ASTR 610: Solutions to Problem Set 1

Problem 1: In a hypothetical universe, $\Omega_\Lambda = 0$, the CMB has a temperature of 10.0 K, and the energy density of neutrinos, which are still relativistic at the present, is 1.5 times higher than that of the photons. What is the redshift of matter-radiation equality in units of $\Omega_{\rm m,0}\, h^2$?

ANSWER: The energy density of radiation is

$$\rho_{\rm r}(z) = \rho_{\rm r,0}(1+z)^4 = 2.5\rho_{\gamma,0}(1+z)^4 = \frac{10\sigma_{\rm SB}}{c^3}\, T_{\gamma,0}^4$$

The energy density of matter, on the other hand, is

$$\rho_{\rm m}(z) = \rho_{\rm m,0}(1+z)^3 = \Omega_{\rm m,0}\frac{3H_0^2}{8\pi G}(1+z)^3$$

The redshift of equality, $z_{\rm eq}$ is defined by $\rho_{\rm r}(z_{\rm eq}) = \rho_{\rm m}(z_{\rm eq})$, which implies

$$(1+z_{\rm eq}) = \frac{3H_0^2 c^3}{80\pi G \sigma_{\rm SB} T_{\gamma,0}^4}\Omega_{\rm m,0}$$

Using that $1/H_0 = 9.78h^{-1}$Gyr (see MBW Appendix E), one easily obtains that

$$(1+z_{\rm eq}) = 89\,\Omega_{\rm m,0}h^2$$

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Problem 2: Show that the equation of state (EoS) parameter of the non-relativistic matter component can be written as

$$w = w(T) = \frac{k_{\rm B}T}{\mu m_{\rm p} c^2}\left[1 + \frac{1}{\gamma - 1}\frac{k_{\rm B}T}{\mu m_{\rm p} c^2}\right]^{-1}$$

ANSWER: The energy density of non-relativistic matter is conveniently written as

$$\rho c^2 = \rho_{\rm m} c^2 + \rho_{\rm m}\varepsilon$$

where ε is the specific, internal energy of the matter (see MBW § 3.1.5). One can rewrite this as

$$\frac{\rho}{\rho_{\rm m}} = 1 + \frac{\varepsilon}{c^2}$$

which we will use below. The EoS for non-relativistic matter is well approximated by that of an ideal gas:

$$P = \frac{k_{\rm B}T}{\mu m_{\rm p}}\rho_{\rm m}$$

Using that the EoS parameter w is defined by $P = w\rho c^2$, we write this as

$$P = \frac{k_{\rm B}T}{\mu m_{\rm p}c^2}\frac{\rho_{\rm m}}{\rho}\rho c^2$$

from which it is immediately evident that

$$w = \frac{k_{\rm B}T}{\mu m_{\rm p}c^2}\frac{\rho_{\rm m}}{\rho}$$

Using that the specific, internal energy of an ideal gas is given by

$$\varepsilon = \frac{1}{\gamma - 1}\frac{k_{\rm B}T}{\mu m_{\rm p}}$$

(see MBW Appendix B), we have that

$$\frac{\rho}{\rho_{\rm m}} = 1 + \frac{\varepsilon}{c^2} = 1 + \frac{1}{\gamma - 1}\frac{k_{\rm B}T}{\mu m_{\rm p}c^2}$$

Combining these results it is clear that we can write

$$w = w(T) = \frac{k_{\rm B}T}{\mu m_{\rm p}c^2}\left[1 + \frac{1}{\gamma - 1}\frac{k_{\rm B}T}{\mu m_{\rm p}c^2}\right]^{-1}$$

Problem 3: Using that, in a homogeneous and isotropic expanding universe, the equation of motion of a shell of matter is given by

$$\ddot{R} = -\frac{GM}{R^2}$$

with $M = M(< R)$ ythe mass enclosed by that shell. Show that the Laplacian of $\Phi \equiv \phi + a\ddot{a}x^2/2$ with respect to the comoving coordinate $\vec{x}$ is equal to $4\pi G\bar{\rho}a^2\delta$, indicating that Φ is only 'sourced' by the density contrast $\bar{\rho}\delta = \rho - \bar{\rho}$. Here ϕ is the Newtonian gravitational potential, and $a = a(t)$ is the scale factor.

