ASTR 610
Theory of Galaxy Formation

Summary Slides

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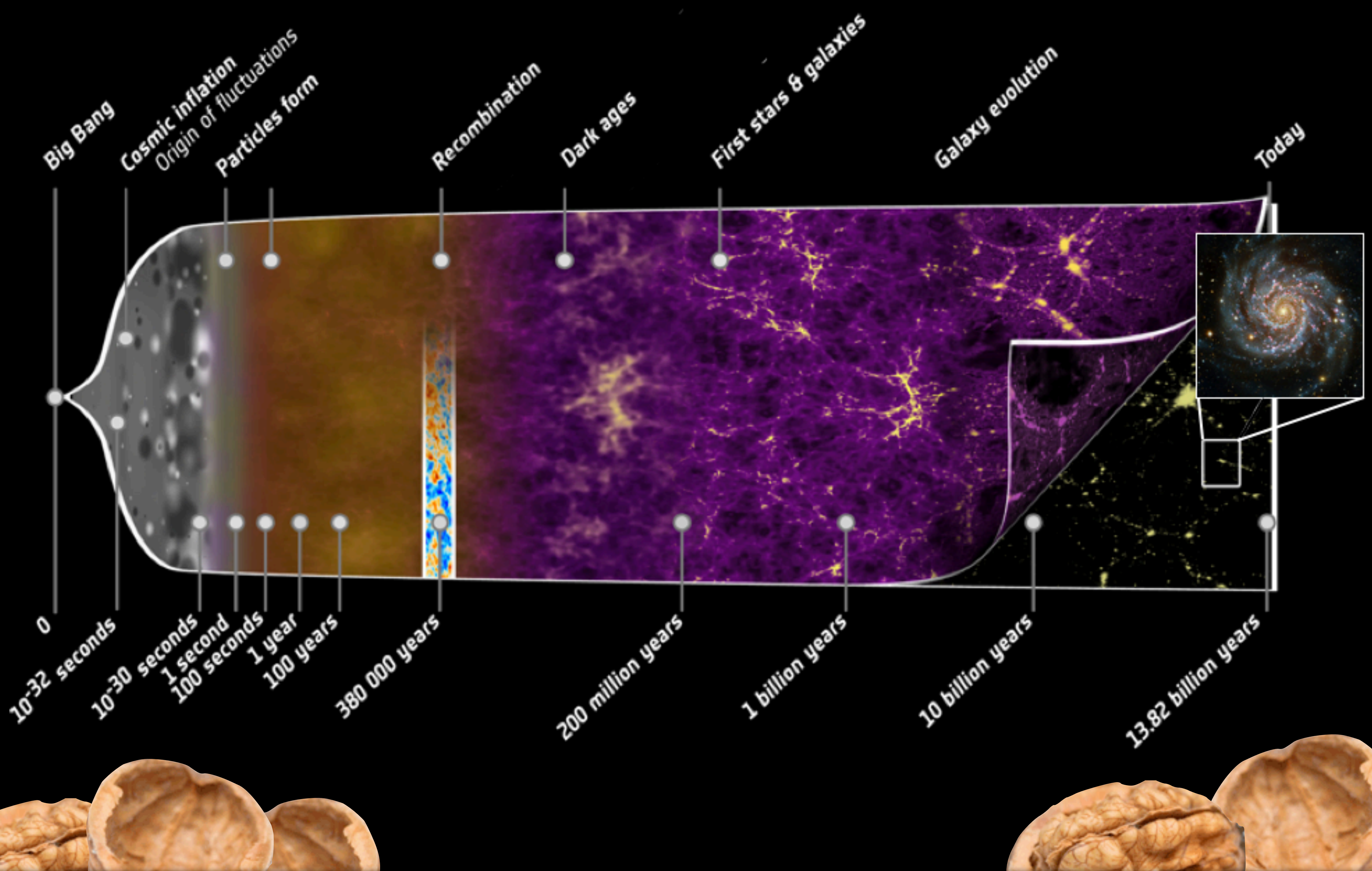
Yale University, Spring 2024

LUX ET VERITAS

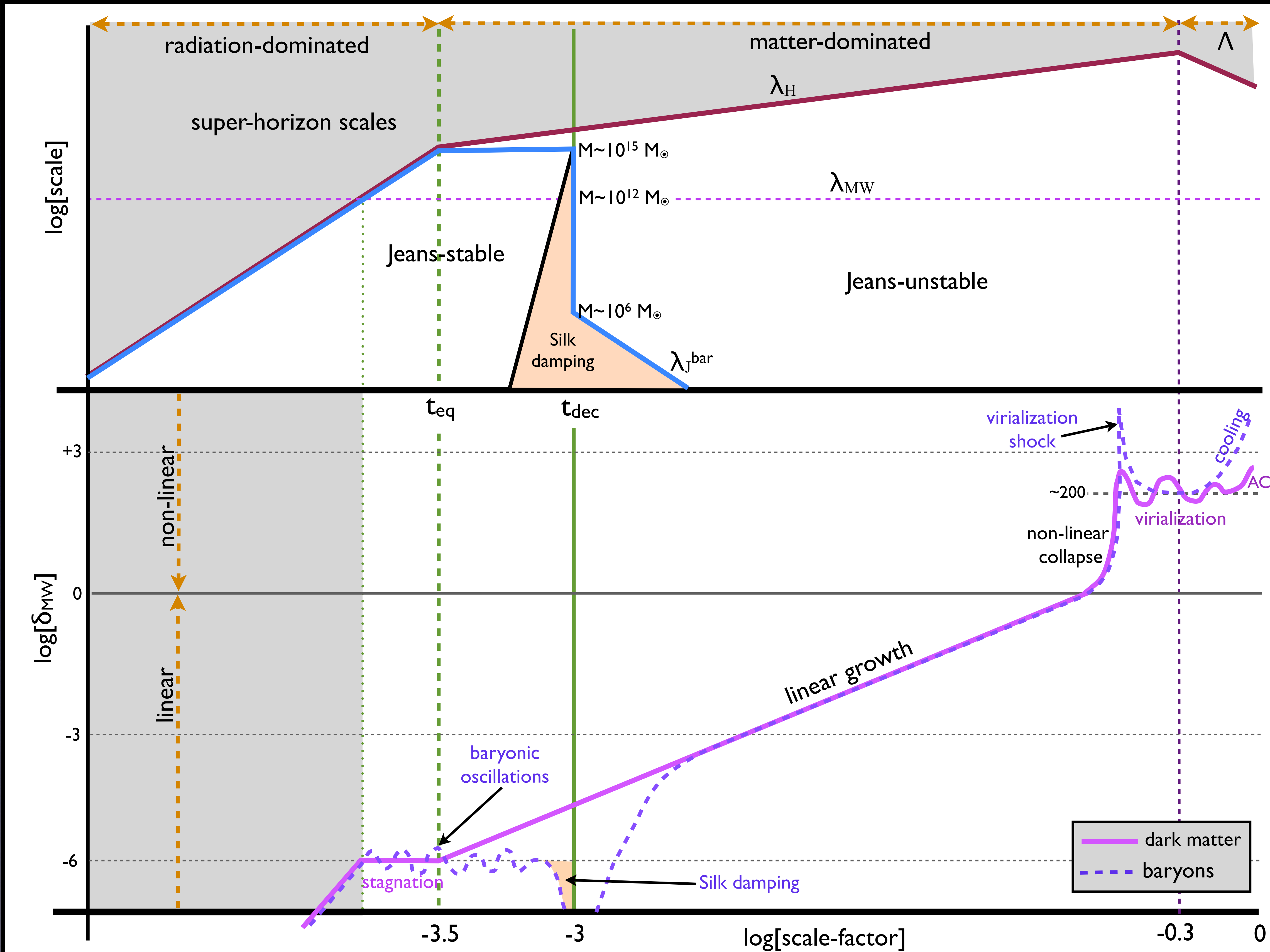
Lecture 1

Introduction

Galaxy Formation in a Nutshell



Galaxy Formation in a Nutshell



Lecture 2

Overview of Cosmology I

(Riemannian Geometry & FRW metric)

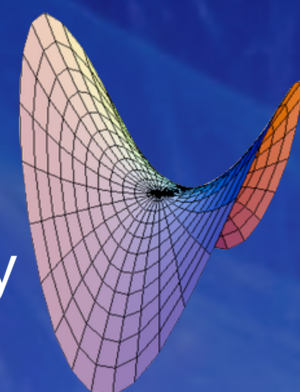
Cosmology in a Nutshell

Lecture 2

Cosmological Principle

$\boxed{\text{Universe is homogeneous \& Isotropic}}$

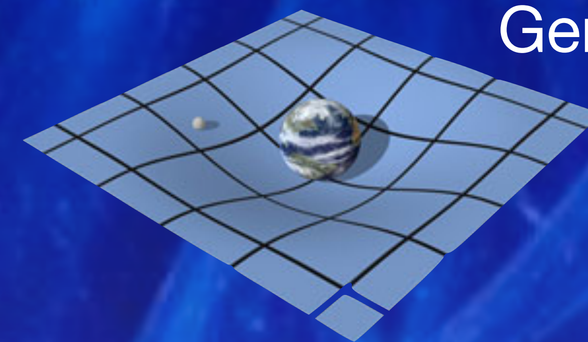
Riemannian Geometry



Friedmann-Robertson-Walker Metric

$$\boxed{\mathrm{d}s^2 = a^2(\tau)\left[\mathrm{d}\tau^2 - \mathrm{d}\chi^2 - f_K^2(\chi)\,\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right]}$$

