ASTR 610: Solutions to Problem Set 2

Problem 1: Mass Variance
Define

$$M(\vec{x};R) \equiv V(R) \int \rho(\vec{x}')\, W(\vec{x}-\vec{x}';R)\, \mathrm{d}^3\vec{x}'$$

with $V(R)$ the volume associated with filter $W(\vec{x};R)$, and $\bar{M}(R) \equiv \langle M(\vec{x};R)\rangle$. Show that

$$\left\langle \left(\frac{M(\vec{x};R) - \bar{M}(R)}{\bar{M}(R)} \right)^2 \right\rangle = \sigma^2(M) \equiv \frac{1}{2\pi^2} \int P(k)\, \tilde{W}^2(kR)\, k^2\, \mathrm{d}k$$

`Hint:` Use that $\sigma^2(M) = \langle \delta^2(\vec{x};R)\rangle$.

ANSWER: Using that $\rho(\vec{x}) = \bar{\rho}\,[1+\delta(\vec{x})]$ we can write

$$\begin{aligned} M(\vec{x};R) &= V(R) \int [\bar{\rho} + \bar{\rho}\delta(\vec{x})]\, W(\vec{x}-\vec{x}';R)\, \mathrm{d}^3\vec{x} \\ &= V(R)\,\bar{\rho} + V(R)\,\bar{\rho} \int \delta(\vec{x}) W(\vec{x}-\vec{x}';R)\, \mathrm{d}^3\vec{x} \\ &= V(R)\,\bar{\rho} + V(R)\,\bar{\rho}\,\delta(\vec{x};R) \end{aligned}$$

where we have used the normalization condition

$$\int W(\vec{x};R)\, \mathrm{d}^3\vec{x} = 1$$

Using the above, we infer that

$$\bar{M}(R) \equiv \langle M(\vec{x};R)\rangle = V(R)\bar{\rho} + V(R)\bar{\rho}\langle\delta(\vec{x};R)\rangle = V(R)\bar{\rho}$$

where we have used that

$$\begin{aligned} \langle\delta(\vec{x};R)\rangle &= \langle \int \delta(\vec{x})\, W(\vec{x}-\vec{x}';R)\, \mathrm{d}^3\vec{x}\rangle \\ &= \int \langle\delta(\vec{x})\rangle W(\vec{x}-\vec{x}';R)\, \mathrm{d}^3\vec{x}\rangle = 0 \end{aligned}$$

The last step follows from the fact that $\langle\delta(\vec{x})\rangle = 0$. Combining, we have that

$$\frac{M(\vec{x};R)-\bar{M}(R)}{\bar{M}(R)} = \frac{V(R)\bar{\rho}\,[1+\delta(\vec{x};R)]-V(R)\bar{\rho}}{V(R)\bar{\rho}} = \delta(\vec{x};R)$$

so that

$$\left\langle\left(\frac{M(\vec{x};R)-\bar{M}(R)}{\bar{M}(R)}\right)^2\right\rangle = \sigma^2(M)$$

Problem 2: Free Streaming
Consider a flat ΛCDM cosmology with $\Omega_{\mathrm{m},0} = 0.3$ and $h = 0.7$. Assume that the dark matter particles decouple at $z_{\mathrm{dec}} = 10^{10}$ and have a mass of 2 Gev.

a) At what redshift do the dark matter particles become non-relativistic?

ANSWER: The dark matter particles become non-relativisty when $3k_{\mathrm{B}}T = mc^2$. Using that $T = T_{\mathrm{CMB}} = 2.7\mathrm{K}(1+z)$ we have that

$$(1+z_{\mathrm{NR}}) = \frac{mc^2}{3k_{\mathrm{B}}2.7} = \frac{2\times10^9\mathrm{eV}\times1.6\times10^{-12}\mathrm{erg\,eV}^{-1}}{3\times2.7\mathrm{K}\times1.381\times10^{-16}\mathrm{ergK}^{-1}} = 2.9\times10^{12}$$

