameter $\Gamma = \Omega_{m,0}\, h$ characterizes the horizon scale at matter-radiation equality.

In hierarchical models, such as CDM-based cosmologies, the variance is a monotonically decreasing function of the filter size R (or M). In top-down cosmogonies, such as HDM, however, the lack of small scale structure introduces a characteristic scale where the variance is maximum.

How to Assign (Halo) Mass to Collapsed Regions?

We now return to our main question of interest:

According to SC model, regions in the linear density field with $\delta > \delta_c$ have collapsed to produce virialized dark matter haloes. How can we associate a mass to those haloes, and how can we use the statistics of the linear density field to infer the halo mass function, i.e., the (comoving) number density of haloes as a function of halo mass?

Bill Press

FORMATION OF GALAXIES AND CLUSTERS OF GALAXIES BY SELF-SIMILAR GRAVITATIONAL CONDENSATION*

WILLIAM H. PRESS AND PAUL SCHECHTER
California Institute of Technology
Received 1973 August 1

ABSTRACT

We consider an expanding Friedmann cosmology containing a "gas" of self-gravitating masses. The masses condense into aggregates which (when sufficiently bound) we identify as single particles of a larger mass. We propose that after this process has proceeded through several scales, the mass spectrum of condensations becomes "self-similar" and independent of the spectrum initially assumed. Some details of the self-similar distribution, and its evolution in time, can be calculated with the linear perturbation theory. Unlike other authors, we make no ad hoc assumptions about the spectrum of long-wavelength initial perturbations: the nonlinear N-body interactions of the mass points randomize their positions and generate a perturbation to all larger scales; this should fix the self-similar distribution almost uniquely. The results of numerical experiments on 1000 bodies are presented; these appear to show new nonlinear effects: condensations can "bootstrap" their way up in size faster than the linear theory predicts. Our self-similar model predicts relations between the masses and radii of galaxies and clusters of galaxies, as well as their mass spectra. We compare the predictions with available data, and find some rather striking agreements. If the model is to explain galaxies, then isothermal "seed" masses of $\sim 3 \times 10^7 M_\odot$ must have existed at recombination. To explain clusters of galaxies, the only necessary seeds are the galaxies themselves. The size of clusters determines, in principle, the deceleration parameter q_0; presently available data give only very broad limits, unfortunately.

Subject headings: cosmology — galaxies — galaxies, clusters of

Paul Schechter

The Press-Schechter Mass Function

Press & Schechter (1974) postulated that:

"the probability that $\delta_M > \delta_c(t)$ is the same as the mass fraction that at time t is contained in halos with mass greater than M"

For a Gaussian random field, one has that

$$\mathcal{P}(\delta_M > \delta_c) = \frac{1}{\sqrt{2\pi}\,\sigma_M} \int_{\delta_c}^{\infty} \exp\left[-\frac{\delta_M^2}{2\sigma_M^2}\right] \mathrm{d}\delta_M = \frac{1}{2}\mathrm{erfc}\left[\frac{\delta_c}{2\sigma_M}\right]$$

Here $\mathrm{erfc}(x) = 1 - \mathrm{erf}(x)$ is the complimentary error function, and we consider it understood that $\delta_c = \delta_c(t)$. According to the PS postulate, we thus have that

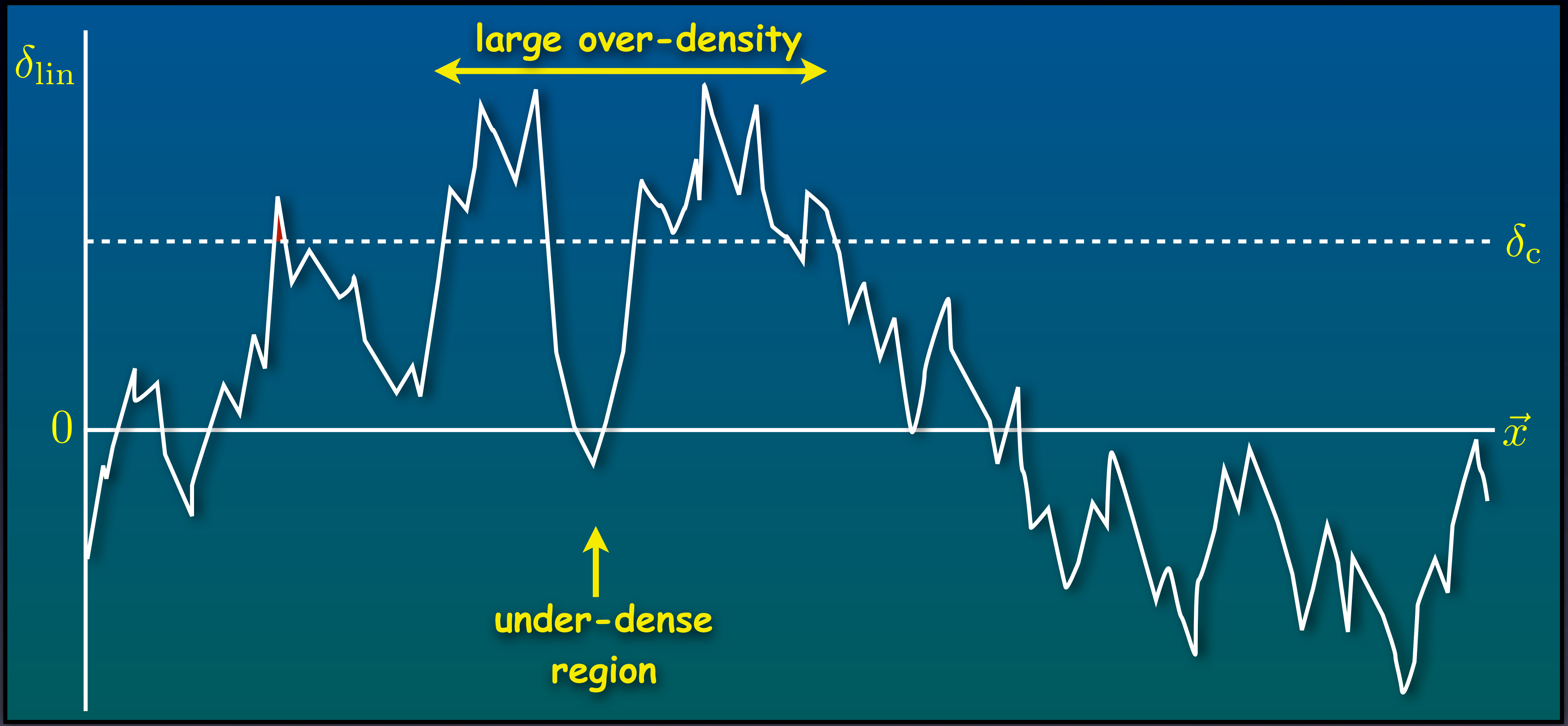
$$F(> M, t) = \frac{1}{2}\,\mathrm{erfc}\left[\frac{\delta_c}{2\,\sigma_M}\right]$$

Note: since $\lim_{x\to\infty} \mathrm{erfc}(x) = 0$ and $\mathrm{erfc}(0) = 1$ we see that the PS postulate predicts that never more than 1/2 of all matter in Universe is locked-up in collapsed haloes...

The Press-Schechter Mass Function

This may seem logical from the fact that $\mathcal{P}(\delta < 0) = \frac{1}{2}$; i.e., only regions that are initially overdense end up in collapsed objects...

However, underdense regions can be enclosed within larger overdense regions, giving them a finite probability of being included in some larger collapsed object (see illustration)



Press & Schechter `solved' this problem by simply introducing a fudge factor two:

$$F(> M, t) = 2\,\mathcal{P}\left[\delta_M > \delta_{\rm c}(t)\right]$$

The Press-Schechter Mass Function

We are now ready to write down the PS halo mass function:

We define the mass function as $n(M,t)\,\mathrm{d}M$, which is the number of haloes with masses in the range $[M, M+\mathrm{d}M]$ per (comoving) volume. Hence, $n(M,t) = \frac{\mathrm{d}n}{\mathrm{d}M} = M\,\frac{\mathrm{d}n}{\mathrm{d}\ln M}$.

We have that $\frac{\partial F(>M)}{\partial M}\,\mathrm{d}M$ is equal to the fraction of mass that is locked up in haloes with masses in the range $[M, M+\mathrm{d}M]$.

Multiplying by $\bar{\rho}$ yields the total mass per unit volume that is locked up in those haloes.

Hence, the halo mass function is simply given by $n(M,t)\,\mathrm{d}M = \frac{\bar{\rho}}{M}\,\frac{\partial F(>M)}{\partial M}\,\mathrm{d}M$

Using the Press-Schechter ansatz plus fudge factor we thus obtain:

$$n(M,t)\,\mathrm{d}M = 2\,\frac{\bar{\rho}}{M}\,\frac{\partial \mathcal{P}(>\delta_{\rm c})}{\partial M}\,\mathrm{d}M = \sqrt{\frac{2}{\pi}}\,\frac{\bar{\rho}}{M^2}\,\frac{\delta_{\rm c}}{\sigma_M}\exp\left(-\frac{\delta_{\rm c}^2}{2\sigma_M^2}\right)\left|\frac{\mathrm{d}\ln\sigma_M}{\mathrm{d}\ln M}\right|\,\mathrm{d}M$$

where we have used that $\partial\mathcal{P}/\partial M = \partial\mathcal{P}/\partial\sigma_M \times |\mathrm{d}\sigma_M/\mathrm{d}M|$.

The Press-Schechter Mass Function

Upon defining the variable $\nu \equiv \delta_c(t)/\sigma(M)$ we can write the PS mass function in a more compact form:

$$n(M,t)\,\mathrm{d}M = \frac{\bar{\rho}}{M^2}\, f_{\rm PS}(\nu) \left|\frac{\mathrm{d}\ln\nu}{\mathrm{d}\ln M}\right| \mathrm{d}M \quad \text{where} \quad f_{\rm PS}(\nu) = \sqrt{\frac{2}{\pi}}\,\nu\, e^{-\nu^2/2}$$

$f_{\rm PS}(\nu)$ is called the multiplicity function and gives the mass fraction associated with haloes in a unit range of $\ln\nu$. Note that time enters only through $\delta_c(t) \simeq 1.686/D(t)$

If we define a characteristic mass, M^*, by $\sigma(M^*) = \delta_c(t)$ (i.e., by $\nu(M^*) = 1$) then:

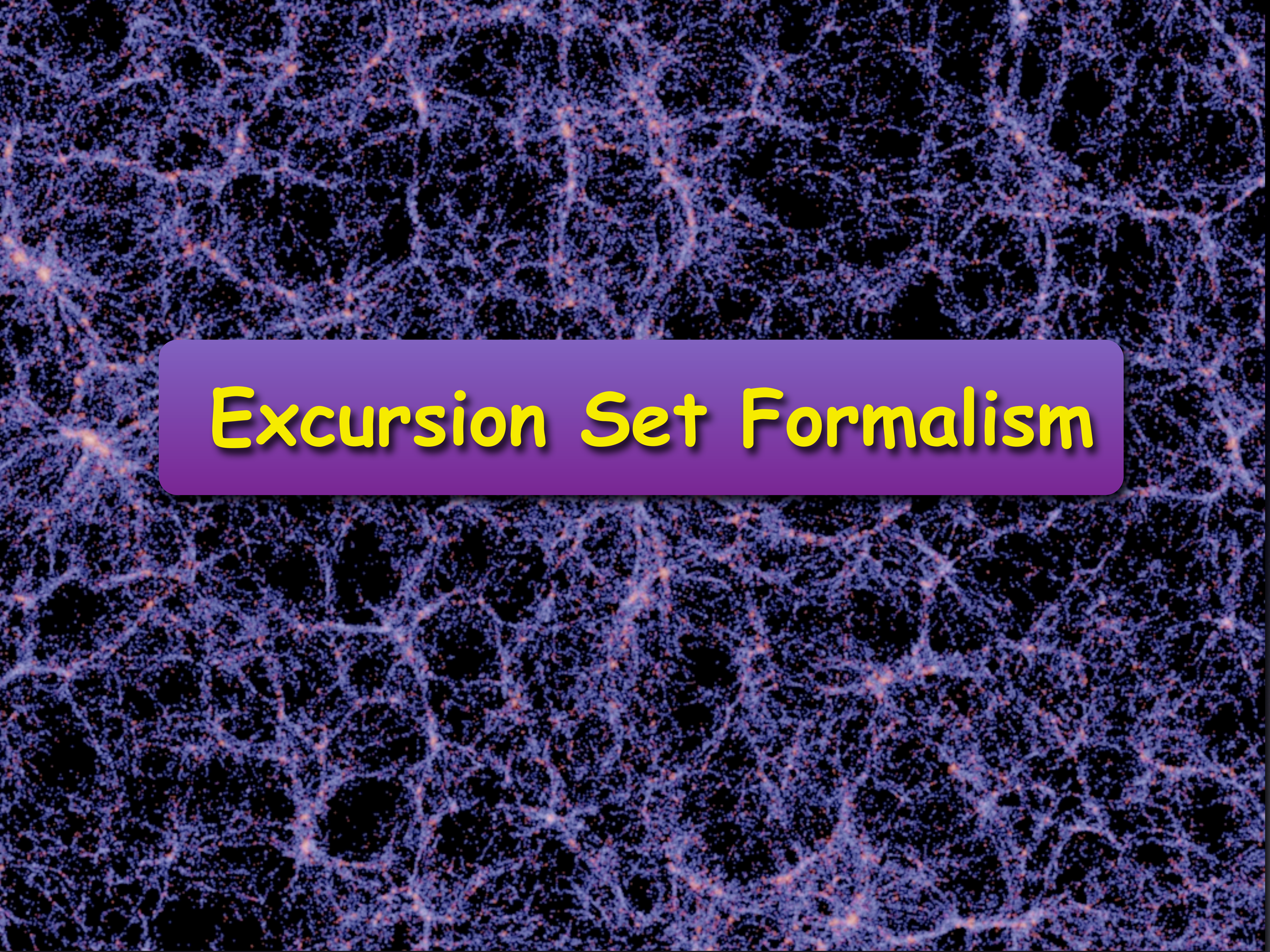
- For $M \ll M^*$ we have that $n(M,t) \propto M^{-(2+\alpha)}$, where $\alpha = \mathrm{d}\ln\sigma/\mathrm{d}\ln M$

  For a CDM cosmology $\alpha \sim 0$ at low mass end so that $n(M) \propto M^{-2}$

- For $M \gg M^*$ the abundance of haloes is exponentially suppressed.