Lecture 3



General Relativity

Einstein's Field Equation

$$\boxed{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - g_{\mu\nu}\Lambda = \frac{8\pi G}{c^4} T_{\mu\nu}}$$

Friedmann Equations

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{Kc^2}{a^2} + \frac{\Lambda c^2}{3}}$$

Summary: key words & important facts

Key words	
Fundamental observer	Cosmological Principle
Proper time *vs*. conformal time	FRW metric
Comoving *vs.* proper distance	Hubble parameter
Angular diameter distance	redshift
Luminosity distance	peculiar velocity

- Physical laws can be made manifest invariant by writing them in tensor form.
- The geometry of space-time is described by the metric $g_{\mu\nu} = g_{\mu\nu}(x^\alpha)$
- The FRW-metric is the most general metric consistent with the cosmological principle, that the Universe is homogeneous and isotropic (on large scales).
- Due to the expansion, the peculiar velocities of particles that do not experience an external force decay with time as $v_{\rm pec} \propto a^{-1}$
- Since energy densities of baryons & dark matter evolve in the same way, it is sufficient to describe the (non-relativistic) matter as one component .
- Since energy densities of radiation & relativistic matter (i.e., neutrinos) evolve in the same way, it is sufficient to describe them as one component .

Summary: key equations & expressions

Two ways of writing the FRW-metric

$$ds^2 = c^2 dt^2 - a^2(t)\left[\frac{dr^2}{1-Kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right]$$

$$ds^2 = a^2(\tau)\left[d\tau^2 - d\chi^2 - f_K^2(\chi)\,\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right]$$

Thermodynamics $\Rightarrow$ $\frac{d\rho}{da} + 3(1+w)\frac{\rho}{a} = 0$ $\Rightarrow$ $\rho \propto a^{-3(1+w)}$

non-relativistic matter (baryons & dark matter) w = 0
relativistic matter (radiation) w = 1/3
cosmological constant (dark energy) w = -1

Redshift, wavelength, scale-factor & peculiar velocity

$$z \equiv \frac{\lambda_{obs} - \lambda_{em}}{\lambda_{em}} = \frac{a(t_{obs})}{a(t_{em})} - 1$$

$v = \dot{a}\chi + a\dot{\chi} \equiv v_{exp} + v_{pec}$ $\Rightarrow$ $1 + z_{obs} = (1 + z_{cos})\,(1 + z_{pec})$

angular diameter distance
luminosity distance

$$d_A(z) = \frac{a_0\, r}{1+z} \qquad d_L(z) = a_0\, r\,(1+z)$$

Lecture 3

Overview of Cosmology II

(General Relativity & Friedmann Eqs)

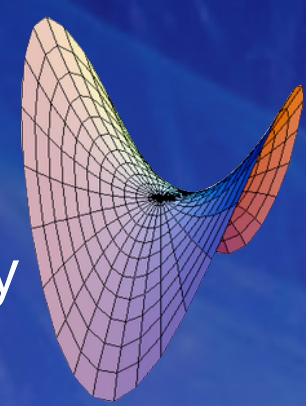
Cosmology in a Nutshell

Lecture 2

Cosmological Principle

Universe is homogeneous & Isotropic

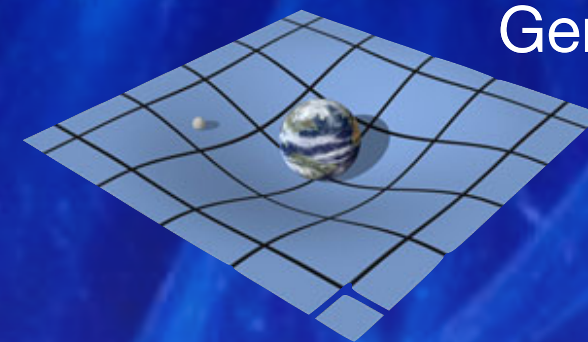
Riemannian Geometry



Friedmann-Robertson-Walker Metric

$$\mathrm{d}s^2 = a^2(\tau)\left[\mathrm{d}\tau^2 - \mathrm{d}\chi^2 - f_K^2(\chi)\,\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right]$$

Lecture 3



General Relativity

Einstein's Field Equation

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - g_{\mu\nu}\Lambda = \frac{8\pi G}{c^4}T_{\mu\nu}$$

Friedmann Equations

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{Kc^2}{a^2} + \frac{\Lambda c^2}{3}$$

Summary: key words & important facts

Key words	
Equivalence Principle	Riemann tensor
Christoffel symbols	Ricci tensor
covariant derivative	Einstein tensor

Why we need GR

- Newtonian gravity only holds in inertial systems, is covariant under Galilean transformations, and moving mass has immediate effect all throughout space.

but
- inertial systems do not exist (you can't shield yourself from gravity)
- SR: inertial systems transform according to Lorentz transformations
- SR: universal speed limit; no information can propagate instantaneously

The Key to GR

- Since gravity is `permanent' (can only be transformed away locally), it must be related to an intrinsic property of space-time itself.
- Space-time of freely falling observer (no gravity) is flat Minkowski space; hence, gravity originates from curvature in space-time (Riemann space)
- Einstein Field equation is the manifest covariant version of Poisson equation

Summary: key equations & expressions

Poisson Equation

$$\nabla^2\Phi = 4\pi G\rho$$

GR

Einstein Field Equation

$$R_{\mu\nu} - \frac{1}{2}R\,g_{\mu\nu} - \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}$$

The Friedmann Equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{Kc^2}{a^2} + \frac{\Lambda c^2}{3}$$

$$H^2(z) = H_0^2\,E^2(z)$$ where

$$E(z) = \left[\Omega_{\Lambda,0} + (1-\Omega_0)\,(1+z)^2 + \Omega_{\mathrm{m},0}(1+z)^3 + \Omega_{\mathrm{r},0}(1+z)^4\right]^{1/2}$$

Density Parameter

$$\Omega(z) - 1 = (\Omega_0 - 1)\,\frac{(1+z)^2}{E^2(z)}$$

Lecture 4

Newtonian Perturbation Theory
I. Linearized Fluid Equations

Summary: key words & important facts

Key words	
Euler equations	Hubble drag
Equation of State	Perturbation analysis
Ideal Gas	Isentropic perturbations
Sound Speed	Isocurvature perturbations

- Dark matter can be described as a collisionless fluid as long as the velocity dispersion of the particles is sufficiently small that particle diffusion can be neglected on the scale of interest. This is true on scales larger than the free-streaming scale.

- In the linear regime, all modes evolve independently (there is no `mode-coupling')

- If evolution is adiabatic, isentropic perturbations remain isentropic.
  If not, the non-adiabatic processes create non-zero ∇S

- Isentropic and isocurvature perturbations are orthogonal;
  any perturbation can be written as a linear combination of both.

Summary: key equations & expressions

continuity equation $\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho \nabla_r \cdot \vec{u} = 0$

Euler equations $\frac{\mathrm{D}\vec{u}}{\mathrm{D}t} = -\frac{\nabla_r P}{\rho} - \nabla_r \phi$

Poisson equation $\nabla_r^2 \phi = 4\pi G \rho$

ideal gas $P = \frac{k_{\mathrm{B}} T}{\mu m_{\mathrm{p}}} \rho$ $\quad \varepsilon = \frac{1}{\gamma - 1} \frac{k_{\mathrm{B}} T}{\mu m_{\mathrm{p}}}$

sound speed $c_{\mathrm{s}} = (\partial P / \partial \rho)_S^{1/2}$

Perturbation analysis in expanding space-time

$$\frac{\partial^2 \delta}{\partial t^2} + 2 \frac{\dot{a}}{a} \frac{\partial \delta}{\partial t} = 4\pi G \bar{\rho} \delta + \frac{c_{\mathrm{s}}^2}{a^2} \nabla^2 \delta + \frac{2}{3} \frac{\bar{T}}{a^2} \nabla^2 S$$

Fourier Transform

“Master equation”

$$\frac{\mathrm{d}^2 \delta_{\vec{k}}}{\mathrm{d}t^2} + 2 \frac{\dot{a}}{a} \frac{\mathrm{d}\delta_{\vec{k}}}{\mathrm{d}t} = \left[4\pi G \bar{\rho} - \frac{k^2 c_{\mathrm{s}}^2}{a^2} \right] \delta_{\vec{k}} - \frac{2}{3} \frac{\bar{T}}{a^2} k^2 \, S_{\vec{k}}$$

$$\delta_S = \frac{3}{4} \delta_{\mathrm{r}} - \delta_{\mathrm{m}}$$

isentropic perturbations $\delta_{\mathrm{r}} = (4/3) \delta_{\mathrm{m}}$

isocurvature perturbations $\delta_{\mathrm{r}} / \delta_{\mathrm{m}} = -(a / a_{\mathrm{eq}})$

Lecture 5

Newtonian Perturbation Theory
II. Baryonic Perturbations

Summary: key words & important facts

Key words	
Jeans criterion	Linear growth rate
Jeans length	Silk damping
Horizons (particle vs. event)	Radiation drag

- Perturbations below the Jeans mass do not grow, but cause acoustic oscillations.