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b) Show that the comoving free-streaming length at matter-radiation equality can be written as

$$\lambda_{\mathrm{fs}}(t_{\mathrm{eq}}) = \frac{2ct_{\mathrm{NR}}}{a_{\mathrm{NR}}}\left[\left(\frac{a_{\mathrm{dec}}}{a_{\mathrm{NR}}}\right)^{1/2}\left\{2+\ln\left(\frac{a_{\mathrm{eq}}}{a_{\mathrm{dec}}}\right)\right\}-1\right]$$

Hint: use that, during the radiation dominated era $a = a_{\mathrm{NR}}(t/t_{\mathrm{NR}})^{1/2|}$

ANSWER: The comoving free streaming length is given by

$$\lambda_{\mathrm{fs}} = \int_0^{t_{\mathrm{eq}}}\frac{v(t)}{a(t)}\mathrm{d}t = \int_0^{t_{\mathrm{NR}}}\frac{v(t)}{a(t)}\mathrm{d}t + \int_{t_{\mathrm{NR}}}^{t_{\mathrm{dec}}}\frac{v(t)}{a(t)}\mathrm{d}t + \int_{t_{\mathrm{dec}}}^{t_{\mathrm{eq}}}\frac{v(t)}{a(t)}\mathrm{d}t \equiv I_1 + I_2 + I_3$$

Here we have split the integral in three parts corresponding to the following periods:

$$\begin{array}{rll} t < t_{\rm NR} & \text{for which} & v(t) = c \\ t_{\rm NR} < t < t_{\rm dec} & \text{for which} & v(t) = c\left(\dfrac{a_{\rm NR}}{a}\right)^{1/2} \\ t_{\rm dec} < t < t_{\rm eq} & \text{for which} & v(t) = c\left(\dfrac{a_{\rm NR}}{a_{\rm dec}}\right)^{1/2}\left(\dfrac{a_{\rm dec}}{a}\right) \end{array}$$

Using that for $t < t_{\rm eq}$ the scale radius evolves with time as

$$a(t) = a_{\rm NR}\left(\frac{t}{t_{\rm NR}}\right)^{1/2}$$

we have that

$$\frac{{\rm d}a}{{\rm d}t} = \frac{1}{2}a_{\rm NR}\left(\frac{t}{t_{\rm NR}}\right)^{-1/2}\frac{1}{t_{\rm NR}} = \frac{1}{2}\frac{a_{\rm NR}^2}{a(t)\,t_{\rm NR}}$$

This allows us to write that

$$\frac{{\rm d}t}{a(t)} = \frac{2t_{\rm NR}}{a_{\rm NR}^2}{\rm d}a$$

Using this it is straightforward to compute the above three integrals:

$$I_1 = \int_0^{t_{\rm NR}}\frac{c}{a(t)}\,{\rm d}t = \frac{2ct_{\rm NR}}{a_{\rm NR}^2}\int_0^{a_{\rm NR}}{\rm d}a = \frac{2ct_{\rm NR}}{a_{\rm NR}}$$

$$\begin{aligned} I_2 &= c\int_{t_{\rm NR}}^{t_{\rm dec}}\left(\frac{a_{\rm NR}}{a(t)}\right)^{1/2}\frac{2t_{\rm NR}}{a_{\rm NR}^2}\,{\rm d}a = \frac{2ct_{\rm NR}}{a_{\rm NR}^{3/2}}\int_{a_{\rm NR}}^{a_{\rm dec}}\frac{{\rm d}a}{a^{1/2}} \\ &= \frac{4ct_{\rm NR}}{a_{\rm NR}^{3/2}}\left[a_{\rm dec}^{1/2} - a_{\rm NR}^{1/2}\right] = \frac{4ct_{\rm NR}}{a_{\rm NR}^{3/2}}\left[\left(\frac{a_{\rm dec}^{1/2}}{a_{\rm NR}^{1/2}}\right)^{1/2} - 1\right] \end{aligned}$$