- Since $\delta_c(t)$ decreases with time, the characteristic halo mass grows as function of time; as time passes more and more massive haloes will start to form...

Excursion Set Formalism



The Excursion Set Formalism

Bond et al. (1991) came up with an alternative derivation of the halo mass function that does not suffer from a `fudge-factor problem'


Dick Bond

EXCURSION SET MASS FUNCTIONS FOR HIERARCHICAL GAUSSIAN FLUCTUATIONS

J. R. Bond,[1] S. Cole,[2] G. Efstathiou,[3] and N. Kaiser[1]

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ABSTRACT

Most schemes for determining the mass function of virialized objects from the statistics of the initial density perturbation field suffer from the "cloud-in-cloud" problem of miscounting the number of low-mass clumps, many of which would have been subsumed into larger objects. We propose a solution based on the theory of the excursion sets of $F(\mathbf{r}, R_f)$, the four-dimensional initial density perturbation field smoothed with a continuous hierarchy of filters of radii R_f. We identify the mass fraction of matter in virialized objects with mass greater than M with the fraction of space in which the initial density contrast lies above a critical overdensity when smoothed on some filter of radius greater than or equal to $R_f(M)$. The differential mass function is then given by the rate of first upcrossings of the critical overdensity level as one decreases R_f at constant position $\mathbf{r}$. The shape of the mass function depends on the choice of filter function. The simplest case is "sharp k-space" filtering, in which the field performs a Brownian random walk as the resolution changes. The first upcrossing rate can be calculated analytically and results in a mass function identical to the formula of Press and Schechter—complete with their normalizing "fudge factor" of 2. For general filters (e.g., Gaussian or "top hat") no analogous analytical result seems possible, though we derive useful analytical upper and lower bounds. For these cases, the mass function can be calculated by generating an ensemble of field trajectories numerically. We compare the results of these calculations with group catalogs found from N-body simulations. Compared to the sharp k-space result, less spatially extended filter functions give fewer large-mass and more small-mass objects. Over the limited mass range probed by the N-body simulations, these differences in the predicted abundances are less than a factor of 2 and span the values found in the simulations. Thus the mass functions for sharp k-space and more general filtering all fit the N-body results reasonably well. None of the filter functions is particularly successful in identifying the particles which form low-mass groups in the N-body simulations, illustrating the limitations of the excursion set approach. We have extended these calculations to compute the evolution of the mass function in regions that are constrained to lie within clusters or underdensities at the present epoch. These predictions agree well with N-body results, although the sharp k-space result is slightly preferred over the Gaussian or top hat results.

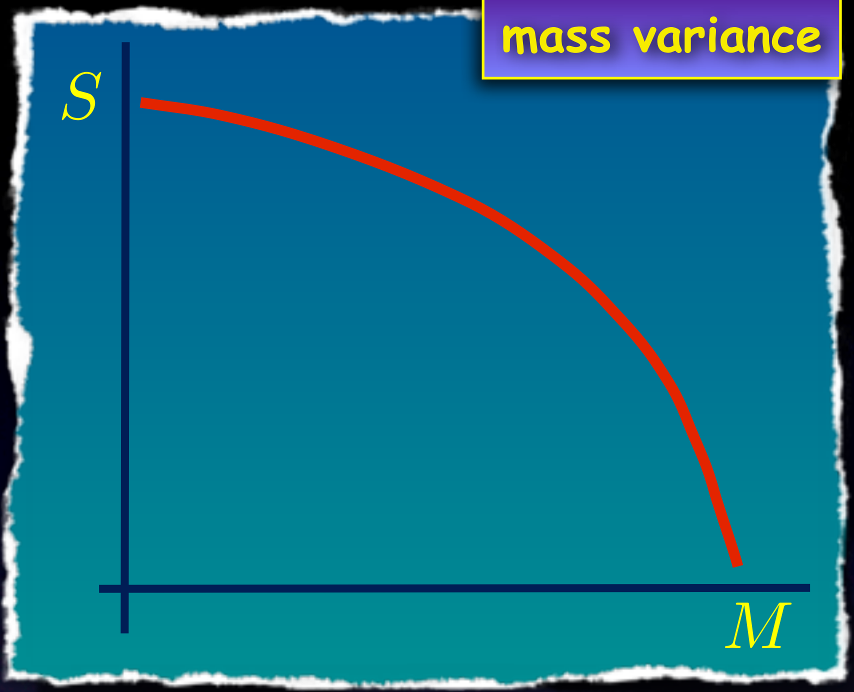
Subject headings: cosmology — galaxies: clustering — numerical methods

The Excursion Set Formalism

In what follows we adopt $S \equiv \sigma^2(M)$ as our mass variable. For a hierarchical cosmogony such as CDM, S is a monotonically declining function of halo mass, so that there is a clear, one-to-one relation between S and M.

Consider a point $\vec{x}$, for which the overdensity, linearly extrapolated to the present day is $\delta_0(\vec{x})$. For each value of the filtering mass M , i.e. for each value of S, the smoothed overdensity $\delta_S = \delta_M(\vec{x})$ will have a different value.

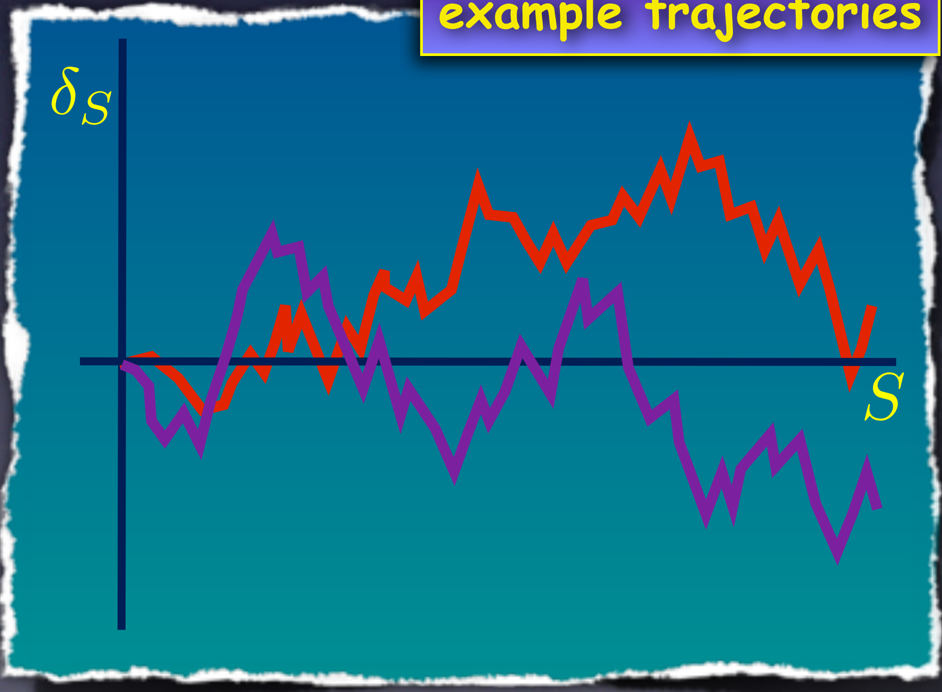
mass variance



With each point $\vec{x}$ corresponds a trajectory δ_S

- For $S \rightarrow 0$ we have that $M \rightarrow \infty$, and thus $\delta_S \rightarrow 0$. Hence, each trajectory starts at $(S, \delta_S) = (0, 0)$
- If the filter is a sharp k-space filter, changing S adds new (and independent) modes. As a consequence, the trajectory is Markovian....

example trajectories



Markovian Random Walks

A **random walk** is a mathematical formalization of a path that consists of a succession of random steps. If the next step depends only on the current state (i.e., has no `memory' of its prior path), the random walk is called Markovian.

For a sharp k-space filter the smoothed density field is given by

$$\delta_S(\vec{x}) = \int \mathrm{d}^3\vec{k}\, \widetilde{W}_{\rm sk}(\vec{k}R)\, \delta_{\vec{k},0}\, e^{i\vec{k}\cdot\vec{x}} = \int_{k<k_c} \mathrm{d}^3\vec{k}\, \delta_{\vec{k},0}\, e^{i\vec{k}\cdot\vec{x}}$$

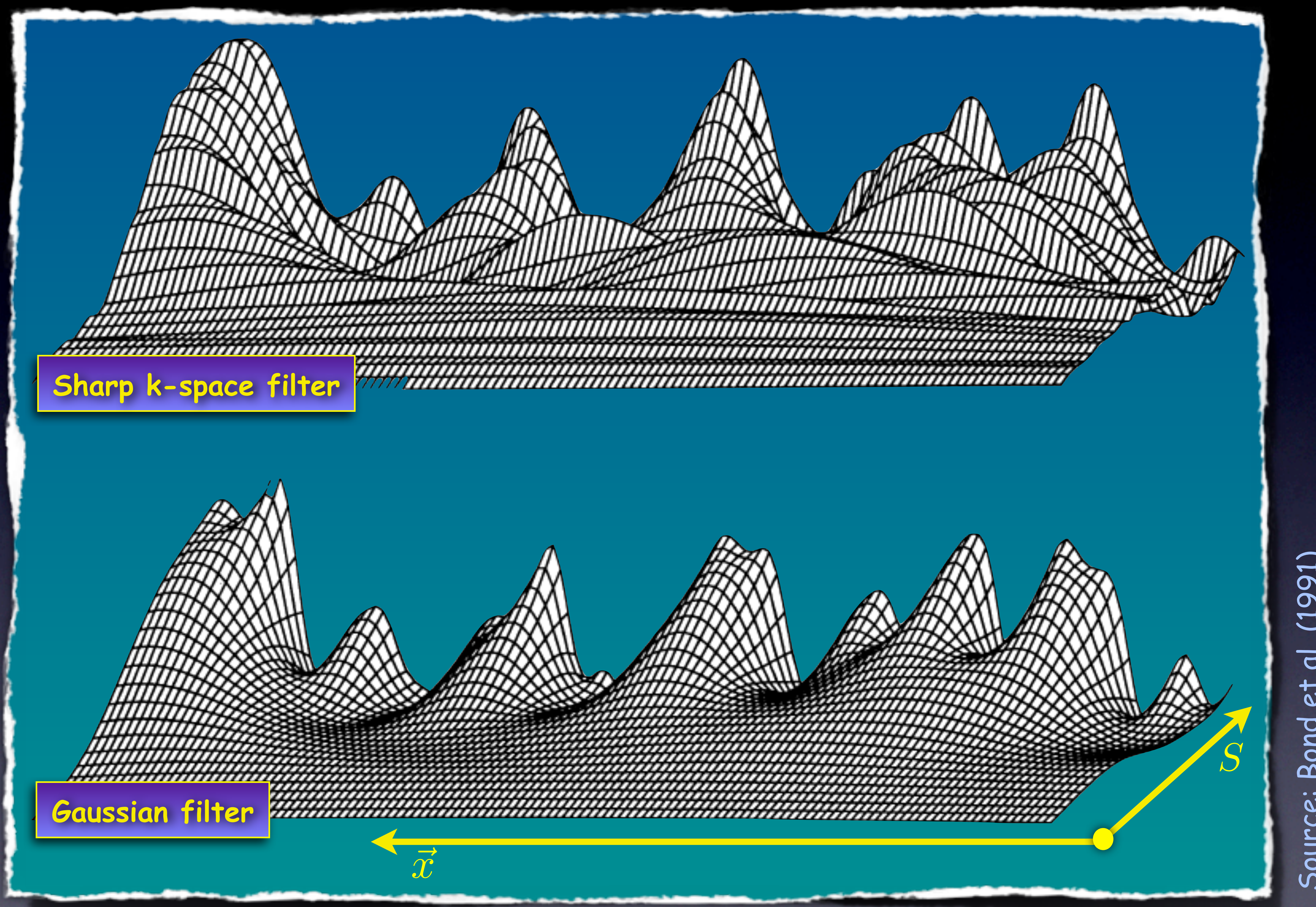
Here $k_c = 1/R$ is the size of the top-hat in k-space, and $\delta_{\vec{k},0}$ are Fourier modes of $\delta_0(\vec{x})$

When increasing S (decreasing R), you add new and independent modes (at least for a Gaussian random field). Since these new and independent modes have random phases, the step $\Delta(\delta_S)$ associated with the change ΔS is Markovian.

NOTE: for any other filter, the trajectories $\delta_S(S)$ will not be Markovian!!