- At recombination photons decouple from baryons → huge drop in the Jeans mass.

- Hubble drag resists perturbation growth → perturbations above the Jeans mass do not grow exponentially, but as a power-law: $\delta_{\vec{k}}(t) \propto t^a$
  The index a depends on cosmology and EoS, as characterized by linear growth rate.

- Growth of super-horizon density-perturbations is governed by conservation of the associated perturbations in the metric.

- If matter is purely baryonic, at recombination Silk damping has erased all perturbations on relevant scales ($M_d \sim 10^{15}\,M_\odot$) → structure formation proceeds in top-down fashion.

Summary: key equations & expressions

prior to recombination: relativistic photon-baryon fluid
$$c_s = \frac{c}{\sqrt{3}}\left[\frac{3}{4}\frac{\rho_b(t)}{\rho_r(t)} + 1\right]^{-1/2}$$

after recombination: baryon fluid is `ideal gas'
$$c_s = (\partial P/\partial \rho)^{1/2} \propto T^{1/2}$$

Jeans length & mass:
$$\lambda_J^{\rm prop} = c_s\sqrt{\frac{\pi}{G\bar\rho}} \qquad M_J = \frac{4\pi}{3}\bar\rho\left(\frac{\lambda_J}{2}\right)^3 = \frac{\pi}{6}\,\bar\rho\,\lambda_J^3$$

comoving particle horizon:
$$\chi_H(a) = \int_0^t \frac{c\,{\rm d}t}{a} = \int_0^a \frac{c\,{\rm d}a}{a\,\dot a} \;\Rightarrow\; \lambda_H^{\rm prop} = \begin{cases} 2\,c\,t & \text{radiation era} \\ 3\,c\,t & \text{matter era}\end{cases}$$

Poisson equation (Fourier space)
$$-k^2\Phi_{\vec k} = 4\pi G a^2\,\bar\rho\delta_{\vec k}$$
$\Rightarrow$ $\Phi_{\vec k}$ is constant implies that $\delta_{\vec k} \propto (\bar\rho a^2)^{-1}$

Lecture 6

Newtonian Perturbation Theory
III. Dark Matter

Summary: key words & important facts

Key words	
Thermal vs. Non-thernal relics	Freeze-out
Cold vs. Hot relics (CDM vs. HDM)	Meszaros effect
Collisionless Boltzmann equation	Free-streaming damping
Jeans equations	ISW effect

- A collisionless fluid with isotropic and homogeneous velocity dispersion is described by the same continuity and momentum equations as a collisional fluid, but with the sound speed $c_{\rm s}$ replaced by $\sigma = \langle v_i^2 \rangle^{1/2}$
- A collisionless fluid does <u>not</u> have an EoS $\Rightarrow$ moment equations are not a closed set
- Collisionless dark matter and baryonic matter have the same linear growth rate.
- Collisional fluid: perturbations below Jeans mass undergo acoustic oscillations
  Collisionless fluid: perturbations below Jeans mass undergo free streaming
- After recombination, baryons fall in DM potential wells, thereby un-doing Silk damping.
- The integrated Sachs-Wolfe effect probes (linear) growth rate of structure.
  In an EdS cosmology $D(a) \propto a$ and the ISW effect is absent.

Summary: key equations & expressions

Collisionless Boltzmann Equation (CBE) $$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + v_i \frac{\partial f}{\partial x_i} - \frac{\partial \Phi}{\partial x_i}\frac{\partial f}{\partial v_i} = 0$$

Moment equations: multiply all terms by v_i^k and integrate over all of velocity space

$k = 0$ ⇒ Continuity equation $$\frac{\partial \delta}{\partial t} + \frac{1}{a}\sum_j \frac{\partial}{\partial x_j}\left[(1+\delta)\langle v_j\rangle\right] = 0$$

$k = 1$ ⇒ Jeans equations $$\frac{\partial \langle v_i\rangle}{\partial t} + \frac{\dot{a}}{a}\langle v_i\rangle + \frac{1}{a}\sum_j \langle v_j\rangle \frac{\partial \langle v_i\rangle}{\partial x_j} = -\frac{1}{a}\frac{\partial \Phi}{\partial x_i} - \frac{1}{\bar{\rho}a(1+\delta)}\sum_j \frac{\partial \rho\sigma_{ij}^2}{\partial x_j}$$

Free-streaming scale $$\lambda_{\rm fs}^{\rm com} = \int_0^{t_{\rm eq}} \frac{v(t')}{a(t')}\mathrm{d}t'$$

Linear growth rate

EdS cosmology	⇒	$D(a) \propto a$
ΛCDM cosmology	⇒	$D(a) \propto a^\gamma \quad (\gamma < 1)$

Poisson equation in Fourier space: $-k^2\Phi_{\vec{k}} = 4\pi G\bar{\rho}a^2\delta_{\vec{k}}$

In matter dominated Universe: $\bar{\rho} \propto a^{-3}$

⇒ $\Phi_{\vec{k}} \propto D(a)/a$

Lecture 7

The Transfer Function &
Cosmic Microwave Background

Summary: key words & important facts

Key words	
ergodic principle	Power spectrum
Gaussian random field	recombination vs. decoupling
two-point correlation function	last scattering surface
Harrison-Zeldovic spectrum	diffusion damping

- The power-spectrum is the Fourier Transform of the two-point correlation function
- A Gaussian random field is completely specified (in statistical sense) by the power-spectrum. The phases of all modes are independent and random.

- CMB dipole reflects our motion wrt last scattering surface (lss)
- Location of first peak in CMB power spectrum ⇒ curvature of Universe
- Ratio of first to second peak in CMB power spectrum ⇒ baryon-to-dark matter ratio
- Finite thickness of lss causes diffusion damping of CMB perturbations

Summary: key equations & expressions

ergodic principle: ensemble average = spatial average

first moment

$$\langle\delta\rangle = \int \delta\, \mathcal{P}(\delta)\, \mathrm{d}\delta = \int \delta(\vec{x})\, \mathrm{d}^3\vec{x} = 0$$

Gaussian random field

$$\mathcal{P}(\delta_1, \delta_2, ..., \delta_N) = \frac{\exp(-Q)}{[(2\pi)^N \det(\mathcal{C})]^{1/2}}$$

$$Q \equiv \frac{1}{2} \sum_{i,j} \delta_i\, (\mathcal{C}^{-1})_{ij} \delta_j$$

$$\mathcal{C}_{ij} = \langle \delta_i \delta_j \rangle = \xi(r_{12})$$

two-point correlation function

$$\langle \delta_1\, \delta_2 \rangle \equiv \xi(\vec{r}_{12}) = \xi(r_{12})$$

cosmological principle: isotropy

$$1 + \xi(r) = \frac{n_{\rm pair}(r \pm \mathrm{d}r)}{n_{\rm random}(r \pm \mathrm{d}r)}$$