$$
\begin{aligned}
I_3 &= c\left(\frac{a_{\rm NR}}{a_{\rm dec}}\right)^{1/2}\int_{t_{\rm dec}}^{t_{\rm eq}}\frac{a_{\rm dec}}{a(t)}\,\frac{2t_{\rm NR}}{a_{\rm NR}^2}\,{\rm d}a = \frac{2ct_{\rm NR}}{a_{\rm NR}^{3/2}}a_{\rm dec}^{1/2}\int_{a_{\rm dec}}^{a_{\rm eq}}\frac{{\rm d}a}{a} \\
&= \frac{2ct_{\rm NR}}{a_{\rm NR}}\left(\frac{a_{\rm dec}}{a_{\rm NR}}\right)^{1/2}\ln\left(\frac{a_{\rm eq}}{a_{\rm dec}}\right)
\end{aligned}
$$

Combining these results, we finally obtain that

$$
\begin{aligned}
\lambda_{\rm fs} &= \frac{2ct_{\rm NR}}{a_{\rm NR}}\left[1+2\left\{\left(\frac{a_{\rm dec}}{a_{\rm NR}}\right)^{1/2}-1\right\}+\left(\frac{a_{\rm dec}}{a_{\rm NR}}\right)^{1/2}\ln\left(\frac{a_{\rm eq}}{a_{\rm dec}}\right)\right] \\
&= \frac{2ct_{\rm NR}}{a_{\rm NR}}\left[\left(\frac{a_{\rm dec}}{a_{\rm NR}}\right)^{1/2}\left\{2+\ln\left(\frac{a_{\rm eq}}{a_{\rm dec}}\right)\right\}-1\right]
\end{aligned}
$$

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c) What is the ratio between $\lambda_{\rm fs}(t_{\rm eq})$ and the comoving particle horizon, $\lambda_{\rm H}$, at $t_{\rm NR}$? Compute the actual, numerical value of $\lambda_{\rm fs}(t_{\rm eq})/\lambda_{\rm H}(t_{\rm NR})$.

ANSWER: The comoving particles horizon at $t_{\rm NR}$ is given by

$$
\lambda_{\rm H} = \int_0^{t_{\rm NR}}\frac{c\,{\rm d}t}{a(t)} = \frac{2\,c\,t_{\rm NR}}{a_{\rm NR}^2}\int_0^{a_{\rm NR}}{\rm d}a = \frac{2\,c\,t_{\rm NR}}{a_{\rm NR}}
$$

Hence, we have that

$$
\frac{\lambda_{\rm fs}(t_{\rm eq})}{\lambda_{\rm H}(t_{\rm NR})} = \left(\frac{a_{\rm dec}}{a_{\rm NR}}\right)^{1/2}\left[2+\ln\left(\frac{a_{\rm eq}}{a_{\rm dec}}\right)\right]-1
$$

Using that

$$
\begin{aligned}
a_{\rm NR} = \frac{1}{1+z_{\rm NR}} &= \frac{1}{2.9\times 10^{12}} \\
a_{\rm dec} = \frac{1}{1+z_{\rm dec}} &= \frac{1}{10^{10}} \\
a_{\rm eq} = \frac{1}{1+z_{\rm eq}} &= \frac{1}{3528}
\end{aligned}
$$

For the latter we have used that $(1+z_{\rm eq}) = 2.4\times10^4\Omega_{\rm m,0}h^2 = 2.4\times10^4\cdot0.3\cdot(0.7)^2 = 3528$. Substituting these values we find that

$$\frac{\lambda_{\rm fs}(t_{\rm eq})}{\lambda_{\rm H}(t_{\rm NR})} = 286$$

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d) What is the free-streaming mass at matter-radiation equality? `Hint:` use eq. (3.80) in MBW.