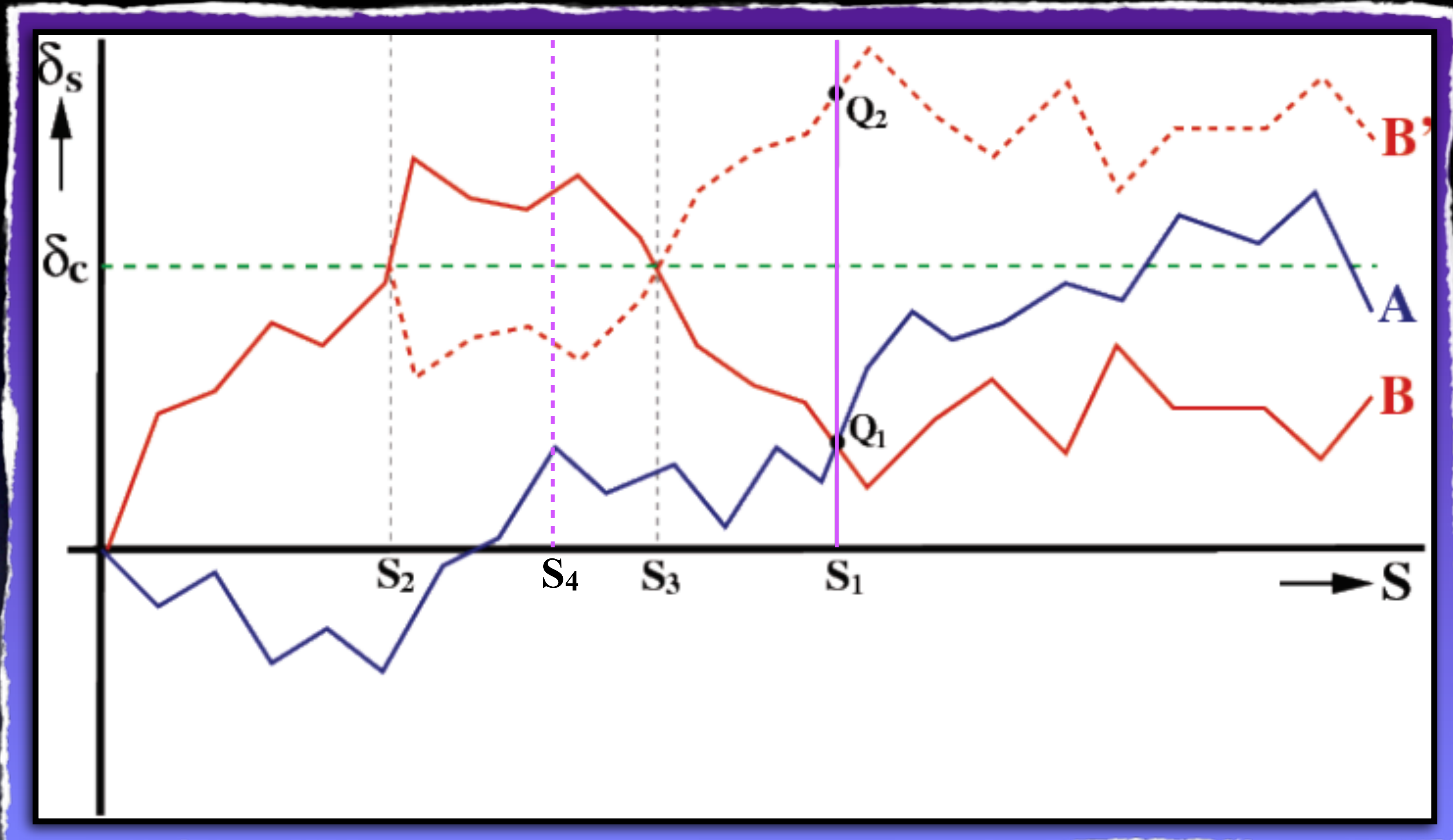
In what follows we will always assume a sharp k-space filter (unless stated otherwise), so that our trajectories can be considered Markovian.

The Excursion Set Formalism



NOTE: for any filter other than sharp k-space filter, the random walks are NOT Markovian

The Excursion Set Formalism



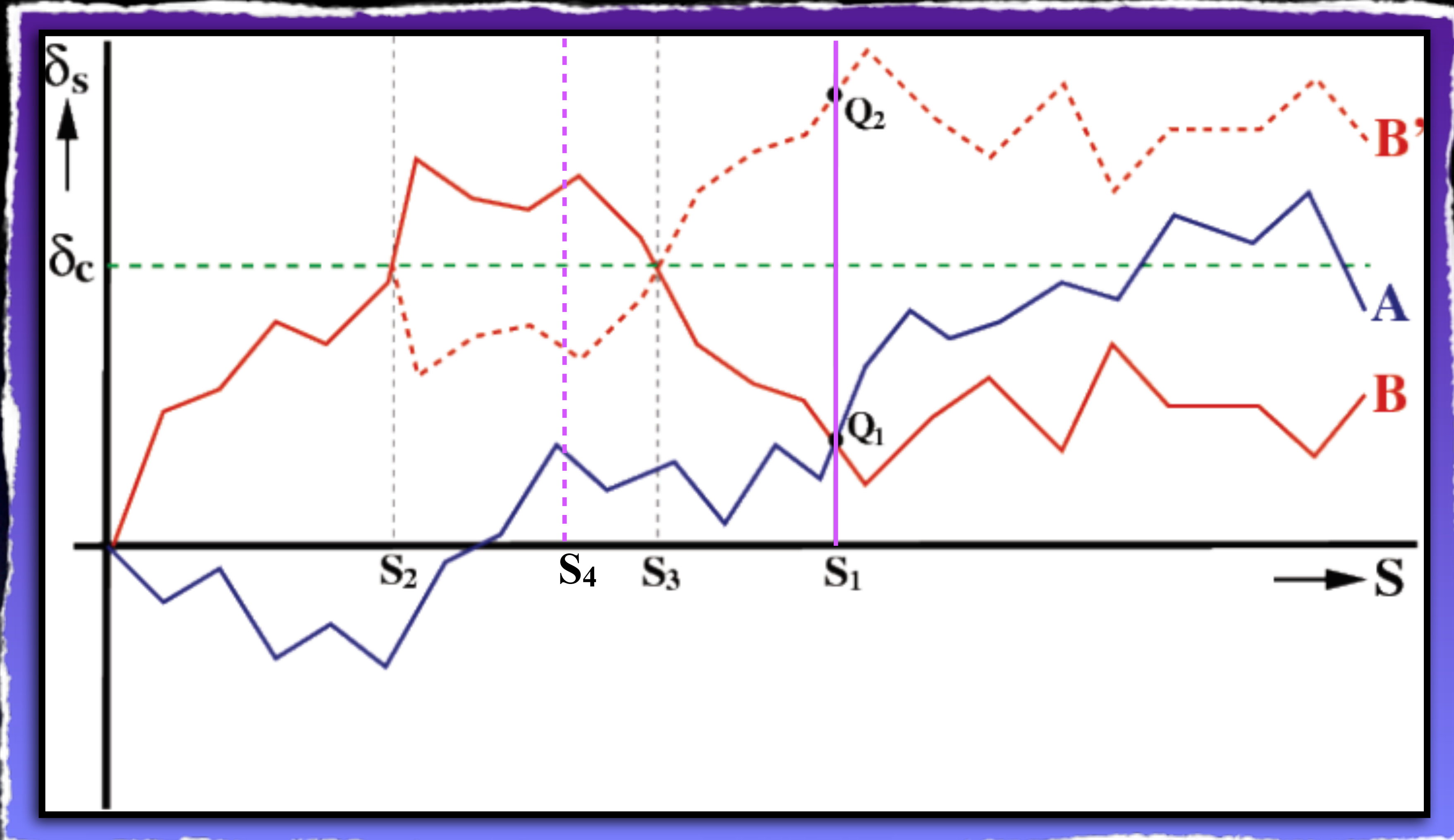
Three trajectories corresponding to three different mass elements in a Gaussian random field. Note that B' is obtained mirroring trajectory B in the line $\delta_S = \delta_c$ for $S \geq S_2$. Since the trajectories are Markovian B and B' are equally likely!

Consider $\delta_0(\vec{x})$ smoothed on a mass scale M_1 corresponding to $S_1 = \sigma^2(M_1)$

According to PS ansatz, mass elements whose trajectory $\delta_S > \delta_c$ at S_1 reside in dark matter haloes with mass $M > M_1$ ⇒ neither A or B are in halo with $M > M_1$

BUT: according to same PS ansatz, mass element associated with trajectory B resides in a halo with $M > M_4 > M_1$: PS ansatz is not self-consistent!!!

The Excursion Set Formalism



Three trajectories corresponding to three different mass elements in a Gaussian random field. Note that B′ is obtained mirroring trajectory B in the line $\delta_S = \delta_c$ for $S \geq S_2$. Since the trajectories are Markovian B and B′ are equally likely!

The problem with the PS ansatz is that it fails to account for trajectories such as B when counting mass elements in haloes with mass $M > M_1$.

Correcting for this is easy though, by realizing that each trajectory B has a mirror version, B′, that is equally likely (as a result of the Markovian nature of the trajectories).

Double-counting trajectories with $\delta_S > \delta_c$ at S_1 corrects for `missed trajectories'.....

A natural explanation for the fudge-factor two in PS formalism!

The Excursion Set Formalism

In the excursion set formalism , also called the Extended Press-Schechter (EPS) formalism, one uses the (statistics of) Markovian random walks (the trajectories of mass elements in (S, δ_S)-space) to infer the halo mass function (and more).

PS ansatz: fraction of mass elements with $\delta_S > \delta_c(t)$ is equal to the mass fraction that at time t resides in haloes with masses $> M$, where S and M are related according to $S = \sigma^2(M)$

EPS ansatz: fraction of trajectories with a first upcrossing (FU) of the barrier $\delta_S = \delta_c(t)$ at $S > S_1 = \sigma^2(M_1)$ is equal to the mass fraction that at time t resides in haloes with masses $M < M_1$

Since, each trajectory is guaranteed to upcross the barrier $\delta_S = \delta_c(t)$ at some (arbitrarily large) S, the EPS ansatz predicts that every mass element is in a halo of some (arbitrarily low) mass

$$F(< M_1) = 1 - F(> M_1)$$

The EPS Mass Function

Based on the EPS ansatz, we can write the EPS mass function as:

$$n(M,t)\,\mathrm{d}M = \frac{\bar{\rho}}{M}\,\frac{\partial F(>M)}{\partial M}\,\mathrm{d}M = -\frac{\bar{\rho}}{M}\,\frac{\partial F(<M)}{\partial M}\,\mathrm{d}M$$
$$= -\frac{\bar{\rho}}{M}\,\frac{\partial F_{\rm FU}(>S)}{\partial S}\,\frac{\mathrm{d}S}{\mathrm{d}M}\,\mathrm{d}M = \frac{\bar{\rho}}{M}\,f_{\rm FU}(S,\delta_{\rm c})\left|\frac{\mathrm{d}S}{\mathrm{d}M}\right|\mathrm{d}M$$

Here $f_{\rm FU}(S,\delta_{\rm c})\,\mathrm{d}S$ is the fraction of trajectories that have their first upcrossing of barrier $\delta_{\rm c}(t)$ between S and $S+\mathrm{d}S$.

Without proof:

$$f_{\rm FU}(\nu) = \frac{1}{\sqrt{2\pi}}\,\frac{\delta_{\rm c}}{S^{3/2}}\,\exp\left[-\frac{\delta_{\rm c}^2}{2S}\right] = \frac{1}{2\,S}\,f_{\rm PS}(\nu)$$

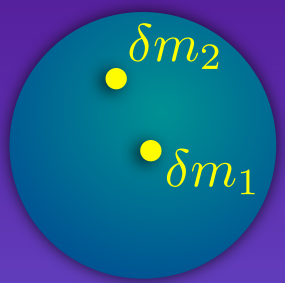
(see MBW §7.2.2 for derivation)

where, as before, we defined $\nu = \delta_{\rm c}(t)/\sigma(M) = \delta_{\rm c}/\sqrt{S}$ and we expressed the result in terms of the PS multiplicity function $f_{\rm PS}(\nu) = \sqrt{2/\pi}\,\nu\,\exp(-\nu^2/2)$

It is straightforward to show that this yields exactly the same halo mass function as before, but this time there has been no need for a fudge factor....

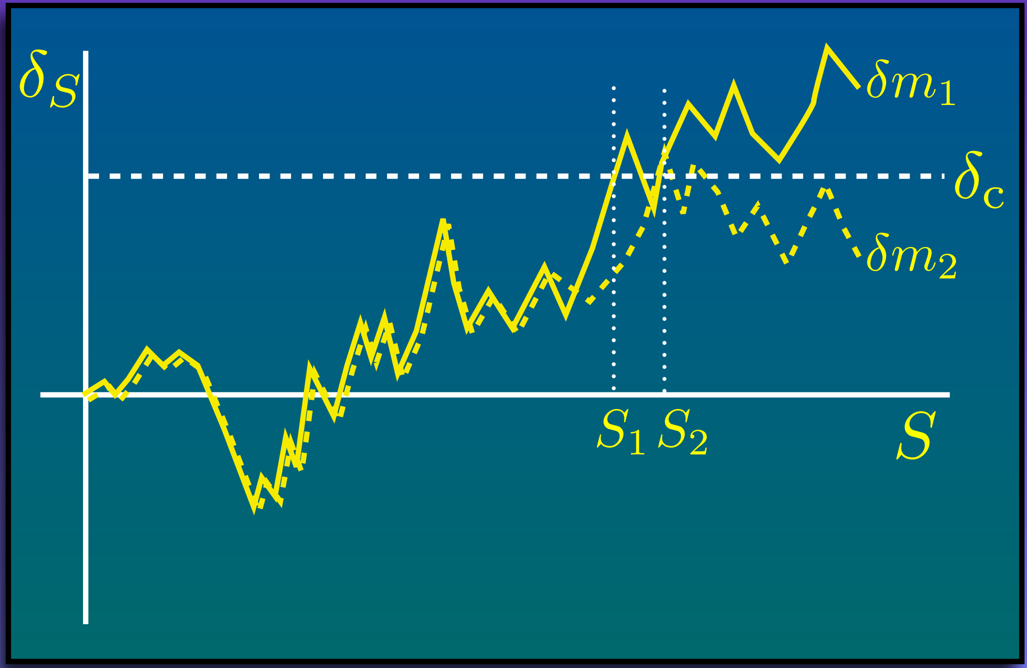
The EPS Mass Function

Although the EPS mass function is used very frequently in modern astronomy, it is important to be aware of its assumptions, shortcomings and pitfalls:



Consider two mass elements (yellow `dots') in the same dark matter halo: one near the center, the other near the outskirts.