Power spectrum & transfer function

$$P(k,t) = P_{\rm i}(k)\, T^2(k)\, D^2(t)$$

$$\Delta^2(k) \equiv \frac{1}{2\pi^2}\, k^3\, P(k)$$

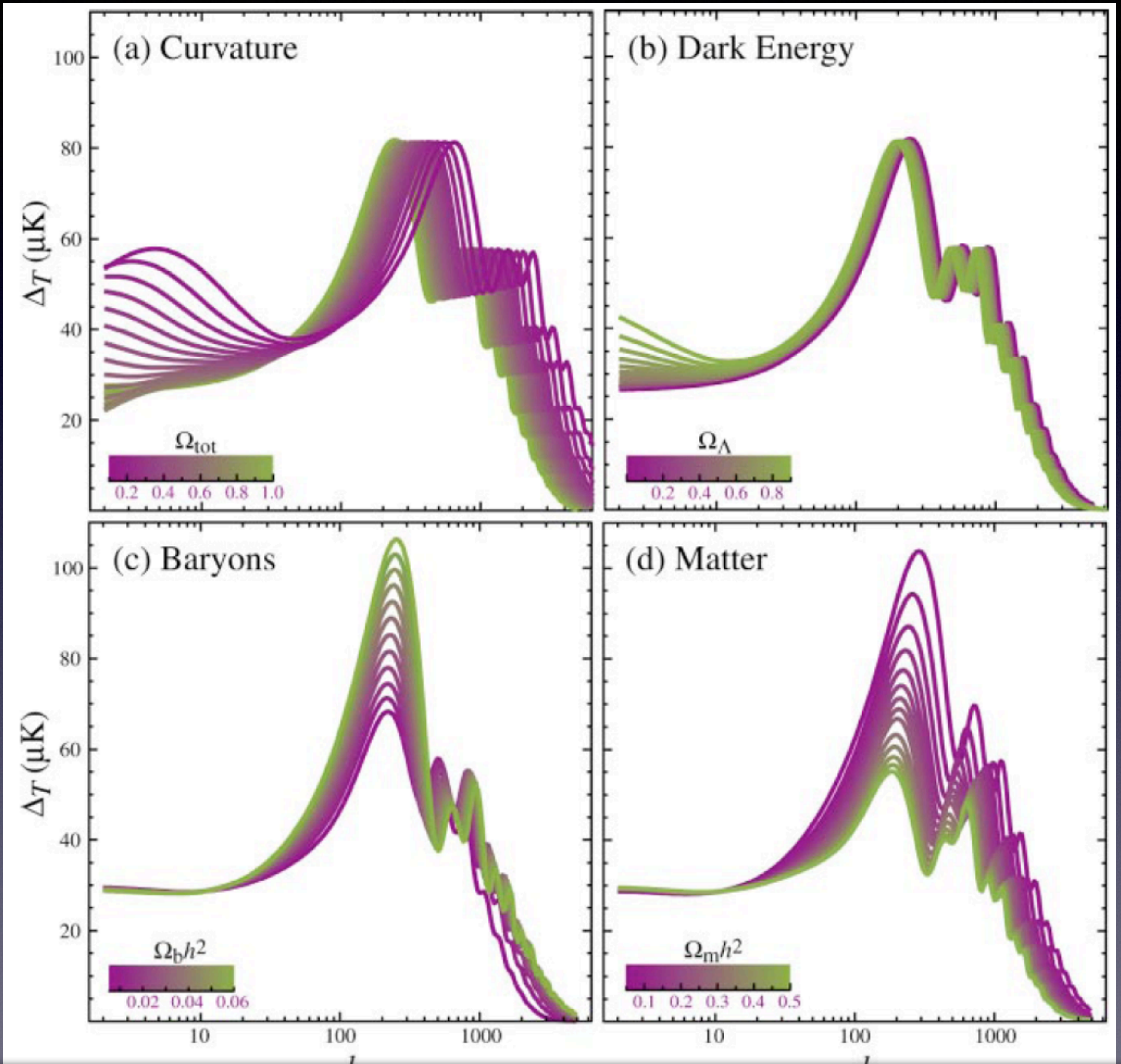
dimensionless power spectrum

$$T(k) = \frac{\Phi_{\vec{k}}(a_{\rm m})}{\Phi_{\vec{k}}(a_{\rm i})}$$

$$P_{\rm i}(k) = \langle |\delta_{\vec{k}}(a_{\rm i})|^2 \rangle = \frac{4}{9} \frac{k^4 \langle |\Phi_{\vec{k},\,{\rm i}}|^2 \rangle}{\Omega_{\rm m,0}^2\, H_0^4} = \frac{4}{9} \frac{k^4\, P_{\Phi,{\rm i}}(k)}{\Omega_{\rm m,0}^2\, H_0^4}$$

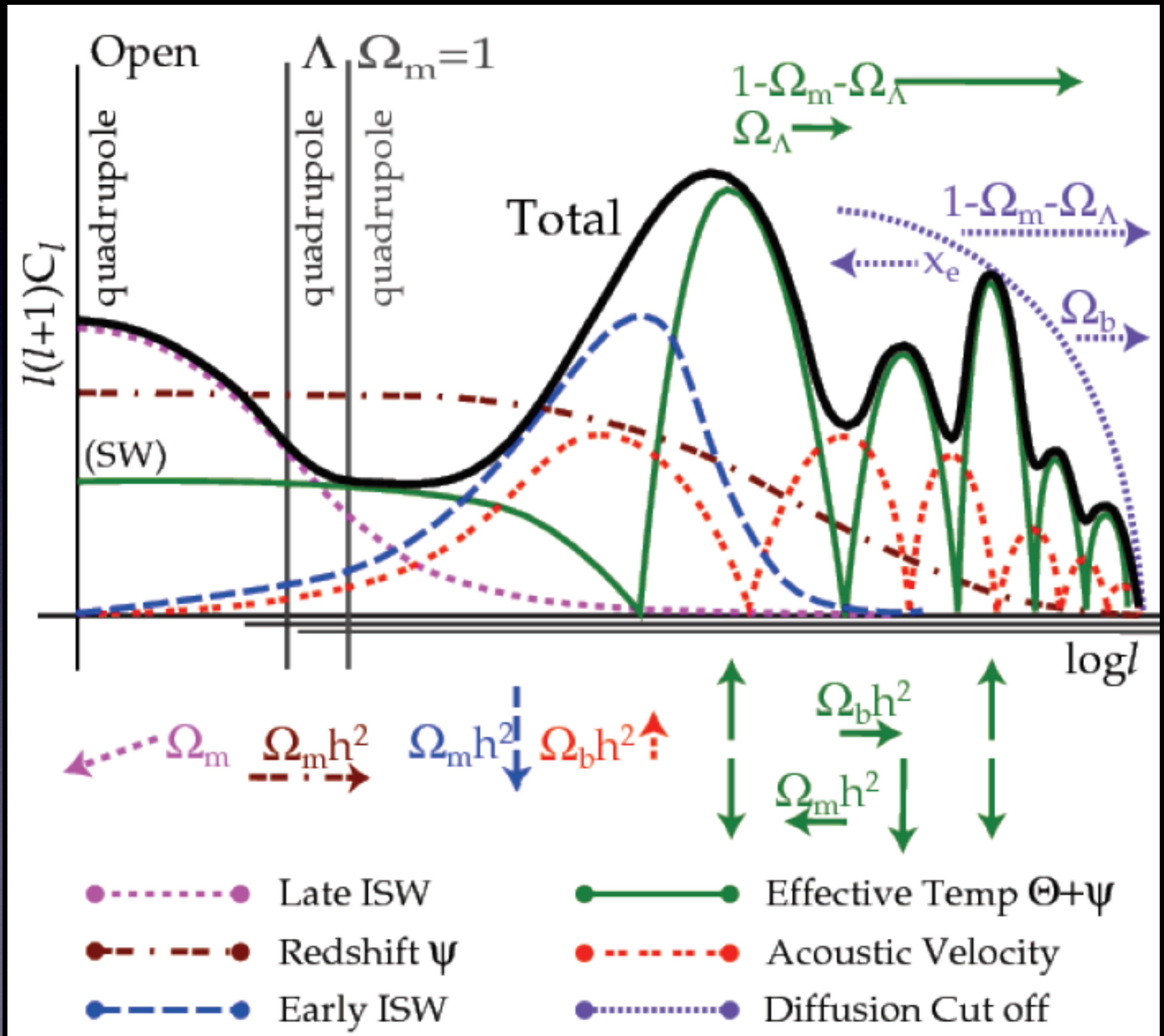
The transfer function $T(k)$ is independent of a_m as long as $\Omega(a_m) \simeq 1$

CMB Summary



Hu & Dodelson 2002

CMB Summary



Open
Λ
$\Omega_m=1$
quadrupole
quadrupole
quadrupole
$l(l+1)C_l$
(SW)
Total
$1-\Omega_m-\Omega_\Lambda$
Ω_Λ
$1-\Omega_m-\Omega_\Lambda$
x_e
Ω_b
$\log l$
Ω_m
$\Omega_m h^2$
$\Omega_m h^2$
$\Omega_b h^2$
$\Omega_b h^2$
$\Omega_m h^2$
Late ISW
Redshift ψ
Early ISW
Effective Temp Θ+ψ
Acoustic Velocity
Diffusion Cut off

Lecture 8

Non-linear Collapse & Virialization

Summary: key words & important facts

Key words	
Spherical/Ellipsoidal collapse	Mode coupling
Secondary Infall model	Violent relaxation
Zel'dovich approximation	Phase Mixing
critical overdensity	Virial Theorem
shell crossing	Two-body relaxation

- In the non-linear regime ($\delta > 1$) perturbation theory is no longer valid. Modes start to couple to each other, and one can no longer describe the evolution of the density field with a simple growth rate: in general, no analytic solutions exist...

- Because of this mode-coupling, the density field looses its Gaussian properties, i.e., in the non-linear regime, density field cannot remain Gaussian.

- Spherical Collapse (SC) model can be used to `identify' when and where collapsed objects will appear. Ellipsoidal Collapse model improves upon SC by accounting for the impact of tides, which typically are more important for less massive objects

- The Zel'dovich approximation is a Lagrangian treatment of the displacement field. It remains accurate in the quasi-linear regime, up to first shell crossing.

