ASTR 610: Problem Set 3

This problem set consists of 4 problems.
Due date: Wed Mar 6, 2024

Problem 1: Spherical Collapse `[6 points]`
According to the SC model, the parametric solution to the evolution of a mass shell is

$$r = A\,(1 - \cos\theta)$$

$$t = B\,(\theta - \sin\theta)$$

where $A^3 = G\,M\,B^2$, which implies that

$$1 + \delta = \frac{9}{2}\frac{(\theta - \sin\theta)^2}{(1 - \cos\theta)^3}$$

Show that at early times (when $\theta \ll 1$) one has that

$$\delta_{\rm i} = \frac{3}{20}\,(6\pi)^{2/3}\,\left(\frac{t_{\rm i}}{t_{\rm max}}\right)^{2/3}$$

Hint: use Taylor series expansions.

Problem 2: The Zel'dovich Approximation
In this problem we seek to characterize the displacement $\psi(t)$ defined by

$$\vec{x}(t) = \vec{x}_{\mathrm{i}} + \psi(t)$$

where $\vec{x}(t)$ is the comoving coordinate of a particle. Obviously we have that

$$\psi(t) = \int_{t_{\mathrm{i}}}^{t} \frac{v(t)}{a(t)}\,\mathrm{d}t$$

where $v(t)$ is the particle's peculiar velocity. Under the Zel'dovich approximation, the gradient of the potential (which defines the direction in which the particle moves), can be written as $\nabla\Phi(t) = f(t)\nabla\Phi_{\mathrm{i}}$, where $f(t)$ is some function (to be determined) of time.

a) `[4 points]` Use the linearized Euler equation for a pressureless fluid to show that

$$\frac{\mathrm{d}}{\mathrm{d}t}(a\vec{v}) = -\nabla\Phi$$

b) `[6 points]` Use the fact that, at early times, the Universe behaves as an EdS cosmology to show that

$$\vec{v} = -\frac{\nabla\Phi_{\mathrm{i}}}{a}\int \frac{D(a)}{a}\,\mathrm{d}t$$

c) `[6 points]` Use the fact that $D(a)$ is a solution of the linearized fluid equation of a pressureless fluid to show that

$$\frac{D(a)}{a} = \frac{1}{4\pi G\bar{\rho}_{\mathrm{i}}}\frac{\mathrm{d}(a^2\dot{D})}{\mathrm{d}t}$$

Hint: you may use that the scale factor is normalized such that $a_{\mathrm{i}} = 1$.

d) `[6 points]` Use the above results to show that the displacement

$$\psi(t) = -\frac{D(a)}{4\pi G\bar{\rho}_{\mathrm{i}}}\nabla\Phi_{\mathrm{i}}$$

Problem 3: The two-point correlation function and σ_8

Let M be the mass inside a top-hat filter. The expectation value for M, i.e., the average value obtained by putting down the top-hat filter at many different locations, is simply $\langle M\rangle = \bar{\rho}\, V$ where V is the volume of the top-hat. Similarly, one can show that

$$\langle M^2\rangle = \langle M\rangle^2 + \frac{\langle M\rangle^2}{V^2}\int_V \xi(|\vec{x}_1 - \vec{x}_2|)\,\mathrm{d}^3\vec{x}_1\,\mathrm{d}^3\vec{x}_2$$

a) `[5 points]` Show that the mass variance, $\sigma^2(M)$, can be written as

$$\sigma^2(M) = \frac{3}{R^3}\int_0^R \xi(r)\,r^2\mathrm{d}r$$

where $R = R(M)$ is the size of the top-hat filter.

b `[6 points]` The first measurements of the two-point correlation function of galaxies revealed a power-law $\xi(r) = (r/r_0)^\gamma$ with $r_0 = 5h^{-1}$Mpc and $\gamma = -1.8$. Under the assumption that galaxies are unbiased tracers of the mass distribution, what does this imply for the value of σ_8?

Problem 4: Power spectrum and Mass variance

Let the matter power spectrum be a pure power-law, $P(k) \propto k^n$.

a) `[5 points]` Using a sharp k-space filter, show that the mass variance $\sigma^2(M) \propto M^\gamma$, and give the relation between γ and n.

b `[3 points]` Repeat the same exersize as under **(a)**, but this time using a Gaussian filter.

c `[3 points]` Give the ratio of the mass variances computed using the Gaussian filter and the sharp k-space filter for the case $n = 1$. Do NOT use mathematica (or similar), but first express your answer in terms of a special function, prior to giving the numerical value of the ratio.