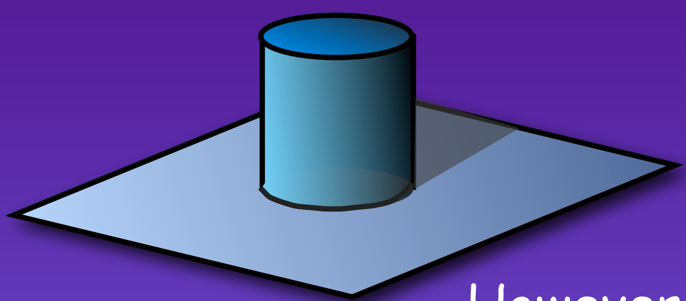
Since both particles have very similar large-scale environments (on scales larger than halo itself), their trajectories are very similar for small S:



Although both particles reside in same halo, their trajectories have first upcrossings at different S: according to EPS formalism, δm_2 resides in a less massive halo than δm_1: excursion set formalism only predicts how much mass ends up in haloes of different mass in a statistical sense....

The EPS Mass Function

Although the EPS mass function is used very frequently in modern astronomy, it is important to be aware of its assumptions, shortcomings and pitfalls:

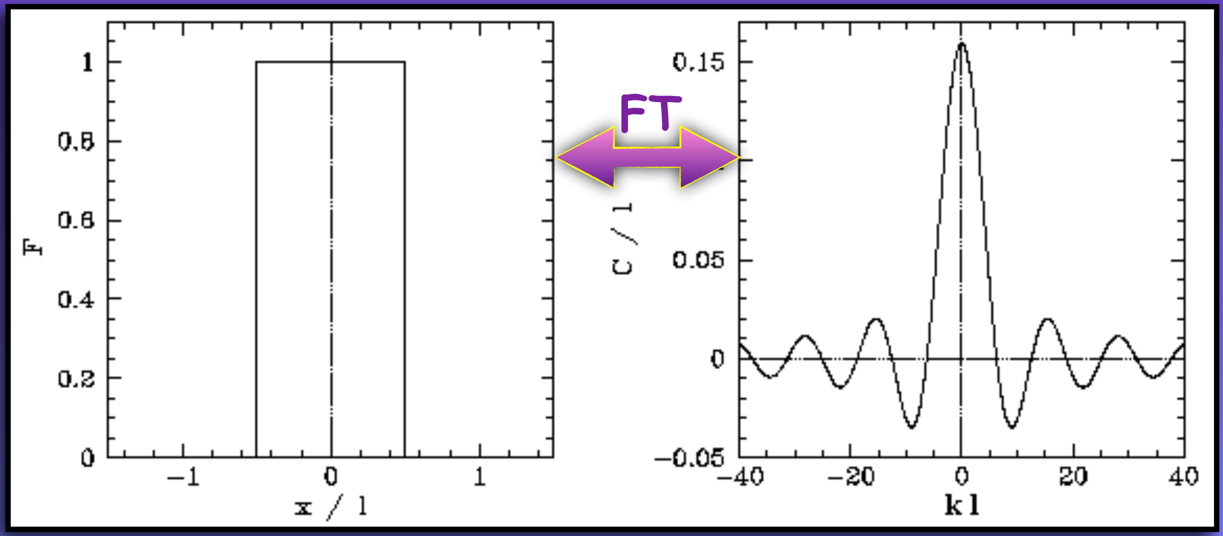


Trajectories have to be constructed with sharp k-space filter in order to guarantee Markovian nature of the random walks.

However, the corresponding real-space filter has complicated (sinc-like) form; difficult to interpret....

In particular, the real-space filter is not spatially localized; it has oscillating wings that extent out to large distances...

Yet, according to EPS formalism, this structure corresponds to a collapsed dark matter halo, which *is* spatially localized...



The EPS Mass Function

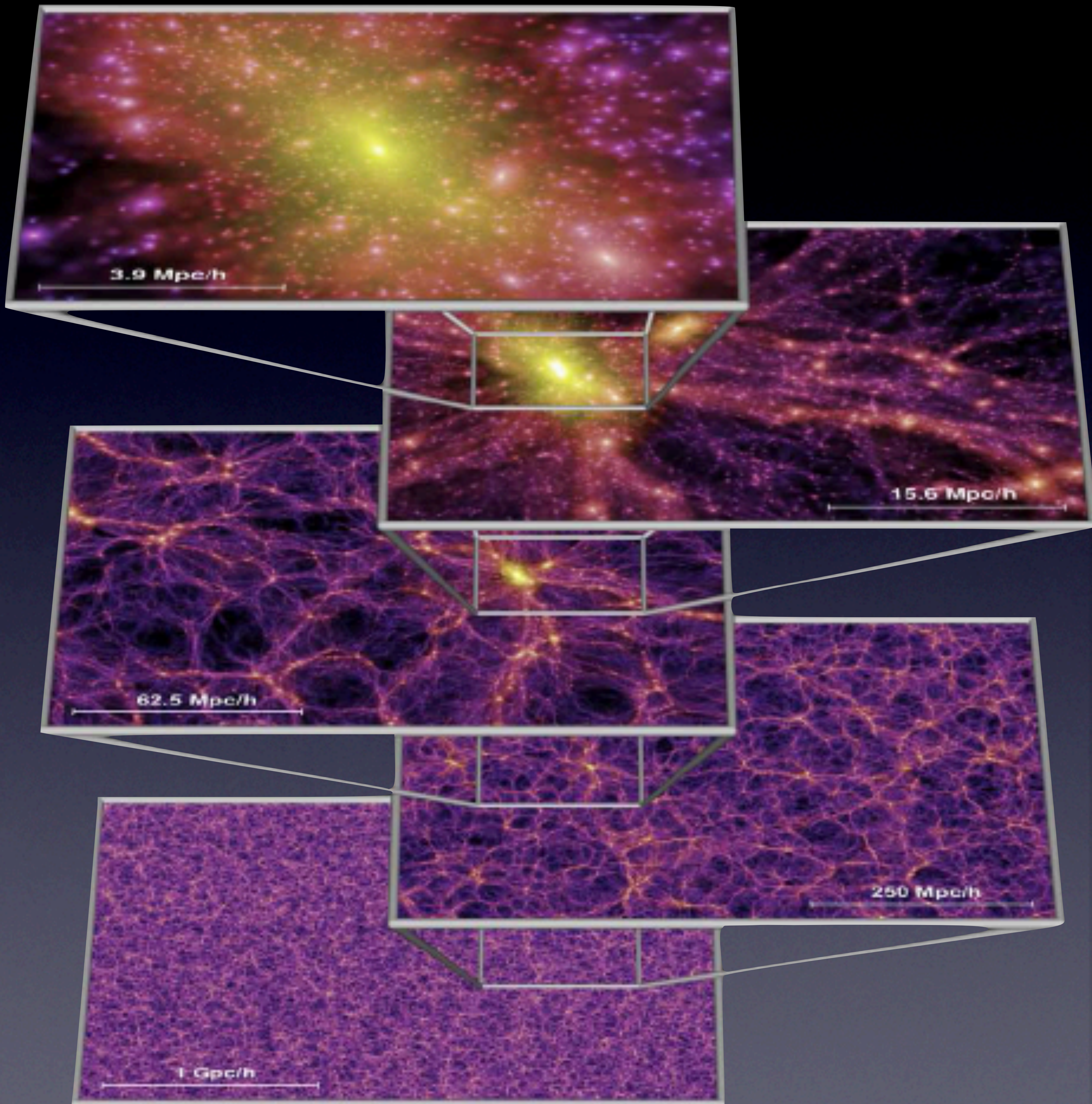
Although the EPS mass function is used very frequently in modern astronomy, it is important to be aware of its assumptions, shortcomings and pitfalls:

The Spherical Cow: The upcrossing barrier used is based on the spherical collapse model; as we have seen collapse is believed to be ellipsoidal instead...

As we will see, though, this can be taken into account...

Finally, the mere idea that one can use the linear density field to identify collapsed structures in the non-linear field constitutes a leap of faith...

Comparison with Numerical Simulation



Given the various crude assumptions underlying the PS & EPS formalisms, it is important to test their predictions for halo mass function against numerical simulations...

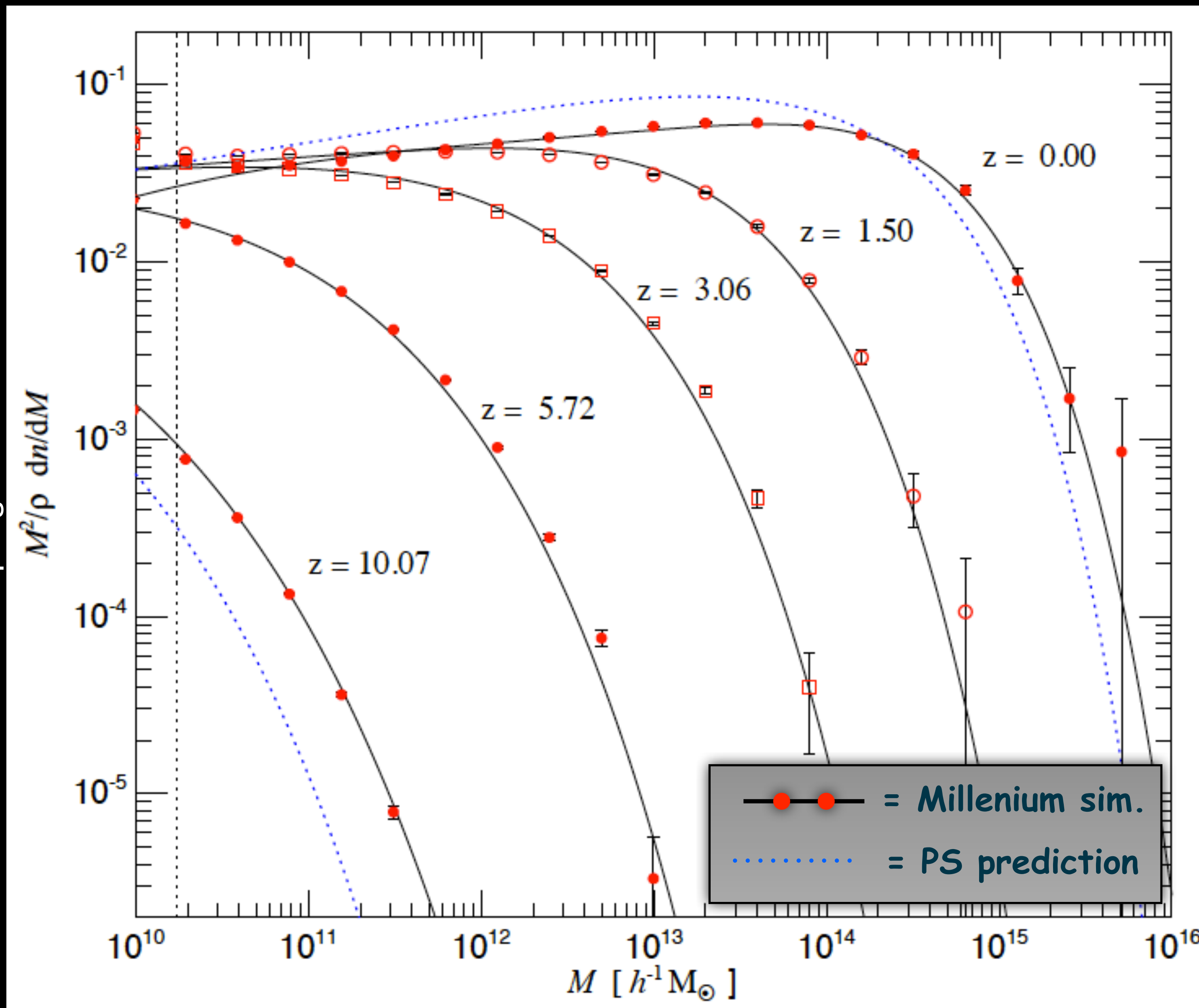
These follow the growth & collapse of structures directly by solving the equations of motion for dark matter particles. However, as will be discussed later, identifying haloes in simulations is a non-trivial task.....

Until end of 1990s, most simulations yielded results in fair agreement with PS predictions....

However, when larger and more accurate simulations became available, it became clear that there where some problems....

Comparison with Numerical Simulation

Source: Millenium Simulation; Springel V., et al. 2005, Nature, 435, 629



The Millenium Simulation followed the evolution of 2160^3 (~10 billion) particles in a periodic box 500 Mpc/h on a side in a ΛCDM cosmology.