ANSWER: The free streaming mass at equality is

$$M_{\rm fs} = \frac{\pi}{6}\,\bar\rho\,(\lambda_{\rm fs}^{\rm prop})^3 = \frac{\pi}{6}\,\bar\rho_0\,(\lambda_{\rm fs}^{\rm com})^3$$

Using that $\bar\rho_0 = \Omega_{\rm m,0}\,\rho_{\rm crit,0}$, with $\rho_{\rm crit,0} = 2.78\times10^{11}h^{-1}\,{\rm M}_\odot/(h^{-1}\,{\rm Mpc})^3$ we find that

$$M_{\rm fs} = 4.36\times10^{10}h^{-1}\,{\rm M}_\odot\left(\frac{\lambda_{\rm fs}^{\rm com}}{h^{-1}\,{\rm Mpc}}\right)^3$$

For the comoving free-streaming length we have that

$$\lambda_{\rm fs}^{\rm com} = 286\frac{2\,c\,t_{\rm NR}}{a_{\rm NR}}$$

Evaluating this quantity requires that we first compute $t_{\rm NR}$. For this we use that

$$a(t) = \left(\frac{32\,\pi\,G\,\rho_{\rm r,0}}{3}\right)^{1/4}\,t^{1/2}$$

[see eq.(3.80) in MBW]. Using that $\Omega_{\rm r,0} = 4.2\times10^{-5}h^{-2}$ and that $z_{\rm NR} = 2.9\times10^{12}$ we find that $t_{\rm NR} = 2.83\times10^{-6}$s. Substitution in the equation for the free-streaming length yields that $\lambda_{\rm fs}^{\rm com} = 45.6\,{\rm pc} = 4.56\times10^{-5}\,{\rm Mpc}$. Substituting this in the expression for the free-streaming mass, and using that $h = 0.7$, we finally find that $M_{\rm fs} = 2.0\times10^{-3}\,{\rm M}_\odot$

Problem 3: Spherical Collapse
According to the SC model, the parametric solution to the evolution of a mass shell is

$$r = A\,(1 - \cos\theta)$$

$$t = B\,(\theta - \sin\theta)$$

where $A^3 = G\,M\,B^2$, which implies that

$$1 + \delta = \frac{9}{2}\frac{(\theta - \sin\theta)^2}{(1 - \cos\theta)^3}$$

Show that at early times (when $\theta \ll 1$) one has that

$$\delta_{\rm i} = \frac{3}{20}\,(6\pi)^{2/3}\,\left(\frac{t_{\rm i}}{t_{\rm max}}\right)^{2/3}$$

Hint: use Taylor series expansions of $\sin\theta$ and $\cos\theta$ and the fact that $t_{\rm max} = \pi B$.

ANSWER: We have that

$$\begin{aligned}
\sin\theta &\simeq \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - ... \\
\cos\theta &\simeq 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - ...
\end{aligned}$$

where we can ignore the higher-order terms, since at early times $\theta \ll 1$. Hence,

$$\begin{aligned}
(\theta - \sin\theta)^2 &= \left(\frac{\theta^3}{6} - \frac{\theta^5}{120}\right)^2 = \frac{\theta^6}{36}\left[1 - \frac{\theta^2}{10} + \frac{\theta^4}{400}\right] \simeq \frac{\theta^6}{36}\left[1 - \frac{\theta^2}{10}\right] \\
(1 - \cos\theta)^3 &= \left(\frac{\theta^2}{2} - \frac{\theta^4}{24}\right)^3 = \frac{\theta^6}{8}\left[1 - \frac{\theta^2}{6} + \frac{\theta^4}{144} - \frac{\theta^2}{12} + \frac{\theta^4}{72} - \frac{\theta^6}{1728}\right] \simeq \frac{\theta^6}{8}\left[1 - \frac{\theta^2}{4}\right]
\end{aligned}$$