Summary: key words & important facts

Key words	
Spherical/Ellipsoidal collapse	Mode coupling
Secondary Infall model	Violent relaxation
Zel'dovich approximation	Phase Mixing
critical overdensity	Virial Theorem
shell crossing	Two-body relaxation

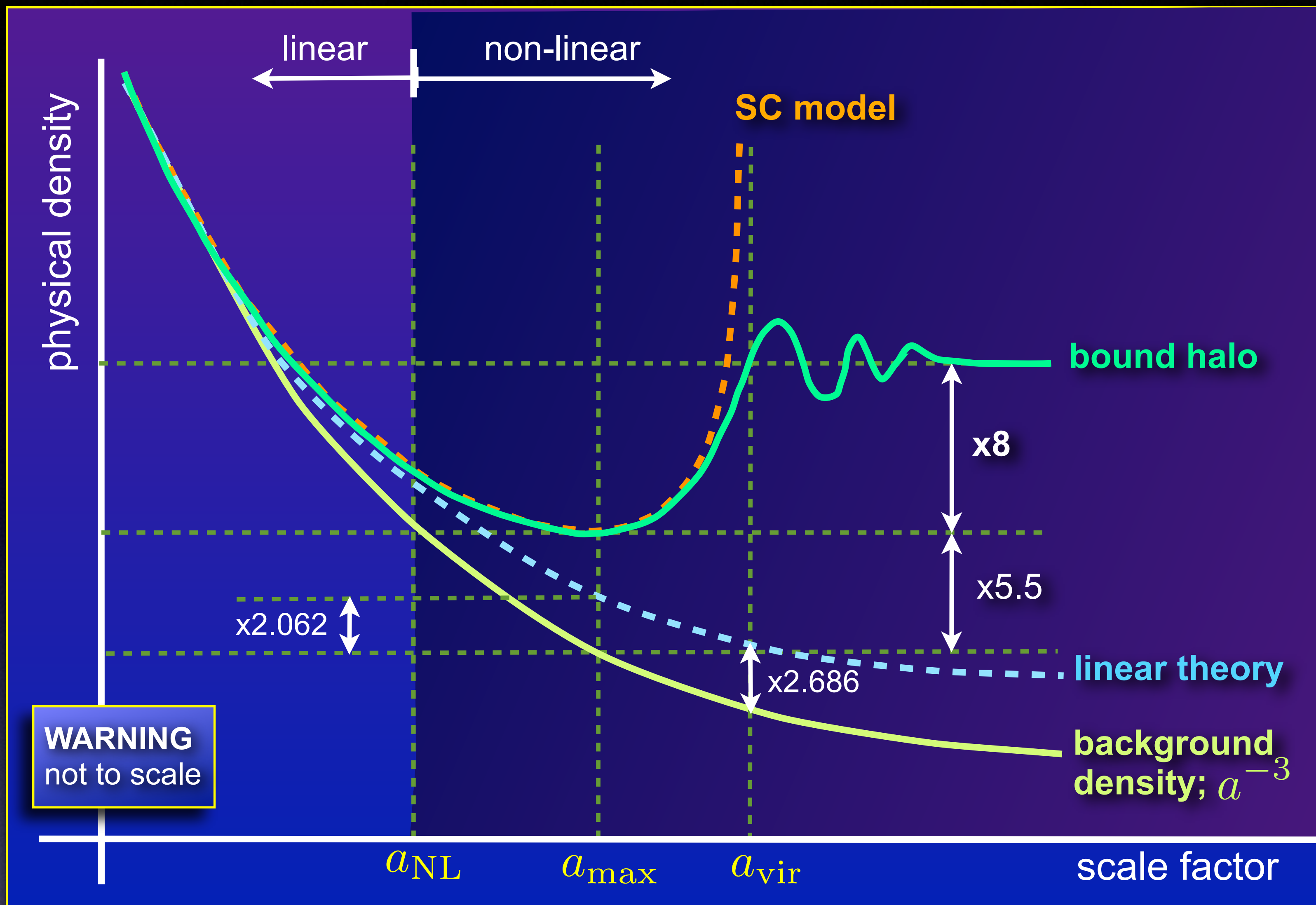
- There are four relaxation mechanisms for collisionless systems:
  - phase mixing
  - chaotic mixing
  - violent relaxation
  - Landau damping

- The only way in which a particle's energy can change in a collisionless system is by having a time-dependent potential.

- Unlike collisional relaxation, violent relaxation does not cause mass segregation

- Violent relaxation operates on the free-fall time, only mixes at the course-grain level of the distribution function, and is self-limiting.

Summary: key equations & expressions



$\delta = \rho/\bar{\rho} - 1$	turn-around	collapse
SC model	4.55	∞
linear model	1.062	1.686

Spherical **C**ollapse model

$$1 + \delta = \frac{\rho}{\bar{\rho}} = \frac{9}{2}\,\frac{(\theta - \sin\theta)^2}{(1-\cos\theta)^3}$$

Linear theory

$$\delta_{\rm lin} = \frac{3}{20}(6\pi)^{2/3}\left(\frac{t}{t_{\rm max}}\right)^{2/3}$$

Virialization:

$$r_{\rm vir} = r_{\rm ta}/2$$
$$1 + \Delta_{\rm vir} = 18\pi^2 \simeq 178 \sim 200$$

Zel'dovich Approximation

$$\vec{x}(t) = \vec{x}_{\rm i} - \frac{D(a)}{4\pi G\bar{\rho}_{\rm i}}\,\vec{\nabla}\Phi_{\rm i}$$

Virial Theorem: $2K + W + \Sigma = 0$

Violent Relaxation: $\mathrm{d}E/\mathrm{d}t = \partial\Phi/\partial t$

two-body relaxation time: $t_{\rm relax} \simeq \frac{N}{10\ln N}\,t_{\rm cross}$

Lecture 9

Press-Schechter Theory

Summary: key words & important facts

Key words	
(Extended) Press-Schechter	Mass Variance
Excursion Set Formalism	Halo Mass Function
Moving Collapse Barrier	Multiplicity Function
Markovian random walk	Characteristic Halo Mass

- Locations in linearly extrapolated density field where $\delta > \delta_c \simeq 1.686$ correspond to collapsed objects (halos)
- If $\delta(x)$ is Gaussian, then so is the smoothed density field $\delta(x;R)$
- Excursion sets are Markovian if, and only if, the density field is smoothed with a sharp-k space filter
- The cosmological parameter σ_8 is defined as the mass variance of the linearly extrapolated density field at $z=0$, smoothed with a Top-Hat filter of size $R=8\ h^{-1}\mathrm{Mpc}$
- The ellipsoidal collapse model gives rise to a moving barrier in excursion set formalism

Summary: key equations & expressions

Mass Smoothing

$$\delta(\vec{x};R) \equiv \int \delta(\vec{x}')\,W(\vec{x}-\vec{x}';R)\,\mathrm{d}^3\vec{x}' \qquad \delta(\vec{k};R) = \delta(\vec{k})\,\widetilde{W}(kR)$$

Mass Variance

$$\sigma^2(M) = \langle\delta^2(\vec{x};R)\rangle = \frac{1}{2\pi^2}\int P(k)\,\widetilde{W}^2(kR)\,k^2\,\mathrm{d}k \qquad M = \gamma_{\rm f}\bar{\rho}R^3$$

(E)PS ansatz

PS $\quad F(>M,t) = 2\,\mathcal{P}[\delta_M > \delta_{\rm c}(t)]$

EPS $\quad F(>M,t) = 1 - F_{\rm FU}(>S) \qquad S = \sigma^2(M)$

Halo Mass Function

$$n(M,t) \equiv \frac{\mathrm{d}n}{\mathrm{d}M} = \frac{1}{M}\,\frac{\mathrm{d}n}{\mathrm{d}\ln M} = \frac{\bar{\rho}}{M}\,\frac{\partial F(>M,t)}{\partial M}$$

EPS + *Gaussian*

$$n(M,t) = \sqrt{\frac{2}{\pi}}\,\frac{\bar{\rho}}{M^2}\,\frac{\delta_{\rm c}}{\sigma_M}\,\exp\left(-\frac{\delta_{\rm c}^2}{2\sigma_{\rm M}^2}\right)\left|\frac{\mathrm{d}\ln\sigma_M}{\mathrm{d}\ln M}\right| = \frac{\bar{\rho}}{M^2}\,f_{\rm PS}(\nu)\left|\frac{\mathrm{d}\ln\nu}{\mathrm{d}\ln M}\right|$$

shorthand $\quad \nu \equiv \delta_{\rm c}(t)/\sigma(M) \qquad f_{\rm PS}(\nu) = \sqrt{2/\pi}\,\nu\,\mathrm{e}^{-\nu^2/2}$

Ellipsoidal Collapse Model

$$\delta_{\rm c}(t) \rightarrow \delta_{\rm c}(t)\left[1 + 0.47\left(\frac{\sigma^2(M)}{\delta_{\rm c}^2(t)}\right)^{0.615}\right]$$

$$f_{\rm PS}(\nu) \rightarrow f_{EC}(\nu) = 0.322\left[1 + \frac{1}{(0.84\nu)^{0.6}}\right] f_{\rm PS}(0.84\nu)$$

Characteristic Mass

$$\sigma^2(M^*) = \delta_{\rm c}(t)$$

Lecture 10

Merger Trees & Halo Bias

Summary: key words & important facts

Key words

Merger Tree	Halo Formation time
Progenitor Mass Function	Halo Bias
Mass Assembly History	Assembly Bias

- Construction of halo merger tree is subject to two conditions
  accurately samples progenitor mass function at all times (self-consistency)
  mass conservation (sum of progenitor masses = descendent mass)
  Different methods for constructing EPS merger trees differ in handling corresponding subtleties..