At the time it was run (2005) it was one of the biggest simulations to date. Because of its superb statistics, it is ideally suited to test the PS mass functions...

At low redshift, the PS mass function under- (over)-predicts the abundance of massive (low mass) haloes. These problems become more pronounced at higher redshifts...

WARNING: this statement is sensitive to how haloes are identified in the simulation box. Here a Friends-Of-Friends (FOF) algorithm has been used....

Extended Press-Schechter with Ellipsoidal Collapse

Ellipsoidal collapse and an improved model for the number and spatial distribution of dark matter haloes

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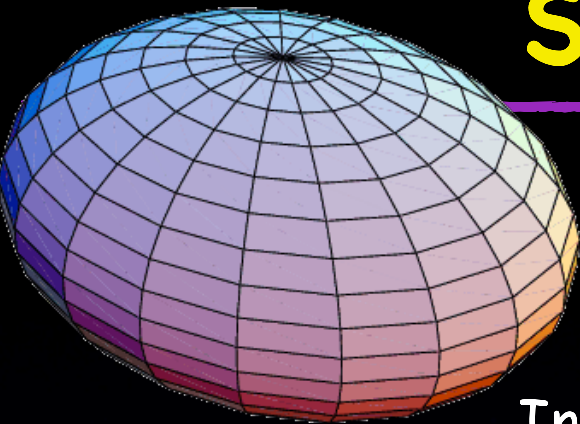
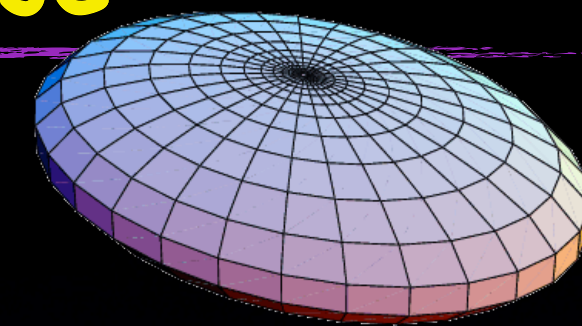
ABSTRACT

The Press–Schechter, excursion set approach allows one to make predictions about the shape and evolution of the mass function of bound objects. The approach combines the assumption that objects collapse spherically with the assumption that the initial density fluctuations were Gaussian and small. The predicted mass function is reasonably accurate, although it has fewer high-mass and more low-mass objects than are seen in simulations of hierarchical clustering. We show that the discrepancy between theory and simulation can be reduced substantially if bound structures are assumed to form from an ellipsoidal, rather than a spherical, collapse. In the original, standard, spherical model, a region collapses if the initial density within it exceeds a threshold value, δ_{sc}. This value is independent of the initial size of the region, and since the mass of the collapsed object is related to its initial size, this means that δ_{sc} is independent of final mass. In the ellipsoidal model, the collapse of a region depends on the surrounding shear field, as well as on its initial overdensity. In Gaussian random fields, the distribution of these quantities depends on the size of the region considered. Since the mass of a region is related to its initial size, there is a relation between the density threshold value required for collapse and the mass of the final object. We provide a fitting function to this $\delta_{ec}(m)$ relation which simplifies the inclusion of ellipsoidal dynamics in the excursion set approach. We discuss the relation between the excursion set predictions and the halo distribution in high-resolution *N*-body simulations, and use our new formulation of the approach to show that our simple parametrization of the ellipsoidal collapse model represents an improvement on the spherical model on an object-by-object basis. Finally, we show that the associated statistical predictions, the mass function and the large-scale halo-to-mass bias relation, are also more accurate than the standard predictions.

Key words: galaxies: clusters: general – cosmology: theory – dark matter.



Spherical vs. Ellipsoidal Collapse



As we have seen, because of the non-zero tidal field, collapse will not be spherical, but ellipsoidal.

In that case, the critical (linear) over density for collapse is given by

$$\frac{\delta_{\rm ec}}{\delta_{\rm sc}} \approx 1 + 0.47\left[5(e^2 \pm p^2)\,\frac{\delta_{\rm ec}^2}{\delta_{\rm sc}^2}\right]^{0.615}$$

Ellipsoidal collapse

Here $\delta_{\rm ec} = \delta_{\rm ec}(e,p)$ is the critical overdensity for ellipsoidal collapse, $\delta_{\rm sc} = \delta_{\rm c} \simeq 1.686$ is the critical overdensity for spherical collapse, and the parameters e and p characterize the asymmetry of the initial tidal field. (see lecture 1)

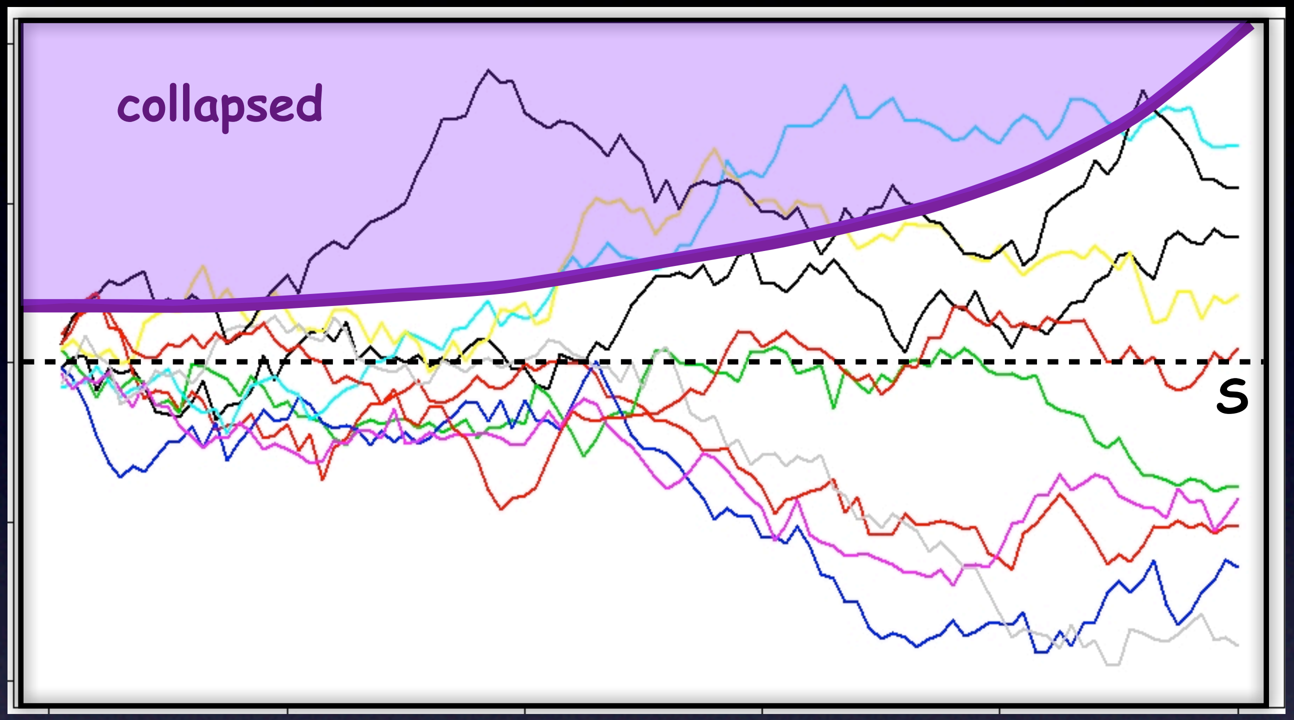
Adopting the most probable values for e and p, Sheth, Mo & Tormen (2001; SMT) showed that the upcrossing boundary for ellipsoidal collapse can be written as:

$$\delta_{\rm ec} \simeq \delta_{\rm ec}(S,t) = \delta_{\rm c}(t)\left[1 + 0.47\left(\frac{S}{\delta_{\rm c}^2(t)}\right)^{0.615}\right]$$

Contrary for spherical collapse, for which the boundary is constant, the boundary for ellipsoidal collapse increases with S (less massive structures need higher overdensity for collapse). Because of this S-dependence, $\delta_{\rm ec}$, is called a "moving barrier".

The EPS Mass Function for Ellipsoidal Collapse

Knowing the critical overdensity for ellipsoidal collapse, we can compute the corresponding PS mass function: all we need to do is to work out the first-upcrossing statistics....



Unfortunately, for a moving barrier one cannot compute this analytically. Rather, one has to resort to Monte Carlo simulations of independent random walks, and register their first upcrossings.

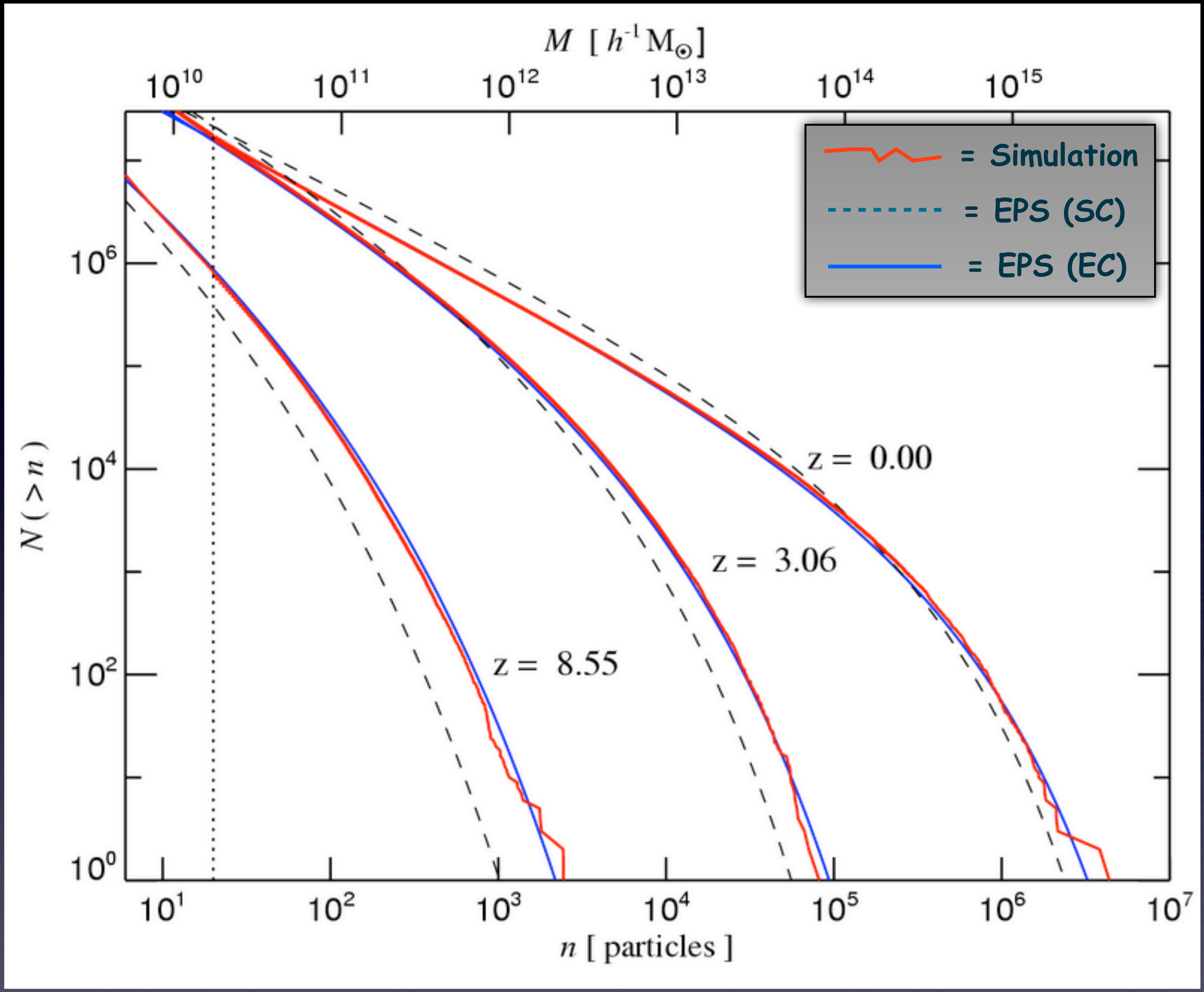
This was done by SMT, who found that the resulting multiplicity function if well approximated by

$$f_{\rm EC}(\nu) = 0.322\left[1 + \frac{1}{\tilde{\nu}^{0.6}}\right] f_{\rm PS}(\tilde{\nu}) \qquad \text{where} \qquad \tilde{\nu} = 0.84\,\nu$$

The normalization 0.322 is set by requiring that $\int_0^\infty n(M)\,M\,{\rm d}M = \bar{\rho}_{\rm m}$, which implies that all matter is in collapsed objects. The PS mass function for ellipsoidal collapse simply follows from replacing $f_{\rm PS}(\nu)$ with $f_{\rm EC}(\nu)$; i.e.,

$$n(M,t)\,{\rm d}M = \frac{\bar{\rho}}{M^2}\,f_{\rm EC}(\nu)\left|\frac{{\rm d}\ln\nu}{{\rm d}\ln M}\right|{\rm d}M$$

Spherical vs. Ellipsoidal Collapse



Source: Millenium Simulation; Springel V., 2004 (MPA research highlight)

The Millenium Simulation followed the evolution of 2160^3 (~10 billion) particles in a periodic box 500 Mpc/h on a side in a ΛCDM cosmology.

