ASTR 610: Problem Set 2

This problem set consists of 4 problems.
Due date: Wednesday Feb 21, 2024

Problem 1: The Sound-Speed of the Photon-Baryon fluid `[5 points]`
Consider a Universe that consists solely of baryons and photons (no dark matter, no dark energy, no neutrinos). Show that, during the radiation era, the sound speed of the photon-baryon fluid can be written as

$$c_{\rm s} = \frac{c}{\sqrt{3}} \left[\frac{3}{4} \frac{\bar{\rho}_{\rm b}(z)}{\bar{\rho}_\gamma(z)} + 1 \right]^{-1/2}$$

where $\bar{\rho}_{\rm b}(z)$ and $\bar{\rho}_\gamma(z)$ are the mean energy densities of baryons and photons at redshift z, and c is the speed of light.

Problem 2: Free Streaming
Consider a flat ΛCDM cosmology with $\Omega_{\rm m,0} = 0.3$ and $h = 0.7$, and with a CMB temperature (at present) of 2.7K. Assume that the dark matter particles decouple at $z_{\rm dec} = 10^{10}$ and have a mass of 2 Gev.

a) `[3 points]` At what redshift do the dark matter particles become non-relativistic?

b) `[5 points]` Show that the comoving free-streaming length at matter-radiation equality can be written as

$$\lambda_{\rm fs}(t_{\rm eq}) = \frac{2ct_{\rm NR}}{a_{\rm NR}} \left[\left(\frac{a_{\rm dec}}{a_{\rm NR}} \right)^{1/2} \left\{ 2 + \ln\left(\frac{a_{\rm eq}}{a_{\rm dec}} \right) \right\} - 1 \right]$$

Hint: use that, during the radiation dominated era $a = a_{\rm NR}(t/t_{\rm NR})^{1/2|}$

c) `[4 points]` What is the ratio between $\lambda_{\rm fs}(t_{\rm eq})$ and the comoving particle horizon, $\lambda_{\rm H}$, at $t_{\rm NR}$? Compute the actual, numerical value of $\lambda_{\rm fs}(t_{\rm eq})/\lambda_{\rm H}(t_{\rm NR})$.

d) `[8 points]` What is the free-streaming mass at matter-radiation equality? `Hint:` use eq. (3.80) in MBW.

Problem 3: Silk Damping

Consider a Universe consisting purely of baryons and radiation (no dark matter or dark energy), and ignore any elements heavier than hydrogen. Assume that $\Omega_{\rm m,0} = 0.3$, $h = 0.7$, and that recombination in this universe happens at a redshift $z_{\rm rec} \simeq 1100$, when the ionization fraction $X_{\rm e} \equiv n_{\rm e}/(n_{\rm p}+n_{\rm H}) = 0.1$. Here $n_{\rm e}$, $n_{\rm p}$ and $n_{\rm H}$ are the number density of free electons, free protons, and hydrogen atoms. A crude estimate for the Silk damping scale at time t, based on kinetic theory, is

$$\lambda_{\rm d} = \left(\frac{ct}{3\sigma_{\rm T} n_{\rm e}} \right)^{1/2}$$

(see lecture 5).

a) `[2 points]` Is this in physical or comoving units? Explain.

b) `[5 points]` Express the mean mass per particle, $\mu m_{\rm p}$, with $m_{\rm p}$ the mass of a proton, in terms of the ionization fraction $X_{\rm e}$

c) `[5 points]` Show that the number density of free electrons at recombination can be written as

$$n_{\rm e} \simeq 1.5 \times 10^4 \, {\rm cm}^{-3} \, X_{\rm e} \, (\Omega_{\rm m,0} h^2)$$

d) `[5 points]` Derive the Silk damping scale, in comoving Mpc, at the epoch of recombination. Use that $X_{\rm e} \sim 0.1$ at that epoch, and that recombination occurs during the matter dominated era, so that

$$t(z) = \frac{2}{3} \frac{1}{H_0} (1+z)^{-3/2}$$

(cf Eq. (3.96) in MBW).

e) `[3 points]` Compute the corresponding Silk damping mass, $M_{\rm d}$, in Solar masses.

Problem 4: The Continuity Equation `[5 points]`

The continuity equation in physical coordinates, $\vec{r}(t)$, in Lagrangian form reads

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho \nabla_r \cdot \vec{u} = 0$$

Show that this equation can be written as

$$\frac{\partial \delta}{\partial t} + \frac{1}{a} \nabla_x \cdot [(1+\delta)\, \vec{v}] = 0$$

where δ is the overdensity, $\vec{x}$ are comoving coordinates and $\vec{v}$ are peculiar velocities. The latter are related to $\vec{r}$ and $\vec{u}$ according to $\vec{r} = a(t)\, \vec{x}$ and $\vec{u} = \dot{a}\, \vec{x} + \vec{v}$. You may use that the (physical) density ρ scales as a^{-3}.