- Even in the limit of infinitesimally small time-step there is a non-zero probability of having more than two progenitors → binary merger tree method fails

- Mass assembly histories of dark matter halos are universal, if scaled appropriately.

- More massive halos assemble later, and are more strongly clustered (i.e., $db_h/dM > 0$)

- Halos that assemble earlier are more strongly clustered than halos of the same mass that assemble later (= halo assembly bias)

Summary: key equations & expressions

Progenitor Mass Function:

$$\boxed{n(M_1,t_1|M_2,t_2)\,\mathrm{d}M_1 = \frac{M_2}{M_1} f_{\rm FU}(S_1,\delta_1|S_2,\delta_2)\left|\frac{\mathrm{d}S_1}{\mathrm{d}M_1}\right|\,\mathrm{d}M_1}$$

Halo Bias

$$\boxed{\delta_{\rm h}(M_1,z_1|M_0,\delta_0) = \delta(z_1) + \frac{\nu_1^2-1}{\delta_1}\delta_0 + \frac{\nu_1^2-1}{\delta_1}\delta_0\delta(z_1)}$$

in linear regime $\delta_{\rm h}(M_1,z_1|\delta_0) \equiv b_{\rm h}(M_1,z_1)\delta(z_1)$ with $b_{\rm h}(M,z) = 1 + \left(\frac{\nu^2-1}{\delta_{\rm c}}\right)$

$$\Rightarrow \boxed{\begin{aligned} \delta_{\rm h}(\vec{x}|M) &= b_{\rm h}(M)\,\delta(\vec{x}) \\ \xi_{\rm hh}(r|M_1,M_2) &= b_{\rm h}(M_1)\,b_{\rm h}(M_2)\,\xi_{\rm mm}(r) \end{aligned}}$$

Lecture 11

Structure of Dark Matter Halos

Summary: key words & important facts

Key words

NFW/Einasto profile	Halo Concentration Parameter
Halo virial relations	Halo Spin Parameter
Cusp-Core controversy	Linear Tidal Torque Theory

- More massive haloes are less concentrated, are more aspherical, and have more substructure
  All these trends are mainly because more massive haloes assemble later
- Both concentration and spin parameter follow log-normal distributions
- The (median) spin parameter is independent of halo mass or redshift
- Dark matter halos have a universal density profile, a universal angular momentum profile, and a universal assembly history
- Subhalos reveal very little segregation by present-day mass, a weak segregation by accretion mass, and strong segregation by accretion redshift and retained mass fraction
- Dark matter haloes acquire angular momentum in the linear regime due to tidal torques from neighboring overdensities...

Summary: key equations & expressions

Halo Virial Relations

$$r_{\rm vir} \simeq 163\,h^{-1}{\rm kpc}\left[\frac{M_{\rm vir}}{10^{12}h^{-1}M_\odot}\right]^{1/3}\left[\frac{\Delta_{\rm vir}}{200}\right]^{-1/3}\Omega_{\rm m,0}^{-1/}$$

$$V_{\rm vir} \simeq 163\,{\rm km/s}\left[\frac{M_{\rm vir}}{10^{12}h^{-1}M_\odot}\right]^{1/3}\left[\frac{\Delta_{\rm vir}}{200}\right]^{1/6}\Omega_{\rm m,0}^{1/6}\,(1+z)^{1/2}$$

Subhalo Mass Function

$$\frac{{\rm d}n}{{\rm d}\ln(m/M)} \propto \left(\frac{m}{M}\right)^{-\gamma}\exp\left[-(m/\beta M)\right]$$

$$\gamma \simeq 0.9 \pm 0.1 \qquad \beta \simeq 0.3$$

Halo Density Profiles

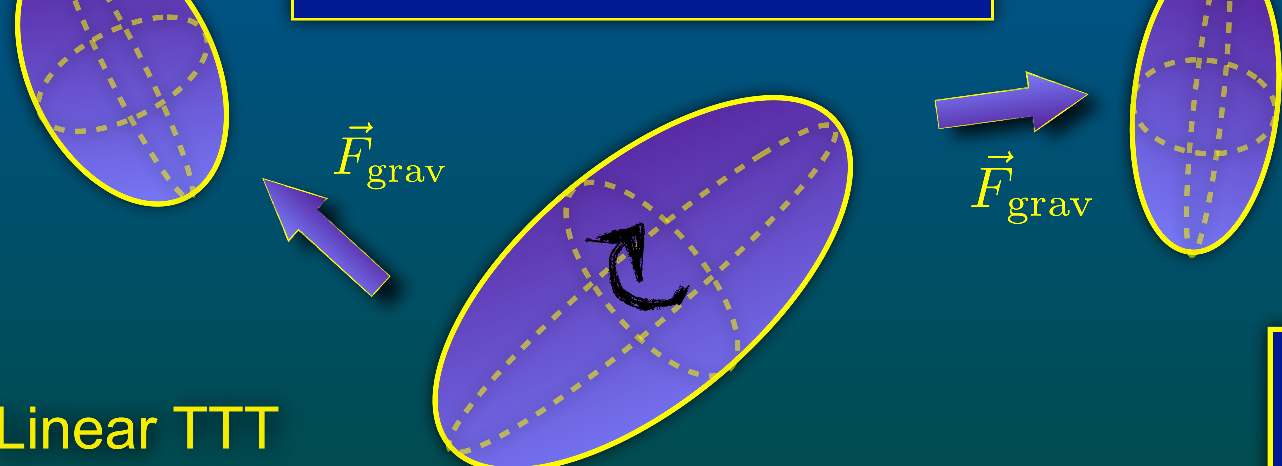
NFW $$\rho(r) = \frac{\rho_{\rm s}}{\left(\frac{r}{r_{\rm s}}\right)\left(1+\frac{r}{r_{\rm s}}\right)^2}$$

concentration parameter $$c = r_{\rm vir}/r_{\rm s}$$

Einasto $$\rho(r) = \rho_{-2}\,\exp\left[\frac{-2}{\alpha}\left\{\left(\frac{r}{r_{-2}}\right)^\alpha - 1\right\}\right] \Rightarrow \frac{{\rm d}\ln\rho}{{\rm d}\ln r} = -2\left(\frac{r}{r_{-2}}\right)^\alpha$$

Linear TTT

$$J_i(t) = a^2(t)\,\dot{D}(t)\,\epsilon_{ijk}T_{jl}\,I_{lk}$$



$$J_{\rm vir} = \int_0^{t_{\rm ta}} J(t)\,{\rm d}t = \epsilon_{ijk}\,T_{jl}\,I_{lk}\int_0^{t_{\rm ta}} a^2(t)\,\dot{D}(t)\,{\rm d}t$$

Halo Spin Parameter

$$\lambda = \frac{J\,|E|^{1/2}}{G\,M^{5/2}} \qquad \lambda' = \frac{J}{\sqrt{2}M\,V\,R}$$

Lecture 12

Large Scale Structure

Summary: key words & important facts

Key words	
reduced/irreducible corr fnc	projected correlation function
Poisson sampling	Redshift space distortions
Wiener-Khinchin theorem	Kaiser effect
Limber equation	Finger-of-God effect

- The reduced (or irreducible) correlation functions express the part of the n-point correlation functions that cannot be obtained from lower-order correlation functions

- For a Gaussian random field, all connected moments (=reduced correlation functions) of $n > 2$ are equal to zero (i.e., $\zeta = \eta = 0$).

  → One can use ζ and η to test whether the density field is Gaussian or not...