Clearly, the EPS mass function based on ellipsoidal collapse is in much better agreement with numerical simulations than the spherical collapse-based model prediction...

WARNING: this statement is sensitive to how haloes are identified in the simulation box. Here a Friends-Of-Friends (FOF) algorithm has been used.....

Halo Merger Trees

Recap: The Halo Mass Function

In the excursion set formulation of PS theory, also called extended Press-Schechter, the halo mass function derives from first-upcrossing statistics of linear density field:

$$n(M,t)\,\mathrm{M} = \frac{\bar{\rho}}{M}\, f_{\rm FU}(S,\delta_{\rm c}) \left|\frac{\mathrm{d}S}{\mathrm{d}M}\right| \mathrm{d}M$$

Here $f_{\rm FU}(S,\delta_{\rm c})\,\mathrm{d}S$ is the fraction of trajectories that have their first upcrossing of barrier $\delta_{\rm c}(t)$ between S and $S+\mathrm{d}S$.

In the case of spherical collapse, the barrier $\delta_{\rm c}(t) \simeq 1.686/D(t)$ is independent of mass, and the upcrossing statistics are analytical:

$$f_{\rm FU}(\nu) = \frac{1}{\sqrt{2\pi}}\,\frac{\delta_{\rm c}}{S^{3/2}}\,\exp\left[-\frac{\delta_{\rm c}^2}{2S}\right] = \frac{1}{2\,S}\, f_{\rm PS}(\nu)$$

where $\nu = \delta_{\rm c}(t)/\sigma(M) = \delta_{\rm c}/\sqrt{S}$ and $f_{\rm PS}(\nu) = \sqrt{\frac{2}{\pi}}\,\nu\, e^{-\nu^2/2}$ is the multiplicity function

In the case of ellipsoidal collapse, Monte Carlo simulations of first-upcrossings with a moving barrier are well fit by

$$f_{\rm FU}(\nu) = \frac{1}{2S}\, f_{\rm EC}(\nu) \quad \text{where} \quad f_{\rm EC}(\nu) = 0.322\left[1 + \frac{1}{\tilde{\nu}^{0.6}}\right] f_{\rm PS}(\tilde{\nu}) \quad \text{with} \quad \tilde{\nu} = 0.84\,\nu$$

Beyond a Halo Mass Function...

An important advantage of EPS over PS is that the excursion set formalism provides a neat way to calculate the properties of the progenitors which give rise to a given class of objects (i.e., haloes of a given mass).

For example, one can calculate the mass function at $z=5$ of those haloes (progenitors) which by $z=0$ end up in a massive halo of 10^{15} solar masses.

These progenitor mass functions, in turn, can be used to describe how dark matter haloes assemble over time (in a statistical sense); in particular, they allow the construction of halo merger trees.

These merger trees are invaluable tools in galaxy formation studies...

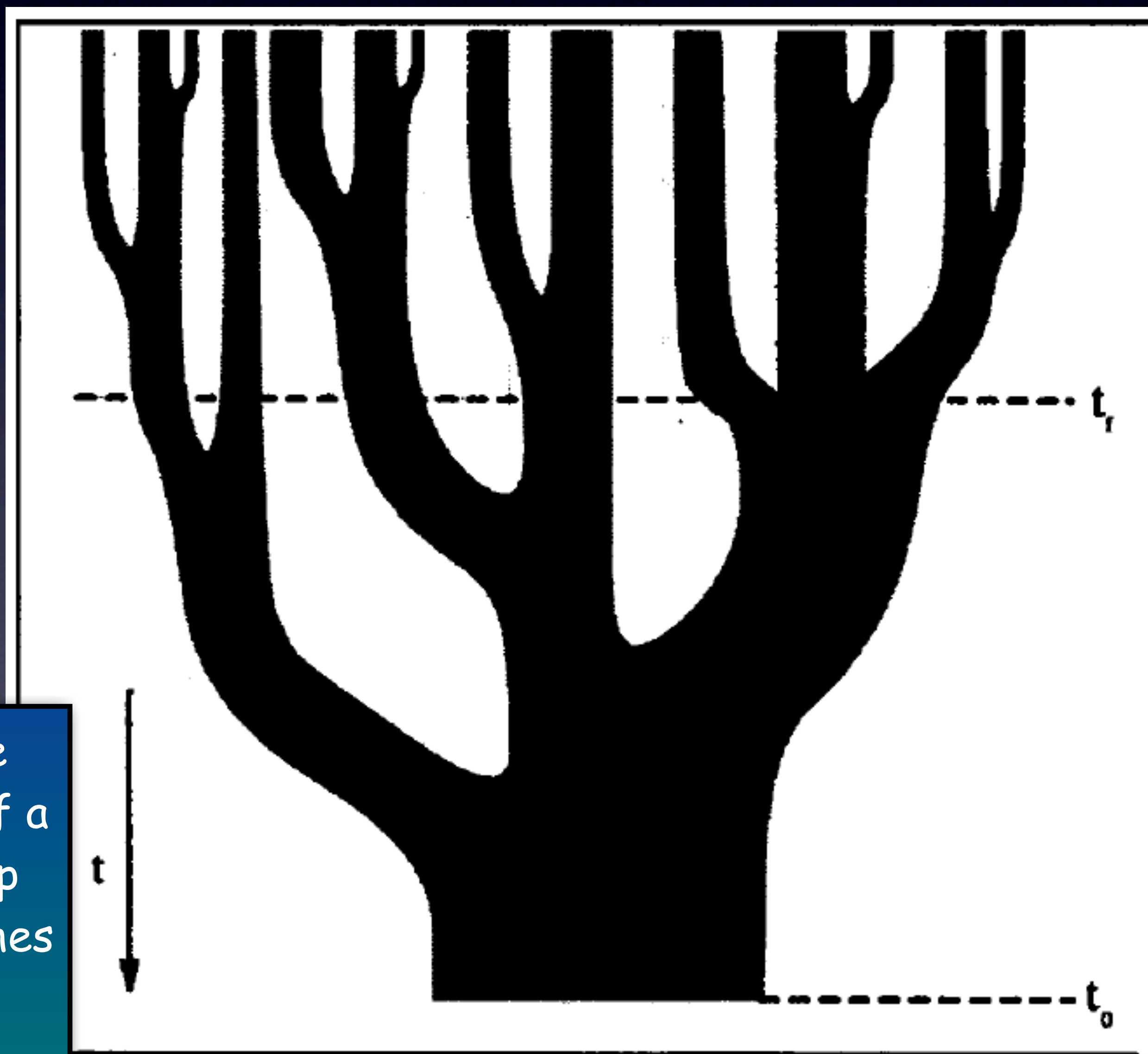


Illustration of a merger tree depicting the growth of a dark matter halo as a result of a series of mergers. Time increases from top to bottom and the width of the tree beaches represents the masses of the individual progenitors...

Source: Lacey & Cole, 1993, MNRAS, 262, 627

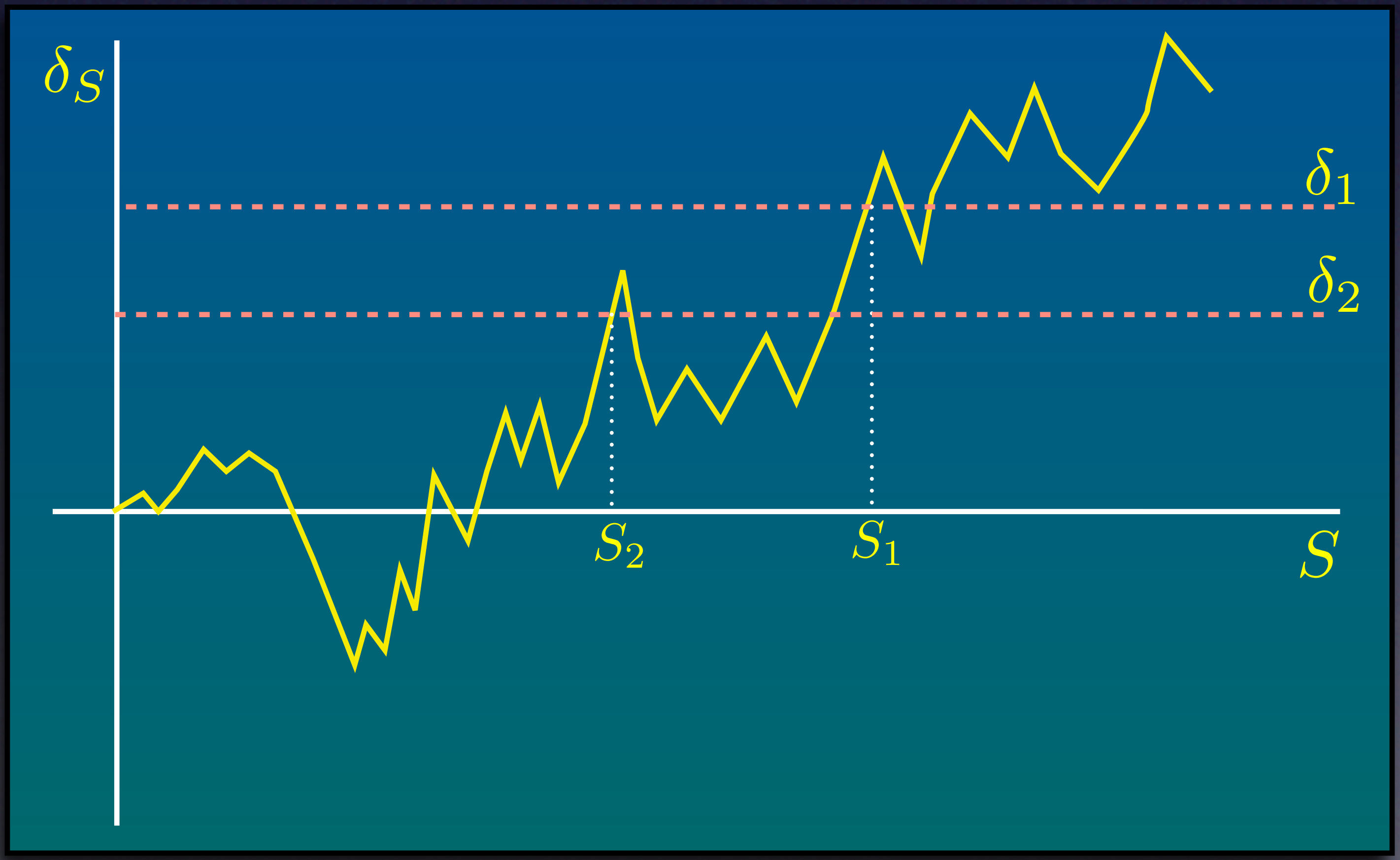
Progenitor Mass Function

Consider a spherical region (a patch) of mass M_2 , corresponding to a mass variance $S_2 = \sigma^2(M_2)$ with linear overdensity $\delta_2 \equiv \delta_c(t_2) = \delta_c/D(t_2)$ so that it forms a collapsed object at time t_2.

We are interested in the fraction of M_2 that at some earlier time $t_1 < t_2$ was in a collapsed object of some mass M_1 .

Within the excursion set formalism this means we want to calculate the probability that a trajectory that upcrosses barrier δ_2 at S_2 has its first upcrossing of barrier $\delta_1 = \delta_c(t_1)$ at $S_1 > S_2$ (see illustration).

This is the same problem as before, except for a translation of the origin in the (S, δ_S)-plane .



Progenitor Mass Function

$$f_{\rm FU}(S, \delta_{\rm c}) = \frac{1}{\sqrt{2\pi}} \frac{\delta_{\rm c}}{S^{3/2}} \exp\left[-\frac{\delta_{\rm c}^2}{2S}\right]$$

translation

$$f_{\rm FU}(S_1, \delta_1 | S_2, \delta_2) = \frac{1}{\sqrt{2\pi}} \frac{\delta_1 - \delta_2}{(S_1 - S_2)^{3/2}} \exp\left[-\frac{(\delta_1 - \delta_2)^2}{2\,(S_1 - S_2)}\right]$$

Converting from mass- to number-weighting

$$n(M_1, t_1 | M_2, t_2)\,{\rm d}M_1 = \frac{M_2}{M_1} f_{\rm FU}(S_1, \delta_1 | S_2, \delta_2) \left|\frac{{\rm d}S_1}{{\rm d}M_1}\right| {\rm d}M_1$$

$n(M_1, t_1 | M_2, t_2)\,{\rm d}M_1$ is the progenitor mass function; it gives the average number of progenitor haloes at time t_1 in the mass range $(M_1, M_1 + {\rm d}M_1)$ that at time $t_2 > t_1$ have merged to form a halo of mass M_2 .