Combining, we find that

$$
\begin{aligned}
1+\delta_{\rm i} &= \frac{9}{2}\frac{\frac{\theta^6}{36}\left[1-\frac{\theta^2}{10}\right]}{\frac{\theta^6}{8}\left[1-\frac{\theta^2}{4}\right]} \\
&\simeq \left[1-\frac{\theta^2}{10}\right]\times\left[1+\frac{\theta^2}{4}\right] \\
&\simeq 1+\frac{3\,\theta^2}{20}
\end{aligned}
$$

from which we see that, to good approximation, $\delta_{\rm i} = 3\theta^2/20$. If we now use that $t = B\,(\theta - \sin\theta) \simeq B\,\theta^3/6$, we see that

$$\theta_{\rm i} \simeq \left(\frac{6\,t_{\rm i}}{B}\right)^{1/3} = \left(\frac{6\,\pi\,t_{\rm i}}{t_{\rm max}}\right)^{1/3}$$

where we have used that $t_{\rm max} = \pi\,B$. Substituting the above expression for $\theta_{\rm i}$ into the expression for $\delta_{\rm i}$, one finally obtains that

$$\delta_{\rm i} = \frac{3}{20}\,(6\pi)^{2/3}\,\left(\frac{t_{\rm i}}{t_{\rm max}}\right)^{2/3}$$

Problem 4: The Zel'dovich Approximation
In this problem we seek to characterize the displacement $\psi(t)$ defined by

$$\vec{x}(t) = \vec{x}_{\rm i} + \psi(t)$$

where $\vec{x}(t)$ is the comoving coordinate of a particle. Obviously we have that

$$\psi(t) = \int_{t_{\rm i}}^{t} \frac{v(t)}{a(t)}\,{\rm d}t$$

where $v(t)$ is the particle's peculiar velocity. Under the Zel'dovich approximation, the gradient of the potential (which defines the direction in which the particle moves), can be written as $\nabla\Phi(t) = f(t)\nabla\Phi_{\rm i}$, where $f(t)$ is some function (to be determined) of time.

a) Use the linearized Euler equation for a pressureless fluid to show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\,(a\vec{v}) = -\nabla\Phi$$

ANSWER: The linearized Euler equations for a pressureless fluid is given by

$$\frac{\partial \vec{v}}{\partial t} + \frac{\dot{a}}{a}\,\vec{v} = -\frac{\nabla\Phi}{a}$$

Using that

$$\frac{\mathrm{d}}{\mathrm{d}t}(a\vec{v}) = a\frac{\partial \vec{v}}{\partial t} + \vec{v}\,\frac{\partial a}{\partial t} = a\left(\frac{\partial \vec{v}}{\partial t} + \frac{\dot{a}}{a}\,\vec{v}\right)$$

Combining this with the linearlized Euler equations, it is immediately evident that

$$\frac{\mathrm{d}}{\mathrm{d}t}\,(a\vec{v}) = -\nabla\Phi$$

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b) `[5 points]` Use the fact that, at early times, the Universe behaves as an EdS cosmology to show that

$$\vec{v} = -\frac{\nabla\Phi_{\mathrm{i}}}{a}\int \frac{D(a)}{a}\,\mathrm{d}t$$

Hint: use that $\Phi_{\vec{k}} \propto D(a)/a$.

ANSWER: The fact that $\Phi_{\vec{k}} \propto D(a)/a$ implies that $\Phi \propto D(a)/a$, and therefore also $\nabla\Phi \propto D(a)/a$. This allows us to write that

$$\nabla\Phi = \frac{D(a)\,a_{\mathrm{i}}}{D(a_{\mathrm{i}})\,a}\,\nabla\Phi_{\mathrm{i}}$$

Since at early times the Universe behaves as an EdS cosmology, for which $D(a) = a$, we have that $D(a_{\mathrm{i}})/a_{\mathrm{i}} = 1$, so that

$$\nabla\Phi = \frac{D(a)}{a}\,\nabla\Phi_{\rm i}$$

Using what we inferred under **a)**, we therefore have that

$$\frac{\mathrm{d}}{\mathrm{d}t}\,(a\,\vec{v}) = -\frac{D(a)}{a}\,\nabla\Phi_{\rm i}$$