- If galaxy formation is a Poisson sampling of the density field, then all n-point correlation functions of the galaxy distribution are identical to those of the matter distribution

  This is not the case though; galaxies are biased tracers of the mass distribution

- On large (linear) scales, redshift space distortions (RSDs) depend on linear growth rate. On small (non-linear) scales, RSDs reveal FoG indicative of virial motion within halos

- Redder and more massive/luminous galaxies are more strongly clustered

Summary: key equations & expressions

n-point correlation function

$$\xi^{(n)} \equiv \langle \delta_1 \, \delta_2 \ldots \delta_n \rangle$$

n-point irreducible correlation function

$$\xi^{(n)}_{\rm red} \equiv \langle \delta_1 \, \delta_2 \ldots \delta_n \rangle_{\rm c}$$

$$\xi(r) = \frac{DD(r)\,\Delta r}{RR(r)\,\Delta r} - 1$$ 2-pt function (discrete)

$$w(\theta) = \frac{DD(\theta)\,{\rm d}\theta}{RR(\theta)\,{\rm d}\theta} - 1$$ angular 2-pt (discrete)

$$P(k) \equiv V\,\langle|\delta_{\vec{k}}|^2\rangle = P_{\rm gg}(k) + \frac{1}{\bar{n}}$$ power spectrum (discrete)

projected correlation function

$$w_{\rm p}(r_{\rm p}) = \int_{-\infty}^{\infty} \xi(r_{\rm p}, r_\pi)\,{\rm d}r_\pi = 2\int_{r_{\rm p}}^{\infty} \xi(r)\,\frac{r\,{\rm d}r}{(r^2 - r_{\rm p}^2)^{1/2}}$$

$$\xi(r) = -\frac{1}{\pi}\int_r^{\infty} \frac{{\rm d}w_{\rm p}}{{\rm d}r_{\rm p}}\,\frac{{\rm d}r_{\rm p}}{(r_{\rm p}^2 - r^2)^{1/2}}$$

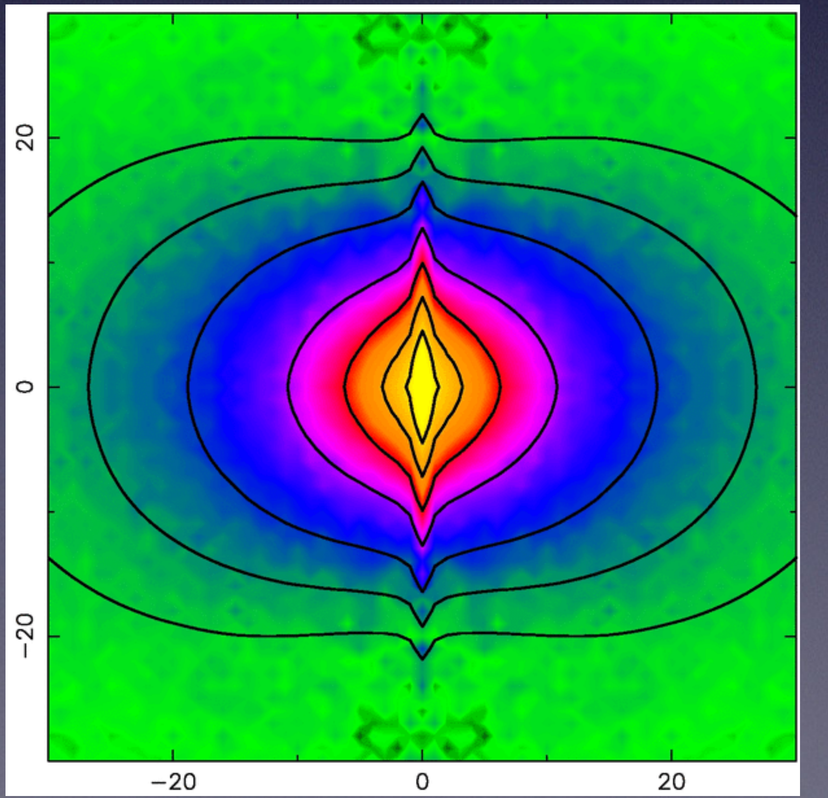
Limber equation

$$w(\theta) = \int_0^{\infty} {\rm d}y\,y^4\,S^2(y)\int_{-\infty}^{\infty} {\rm d}x\,\xi(\sqrt{x^2 + y^2\theta^2})$$

redshift space distortions

$$P^{(s)}(\vec{k}) = \left[1 + \beta\,\mu_{\vec{k}}^2\right]^2\,P(k) \qquad \beta = \frac{1}{b}\,\frac{{\rm d}\ln D}{{\rm d}\ln a} = \frac{f(\Omega_{\rm m})}{b} \simeq \frac{\Omega_{\rm m}^{0.6}}{b}$$

$$1 + \xi(r_{\rm p}, r_\pi) = \int_{-\infty}^{\infty} [1 + \xi_{\rm lin}(r_{\rm p}, r_\pi)]\;f(v_{12}|r)\,{\rm d}v_{12}$$



Lecture 13

The Halo Model & Halo Occupation Statistics

Summary: key words & important facts

Key words	
Halo model	1-halo & 2-halo terms
halo exclusion	Halo Occupation Distribution (HOD)
galaxy-galaxy lensing	Conditional Luminosity Function (CLF)

- The Halo model is an analytical model that describes dark matter density distribution in terms of its halo building blocks, under ansatz that all dark matter is partitioned over haloes.

- In combination with a halo occupation model (HOD or CLF), the halo model can be used to compute galaxy-galaxy correlation function and galaxy-matter cross-correlation function. The latter is related to the excess surface density measured with galaxy-galaxy lensing.

- HOD is mainly used to model clustering of luminosity threshold samples. CLF can be used to model clustering of galaxies of any luminosity (bin).

- It is common to assume that satellite galaxies obey Poisson statistics, such that $\langle N_s(N_s-1)|M\rangle = \langle N_s\rangle^2$, and only the first moment of $P(N_s|M)$ is required. This is not exact and may cause significant errors in the predicted clustering.

Summary: key equations & expressions

halo model

$$P(k) = P^{1h}(k) + P^{2h}(k)$$

$$P^{1h}(k) = \frac{1}{\bar{\rho}^2} \int \mathrm{d}M\, M^2\, n(M)\, |\tilde{u}(k|M)|^2$$

$$P^{2h}(k) = P^{\rm lin}(k) \left[\frac{1}{\bar{\rho}} \int \mathrm{d}M\, M\, b(M)\, n(M)\, \tilde{u}(k|M)\right]^2$$

Galaxy-Galaxy lensing: tangential shear, excess surface density and galaxy-matter cross correlation

$$\gamma_{\rm t}(R)\Sigma_{\rm crit} = \Delta\Sigma(R) = \bar{\Sigma}(< R) - \Sigma(R) \qquad \Sigma(R) = \bar{\rho} \int_0^{D_{\rm s}} [1 + \xi_{\rm g,dm}(r)]\, \mathrm{d}\chi$$

CLF: the link between light and mass

$$\Phi(L) = \int_0^\infty \Phi(L|M)\, n(M)\, \mathrm{d}M \qquad \langle L \rangle_M = \int_0^\infty \Phi(L|M)\, L\, \mathrm{d}L \qquad \langle N_{\rm x} \rangle_M = \int_{L_1}^{L_2} \Phi_{\rm x}(L|M)\, \mathrm{d}L$$

Characteristic examples of CLF and HOD for both centrals and satellites

$$\Phi_{\rm c}(L|M)\mathrm{d}L = \frac{1}{\sqrt{2\pi}\sigma_{\rm c}} \exp\left[-\left(\frac{\ln(L/L_{\rm c})}{\sqrt{2}\sigma_{\rm c}}\right)^2\right] \frac{\mathrm{d}L}{L} \qquad \langle N_{\rm c} \rangle_M = \frac{1}{2}\left[1 + \mathrm{erf}\left(\frac{\log M - \log M_{\rm min}}{\sigma_{\log M}}\right)\right]$$

$$\Phi_{\rm s}(L|M)\mathrm{d}L = \frac{\phi_{\rm s}}{L_{\rm s}} \left(\frac{L}{L_{\rm s}}\right)^{\alpha_{\rm s}} \exp\left[-(L/L_{\rm s})^2\right] \mathrm{d}L \qquad \langle N_{\rm s} \rangle_M = \begin{cases} \left(\frac{M}{M_1}\right)^\alpha & \text{if } M > M_{\rm cut} \\ 0 & \text{if } M < M_{\rm cut} \end{cases}$$

Lecture 14

Galaxy Interactions

Summary: key words & important facts

Key words

Impulse & tidal approximations	dynamical friction
distant encounter approximation	gravitational capture
tidal shock heating	orbital decay
tidal mass stripping	negative heat capacity

- Gravitational encounter results in tidal distortion. If tidal distortion lags perturber, the resulting torque causes a transfer of orbital energy into internal energy of the objects involved.
- An impulsive encounter that results in an (internal) energy increase ΔE that is larger than the system's binding energy does not necessarily result in the system's disruption
- During re-virialization, following an impulsive encounter, the subject converts $2x\Delta E$ from kinetic into potential energy, resulting in the system \`puffing' up.
- Dynamical friction is a global, rather than a local effect. Unlike hydrodynamical friction, the deceleration decreases with increasing velocity, at least at the high-velocity end.
- Dynamical friction is only important for subjects with a mass larger than a few percent of the host halo mass. For less massive subjects, $t_{df} > t_H$
- Dynamical friction does not generally result in orbital circularization. More eccentric orbits decay faster.