Merger Trees

The progenitor mass function allows one to construct halo merger trees using the following algorithm:

For a given host halo mass, M_0 , and a given time step, Δt , draw a set of progenitor masses from the progenitor mass function $n(M_{\rm p}, t_0 + \Delta t | M_0, t_0)$

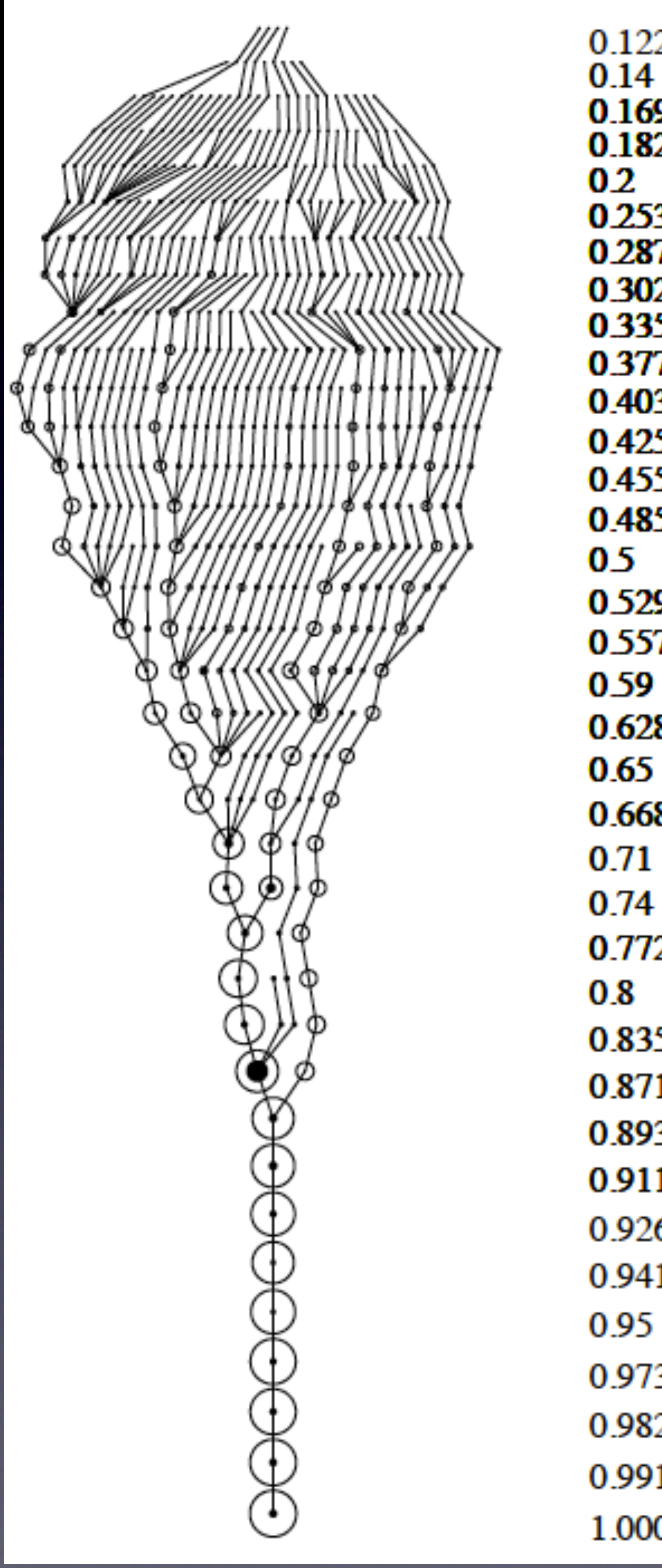
The progenitors must obey the following two conditions:

- accurately sample the progenitor mass function
- mass conservation: $\sum_i M_{{\rm p},i} = M_0$

For each progenitor, repeat above procedure, thus stepping back in time.

Sounds easy.....is not...

Several different methods have been suggested to contruct halo merger trees; none of them is perfect......



Source: Wechsler et al. 2002, ApJ, 568, 52

How to Plant a Healthy Merger Tree?

The problem with how to construct a merger tree can be summarized as follows:

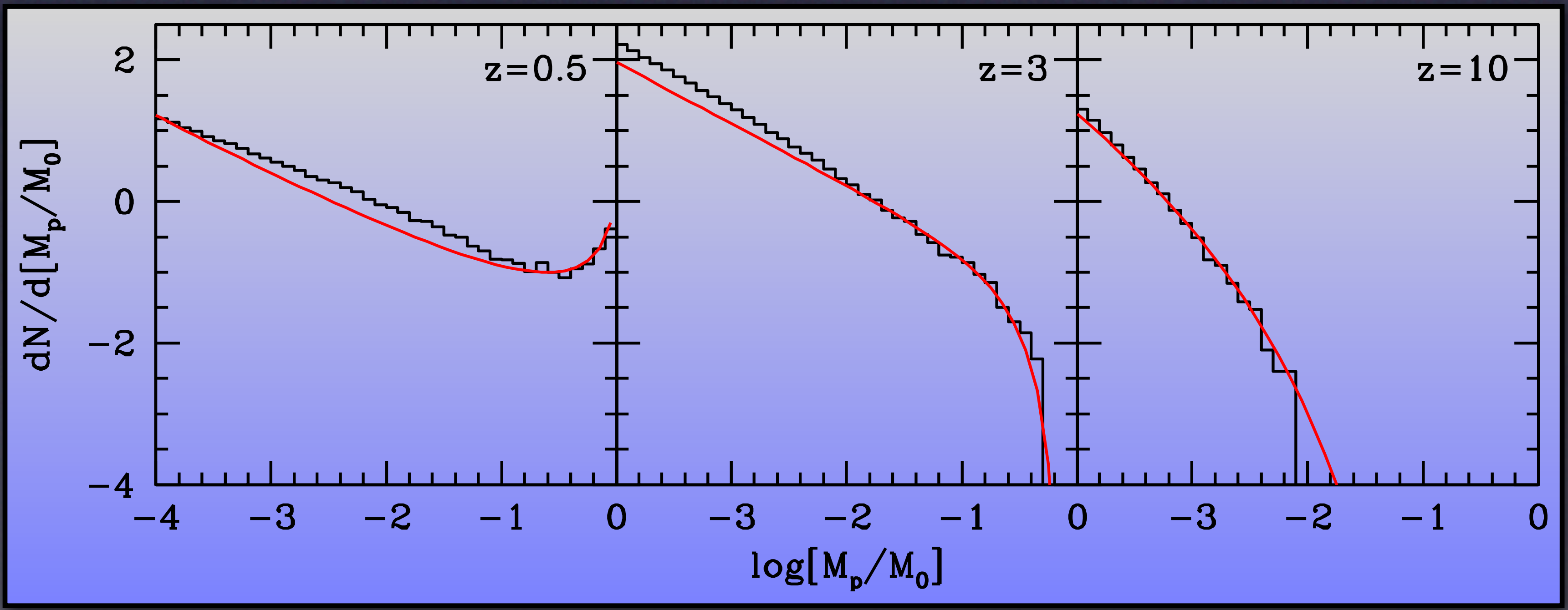
once I have drawn the first progenitor mass, $M_{\rm p,1}$, from the progenitor mass function, $n(M_{\rm p}, t_0 + \Delta t | M_0, t_0)$ mass conservation now implies a constraint on the second progenitor mass: $M_{\rm p,2} \leq M_0 - M_{\rm p,1}$. Unfortunately, there is no analytical method to include this `condition' in the progenitor mass function, i.e., it is not clear how to specify $n(M_{\rm p}, t_0 + \Delta t | M_0, t_0, M_{\rm p,1})$. Different methods for constructing halo merger trees mainly differ in how to deal with this issue...

How to Plant a Healthy Merger Tree?

There are two tests that one can perform to test the accuracy of a merger tree:

1: The Self-Consistency Test

- Construct a larger number of merger trees (using small time steps) for a host halo of a given mass, and compute the average mass function of all progenitors at different redshifts.
- Compare these directly to the EPS progenitor mass functions at those redshifts.
- These need to be in agreement with each other....



Example of a method that fails the Self-Consistency Test:
Black histograms are the progenitor mass functions for a halo of 10^{12} Msun obtained from 2000 merger trees constructed using the N-Branch method with Accretion of Somerville & Kolatt (1998). The red lines are the direct EPS predictions...

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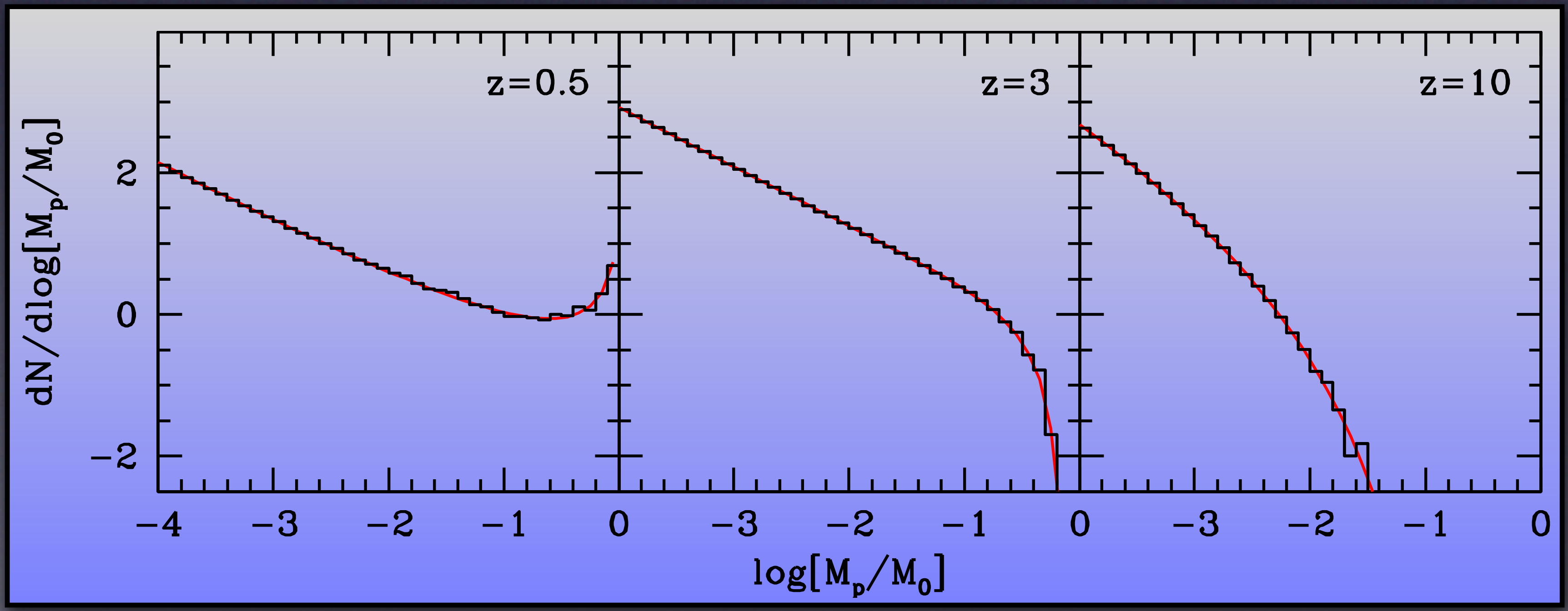


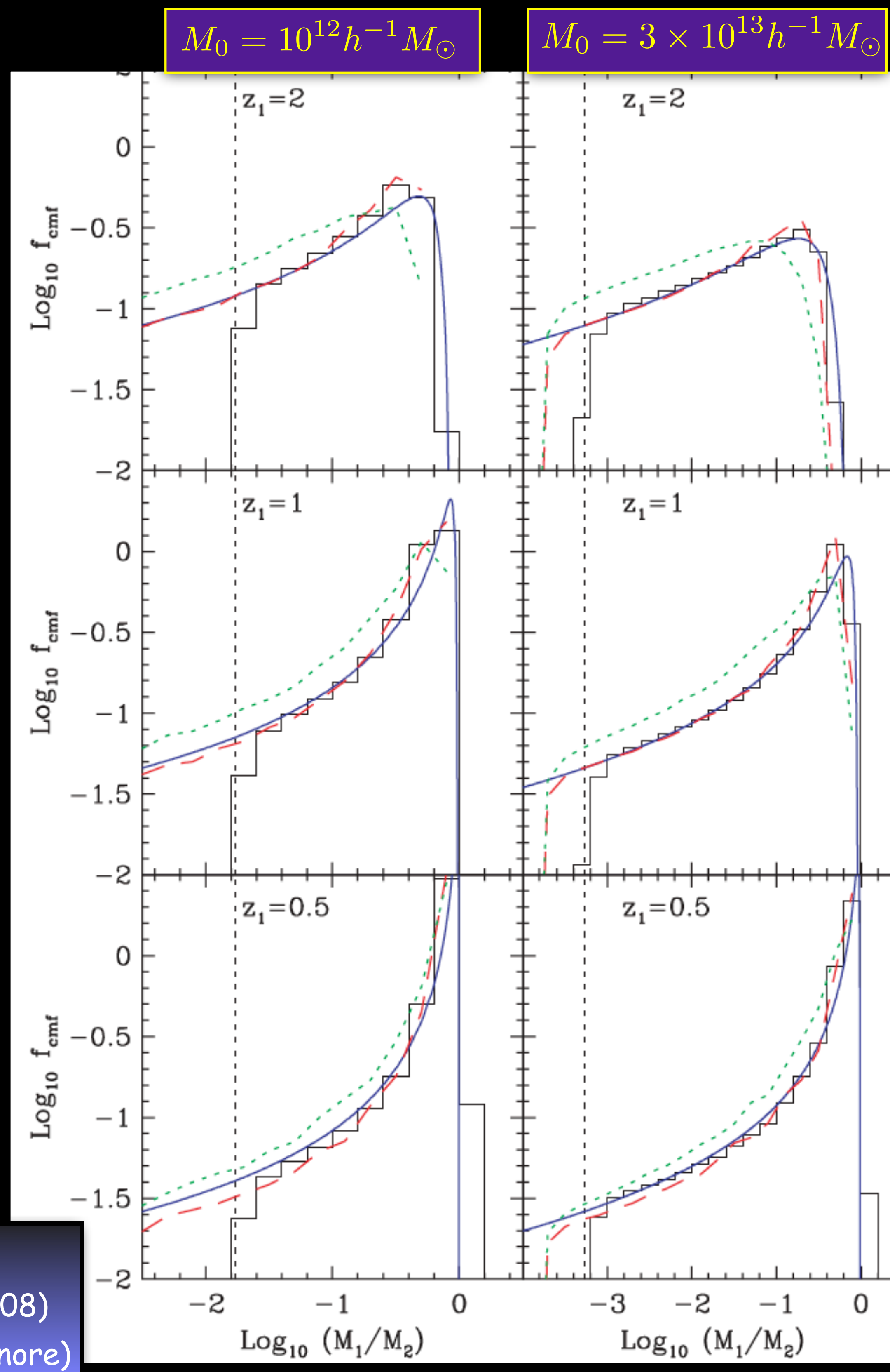
Example of a method that successfully passes the Self-Consistency Test: Black histograms are the progenitor mass functions for a halo of 10^{12} Msun obtained from 2000 merger trees constructed using Method B of Zhang, Fakhouri & Ma (2008). The red lines are the direct EPS predictions, and are in excellent agreement with the merger tree results...