Integrating this equation yields

$$\int \mathrm{d}(a\,\vec{v}) = -\nabla\Phi_{\rm i}\int \frac{D(a)}{a}\,\mathrm{d}t$$

from which it is immediately evident that

$$\vec{v} = -\frac{\nabla\Phi_{\rm i}}{a}\,\int \frac{D(a)}{a}\,\mathrm{d}t$$

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c) `[6 points]` Use the fact that $D(a)$ is a solution of the linearized fluid equation of a pressureless fluid to show that

$$\frac{D(a)}{a} = \frac{1}{4\pi G\bar{\rho}_{\rm i}}\,\frac{\mathrm{d}(a^2\dot{D})}{\mathrm{d}t}$$

Hint: you may use that the scale factor is normalized such that $a_{\rm i} = 1$.

ANSWER: Since $D(a)$ is a solution of the linearized fluid equation for a pressureless fluid, we have that

$$\ddot{D} + 2\,\frac{\dot{a}}{a}\,\dot{D} = 4\pi G\,\bar{\rho}(a)\,D$$

Using that $\bar{\rho}(a) = \bar{\rho}_{\rm i}(a_{\rm i}/a)^3 = \bar{\rho}_{\rm i}a^{-3}$, where we have used that $a_{\rm i} = 1$, the above equation reduces to

$$\ddot{D} + 2\,\frac{\dot{a}}{a}\,\dot{D} = 4\pi G\,\bar{\rho}_{\rm i}\,\frac{D(a)}{a^3}$$

Next we use that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(a^2\dot{D}\right) = a^2\ddot{D} + 2a\dot{a}\dot{D} = a^2\left(\ddot{D} + 2\frac{\dot{a}}{a}\dot{D}\right)$$

to write that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(a^2\dot{D}\right) = a^2\,4\pi G\,\bar{\rho}_{\rm i}\,\frac{D(a)}{a^3} = 4\pi G\bar{\rho}_{\rm i}\,\frac{D(a)}{a}$$

Rearranging shows that

$$\frac{D(a)}{a} = \frac{1}{4\pi G\,\bar{\rho}_{\rm i}}\,\frac{\mathrm{d}(a^2\dot{D})}{\mathrm{d}t}$$

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d) `[5 points]` Use the above results to show that the displacement

$$\psi(t) = -\frac{D(a)}{4\pi G\bar{\rho}_{\rm i}}\,\nabla\Phi_{\rm i}$$

ANSWER: Under **b)** we derived that

$$\vec{v} = -\frac{\nabla\Phi_{\rm i}}{a}\int\frac{D(a)}{a}\,\mathrm{d}t$$

while under **c)** we demonstrated that

$$\frac{D(a)}{a} = \frac{1}{4\pi G\,\bar{\rho}_{\rm i}}\,\frac{\mathrm{d}(a^2\dot{D})}{\mathrm{d}t}$$

Substituting the latter in the former, we find that

$$\vec{v} = -\frac{\nabla\Phi_{\rm i}}{4\pi G\bar{\rho}_{\rm i}a}\int\mathrm{d}(a^2\dot{D}) = -\frac{\nabla\Phi_{\rm i}}{4\pi G\bar{\rho}_{\rm i}}\,a\,\frac{\mathrm{d}D}{\mathrm{d}t}$$

Hence, for the displacement we have that

$$\begin{aligned}
\psi(t) &= \int_{t_{\rm i}}^{t}\frac{v(t)}{a(t)}\,\mathrm{d}t = -\frac{\nabla\Phi_{\rm i}}{4\pi G\bar{\rho}_{\rm i}}\int_{D(a_{\rm i})}^{D(a)}\mathrm{d}D \\
&= -\frac{D(a) - D(a_{\rm i})}{4\pi G\bar{\rho}_{\rm i}}\,\nabla\Phi_{\rm i} \simeq -\frac{D(a)}{4\pi G\bar{\rho}_{\rm i}}\,\nabla\Phi_{\rm i}
\end{aligned}$$

where in the last step we have used that $D(a_{\mathrm{i}}) \ll D(a)$.