Summary: key equations & expressions

Impulse Approximation

$$\Delta E_{\rm S} = \frac{1}{2}\int |\Delta\vec{v}(\vec{r})|^2\,\rho(r)\,{\rm d}^3\vec{r} = \frac{4}{3}G^2\,M_{\rm S}\left(\frac{M_{\rm P}}{v_{\rm P}}\right)^2\frac{\langle r^2\rangle}{b^4}$$

Tidal Radius

Point masses

$$r_{\rm t} = \left(\frac{m}{2M}\right)^{1/3} R$$

+ centrifugal force

$$r_{\rm t} = \left(\frac{m/M}{3+m/M}\right)^{1/3} R$$

+ extended mass distributions

$$r_{\rm t} \simeq \left[\frac{m(r_{\rm t})/M(R_0)}{2+\frac{\Omega^2 R_0^3}{G\,M(R_0)} - \frac{{\rm d}\ln M}{{\rm d}\ln R}\big|_{R_0}}\right]^{1/3} R_0$$

Impulse Approximation

Chandrasekhar dynamical friction force

$$\vec{F}_{\rm df} = M_{\rm S}\frac{{\rm d}\vec{v}_{\rm S}}{{\rm d}t} = -4\pi\left(\frac{GM_{\rm S}}{v_{\rm S}}\right)^2 \ln\Lambda\,\rho(<v_{\rm S})\,\frac{\vec{v}_{\rm S}}{v_{\rm S}}$$

Coulomb logarithm

$$\ln\Lambda = \ln\left(\frac{b_{\rm max}}{b_{90}}\right) \approx \ln\left(\frac{M_{\rm h}}{M_{\rm s}}\right)$$

dynamical friction time scale (isothermal sphere)

$$t_{\rm df} = \frac{1.17}{\ln\Lambda}\left(\frac{r_{\rm i}}{r_{\rm h}}\right)^2\left(\frac{M_{\rm h}}{M_{\rm S}}\right)\frac{r_{\rm h}}{V_{\rm c}}$$

evolution of orbital eccentricity

$$\frac{{\rm d}e}{{\rm d}t} = \frac{\eta}{v}\frac{{\rm d}e}{{\rm d}\eta}\left[1-\left(\frac{v}{V_{\rm c}}\right)^2\right]\frac{{\rm d}v}{{\rm d}t}$$

Lecture 15

Heating & Cooling

Summary: key words & important facts

Key words	
Hydro-static Equilibrium	Overcooling Problem
Accretion Shock	Cold mode vs. Hot mode
Virial Temperature	Ionization equilibrium
Cooling Function	Photo-ionization heating

- Mass estimates based on the assumption of hydrostatic equilibrium need to correct for non-thermal pressure sources (turbulence, magnetic fields, cosmic rays)
- Gas infalling in a halo through an accretion shock is heated to the virial temperature at which the gas is in hydrostatic, virial equilibrium with the halo potential.
- Low mass halos ($M_h < 10^{12}\,M_\odot$) are predicted to experience cold mode accretion (via streams), as they can't support an accretion shock.
- When ignoring photo-ionizations, it is typically assumed that the gas is in collisional ionization equilibrium (CIE) → one uses CIE cooling functions
- In the presence of photo-ionization, the net heating/cooling rate, $(\mathcal{C} - \mathcal{H})/n_{\rm H}^2$, is a function of both temperature and density. This arises because of the competition between photo-ionization & recombination.

Summary: key equations & expressions

Hydrostatic Equilibrium

$$\nabla P(r) = -\rho_{\rm gas} \nabla \Phi(r)$$

Spherical Symmetry

$$\nabla \Phi = \frac{\mathrm{d}\Phi}{\mathrm{d}r} = \frac{G\,M(r)}{r^2}$$

Ideal Gas

$$\nabla P = \frac{\mathrm{d}P}{\mathrm{d}r} = \frac{k_{\rm B}}{\mu m_{\rm p}} \frac{\mathrm{d}}{\mathrm{d}r}(\rho T)$$

$$M(r) = M_{\rm gas}(r) + M_{\rm DM}(r) = -\frac{k_{\rm B}\,T(r)\,r}{\mu m_{\rm p}\,G}\left[\frac{\mathrm{d}\ln \rho_{\rm gas}}{\mathrm{d}\ln r} + \frac{\mathrm{d}\ln T}{\mathrm{d}\ln r}\right]$$

virial temperature

$$T_{\rm vir} = \frac{\mu m_{\rm p}}{2\,k_{\rm B}} V_{\rm vir}^2 \simeq 3.6 \times 10^5\,\mathrm{K} \left(\frac{V_{\rm vir}}{100\,\mathrm{km/s}}\right)^2$$

Photo-ionization heating rate:

$$\mathcal{H} = n_{\rm H_0}\,\varepsilon_{\rm H_0} + n_{\rm He_0}\,\varepsilon_{\rm He_0} + n_{\rm He^+}\,\varepsilon_{\rm He^+}$$

where $$\varepsilon_i = \int_{\nu_i}^{\infty} \frac{4\pi\,J(\nu)}{h_{\rm P}\,\nu}\,\sigma_i(\nu)\,[h_{\rm P}\nu - h_{\rm P}\nu_i]\,\mathrm{d}\nu$$

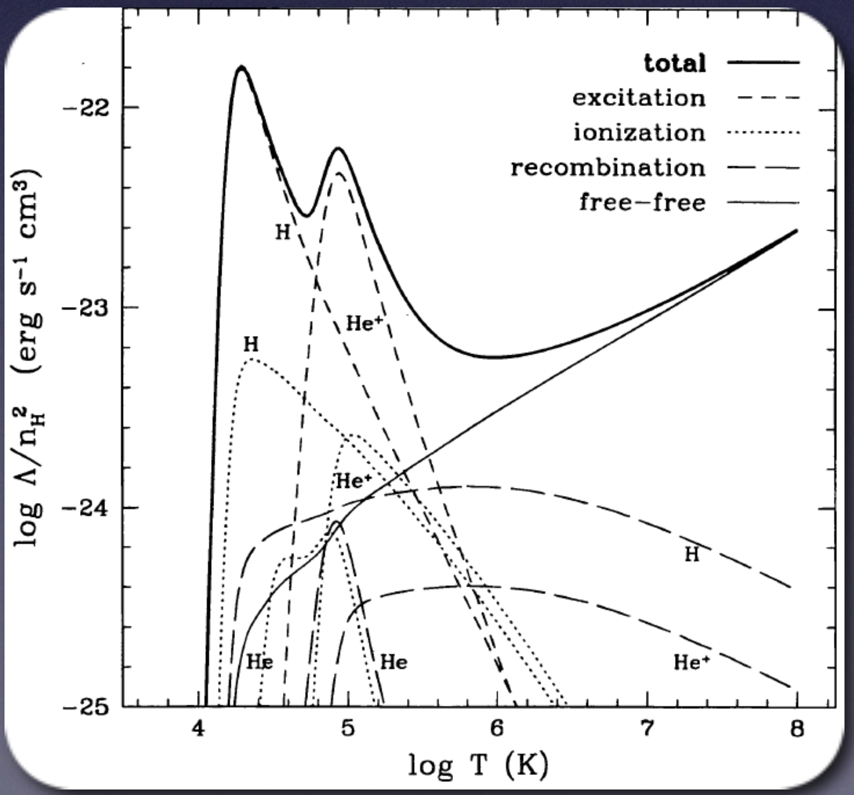
Cooling Time

$$t_{\rm cool} = \frac{3n\,k_{\rm B}T}{2\,n_{\rm H}^2\,\Lambda(T)}$$

Cooling Function

$$\Lambda \equiv \frac{\mathcal{C} - \mathcal{H}}{n_{\rm H}^2} = \Lambda(T, n_{\rm H})$$

in presence of heating



Lecture 16

Star Formation

Summary

- At high gas densities ($\Sigma_{\rm gas} > 10\,M_{\odot}{\rm pc}^{-2}$) conditions are such that self-shielding becomes efficient, and molecular gas forms.

- Various mechanisms trigger instabilities, creating GMCs supported by supersonic turbulence.

- Turbulent compression creates clumps and cores; the latter are Jeans unstable and collapse to form stars.
  Overall SFE per GMC is low $\varepsilon_{\rm SF,GMC} \sim 0.002$

- At low gas densities ($\Sigma_{\rm gas} < 10\,M_{\odot}{\rm pc}^{-2}$) star formation is suppressed, due to inability for gas to self-shield (i.e., form molecules), and due to reduced self-gravity, which enhances stability.

- Mergers and tidal interactions cause efficient transport of angular momentum out (funneling gas in). This boosts efficiency of creating GMCs, so that galaxy enters starburst phase.

- Energy and momentum injection due to star formation process itself is likely to be important regulator of star formation efficiency in GMCs.

Lecture 17

Supernova Feedback

Summary

- SN feedback is essential ingredient of galaxy formation. It helps explain why overall SF efficiency is low, and is invoked to explain why galaxy formation is less efficient in lower mass haloes...
- Unless SN go off in hot, low density medium, almost all SN energy is radiated away during radiative phase of blastwave.
- Efficient SN feedback requires three-phase ISM envisioned by McKee & Ostriker (1977), with most volume (mass) being in hot (cold) phase.
- Radiation pressure, stellar winds, and photo-ionization seem to be crucial ingredients for paving the way for efficient SN feedback.
- Numerical simulations that include wide spectrum of stellar feedback processes yields mass-loading efficiencies that scale similar to momentum-driven winds.
- They predict multi-phase winds; winds properties depend on SFR rate; radiation pressure becomes more important in systems with higher SFRs.