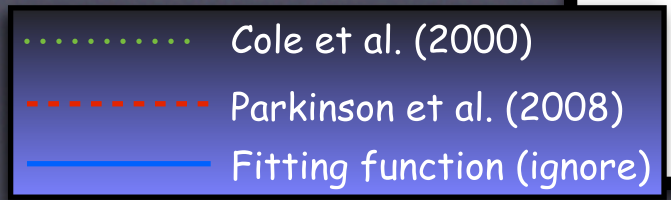
How to Plant a Healthy Merger Tree?

2: Comparison with Numerical Simulations

An important test of EPS merger trees is whether they can reproduce the merger trees obtained from numerical N-body simulations.
We caution, though, that there is no unique way to identify dark matter haloes and their merger histories in numerical simulations, making the comparison non-trivial....

The figure compares progenitor mass fractions in the Millenium simulation (black histograms) to those obtained using two different EPS merger tree algorithms: Cole et al. (2000; green dotted lines), and Parkinson et al. (2008; red dashed lines). The latter is an empirical, ad-hoc modification of Cole et al. (2000) tuned towards better agreement with the simulation results....





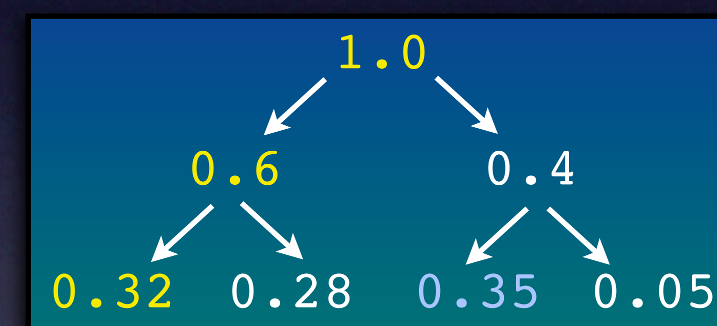
Source: Parkinson et al. 2008, MNRAS, 383, 557

Mass Assembly Histories

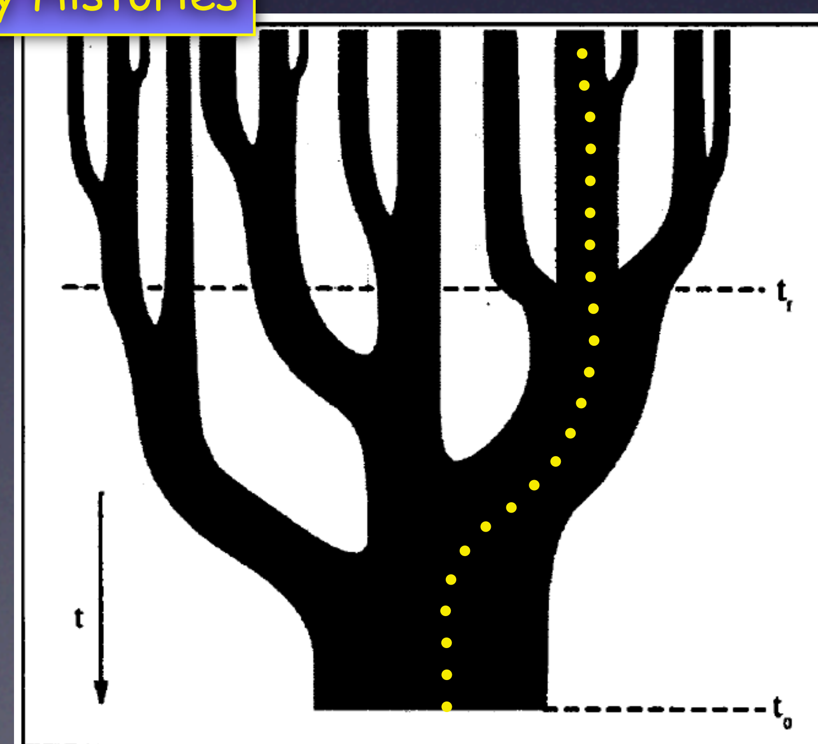
A very useful, reduced characterization of a merger tree is its Mass Assembly History (MAH), also called Mass Accretion History or Main Progenitor History.

The MAH $M(z)$ gives the mass of the main progenitor as a function of redshift; at each time step one associates $M(z)$ with the most massive progenitor, and one follows that progenitor, and that progenitor only, further back in time....

NOTE: the main progenitor is not necessarily also the most massive of all progenitors at a given redshift (see example)...

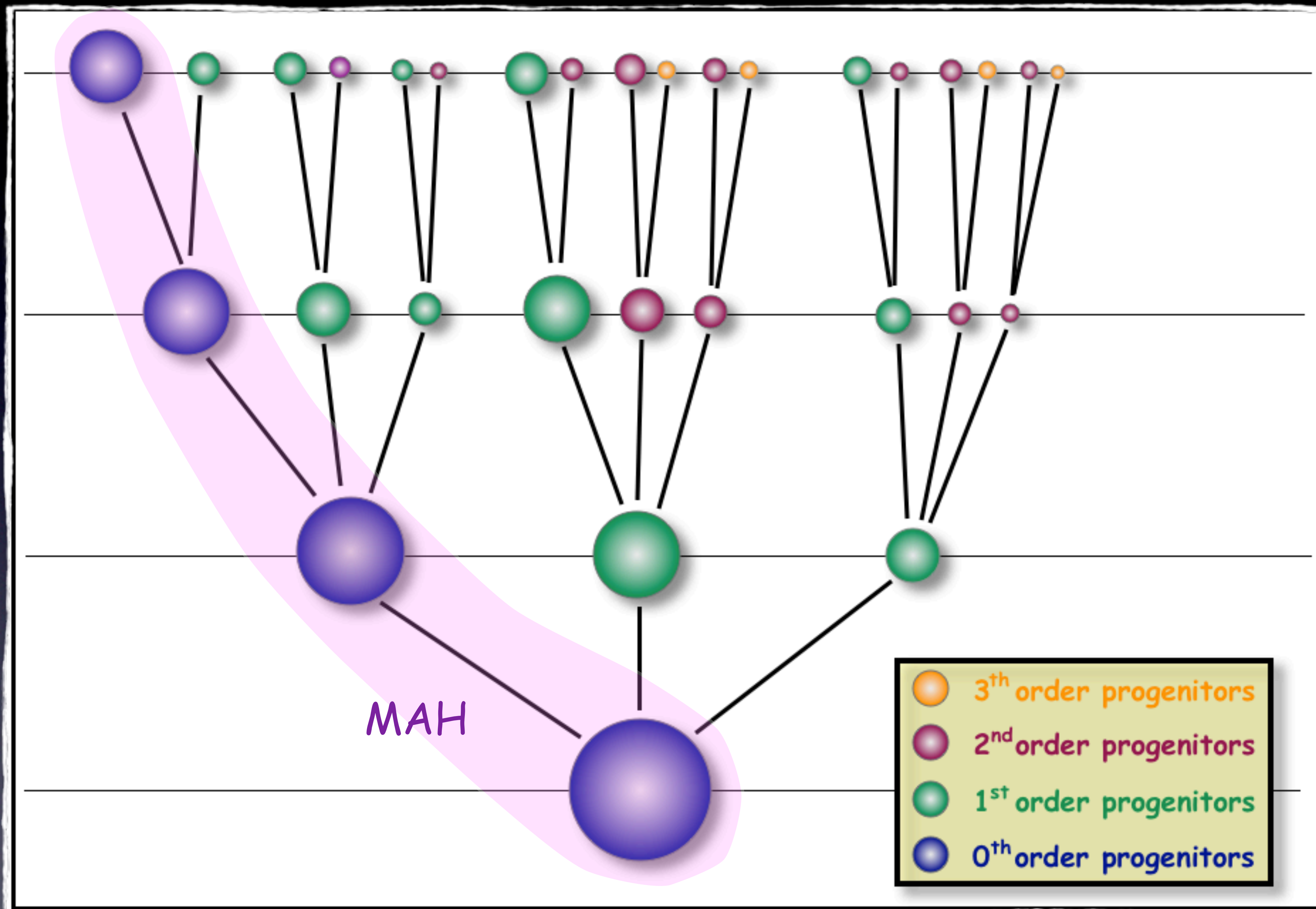


Examples of Mass Assembly Histories



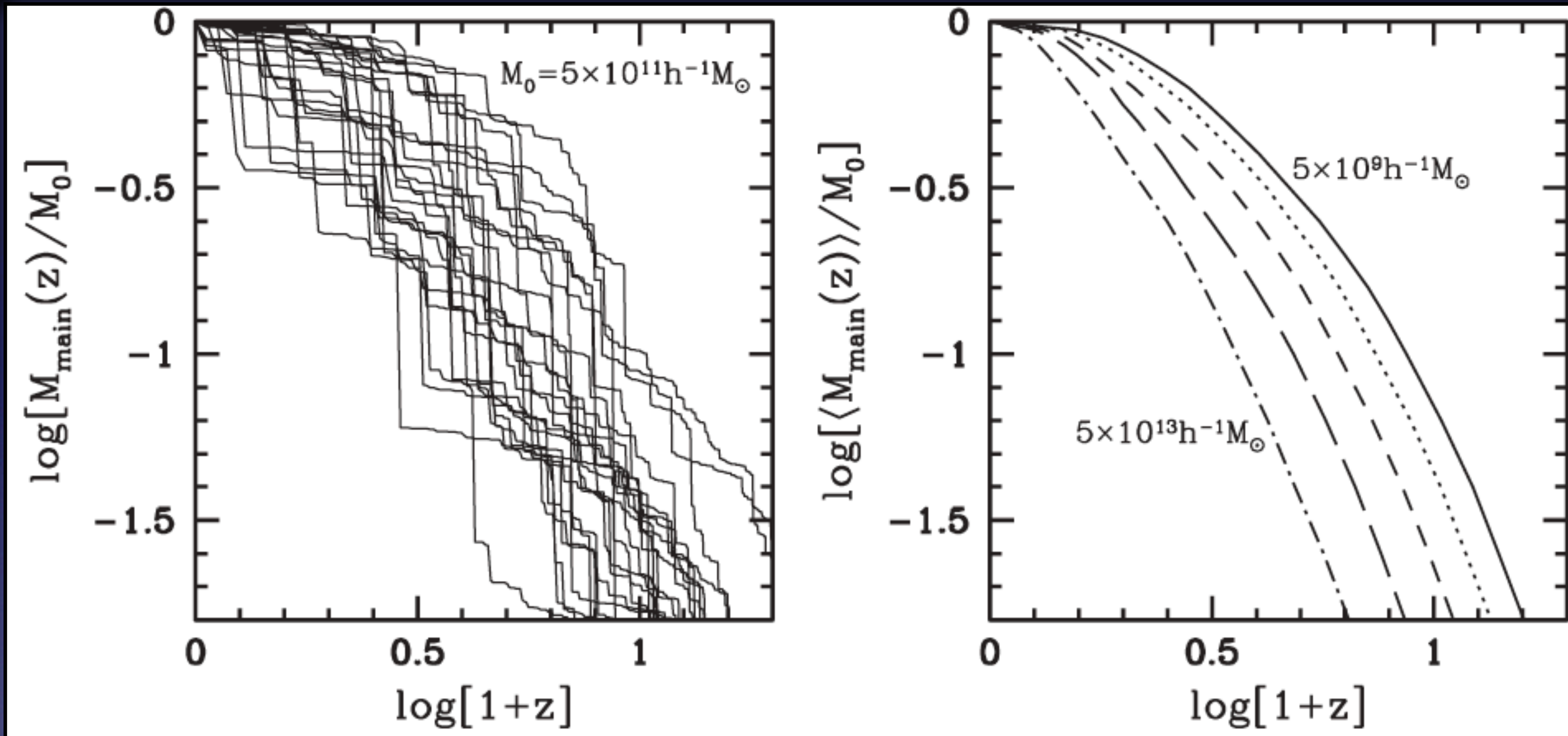
At each branching point in the tree, the MAH follows the most massive branch. Hence, the MAH is sometimes called the main trunk of the merger tree...

Anatomy of a Merger Tree



The MAH is the mass history of the 0^{th} order progenitor...

Mass Assembly Histories



Source: van den Bosch, 2002, MNRAS, 331, 98

A random subset of MAHs for a halo of mass $M_0 = 5 \times 10^{11} h^{-1} M_\odot$ in an EdS Universe. Note the large halo-to-halo variance...

The average MAHs for haloes of different mass in an EdS Universe. Note that more massive haloes assemble later; a clear manifestation of hierarchical structure formation...