ANSWER: The Laplacian is

$$\nabla_x^2\Phi = \nabla_x^2(\phi + a\ddot{a}x^2/2) = a^2\,\nabla_r^2(\phi + a\ddot{a}x^2/2)$$

where $\vec{r} = a\vec{x}$ are the proper coordinates. Using the Poisson equation (which holds in proper coordinates), we can write this as

$$\nabla_x^2\Phi = a^2\left(4\pi G\rho + \frac{\ddot{a}}{2a}\nabla_r^2 r^2\right) = a^2\left(4\pi G\rho + \frac{\ddot{a}}{2a}\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}r^2\right)$$
$$= a^2\left(4\pi G\rho + 3\frac{\ddot{a}}{a}\right) \qquad .$$

Now we use that

$$\ddot{R} = -\frac{GM}{R^2}$$

Using that $M = \frac{4\pi}{3}\bar{\rho}R^3$ and writing that $R = a(t)R_0$, we find that

$$\ddot{a}R_0 = -\frac{4\pi G}{3}\bar{\rho}aR_0$$

which implies that

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\bar{\rho}$$

Substituting this in the equation for our Laplacian we find that

$$\nabla_x^2\Phi = a^2\left(4\pi G\rho - 4\pi G\bar{\rho}\right) = 4\pi G a^2(\rho - \bar{\rho}) = 4\pi G\bar{\rho}a^2\delta$$

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Problem 4a: Use the Friedmann equation to show that before matter-radiation equality

$$\left(\frac{a}{a_0}\right) = \left(\frac{32\pi G\rho_{\mathrm{r},0}}{3}\right)^{1/4} t^{1/2}$$

ANSWER: The Friedmann equation in the radiation dominated era is given by

$$\frac{\dot{a}}{a} = H_0 \left[\Omega_{\rm r,0}(1+z)^4\right]^{1/2} = \sqrt{\frac{8\pi G\rho_{\rm r,0}}{3}} \left(\frac{a}{a_0}\right)^{-2}$$

We can rewrite this as

$$\frac{{\rm d}(a/a_0)}{{\rm d}t} = \sqrt{\frac{8\pi G\rho_{\rm r,0}}{3}}\,\frac{a_0}{a}$$

Integration yields

$$\int_0^{a/a_0} \frac{a}{a_0}{\rm d}(a/a_0) = \sqrt{\frac{8\pi G\rho_{\rm r,0}}{3}} \int_0^t {\rm d}t$$

which reduces to

$$\frac{1}{2}\left(\frac{a}{a_0}\right)^2 = \sqrt{\frac{8\pi G\rho_{\rm r,0}}{3}}\;\; t$$

from which it is immediately clear that

$$\left(\frac{a}{a_0}\right) = \left(\frac{32\pi G\rho_{\rm r,0}}{3}\right)^{1/4} t^{1/2}$$

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b) Use this to show that the proper particle horizon during this era is $\lambda_{\rm H} = 2ct$.

ANSWER: The proper particle horizon is given by

$$\lambda_{\rm H}^{\rm prop} = a\lambda_{\rm H}^{\rm com} = a\int_0^t \frac{c\,{\rm d}t}{a(t)}$$

Using the expression for the scale-factor in the radiation dominated era derived above we have

$$\lambda_{\rm H}^{\rm prop} = \frac{a\,c}{\beta}\int_0^t \frac{{\rm d}t}{t^{1/2}} = \frac{2ac}{\beta}t^{1/2}$$

where we have used the shorthand notation

$$\beta = \left(\frac{32\pi G\rho_{\rm r,0}}{3}\right)^{1/4}$$

Finally, subtituting that $a = \beta t^{1/2}$ yields the required result that $\lambda_{\rm H}^{\rm prop} = 2ct$.

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c) Express the proper Jeans length during this era in units of $\lambda_{\rm H}$.

ANSWER: The definition for the proper Jeans length is

$$\lambda_{\rm J}^{\rm prop} = c_{\rm s}\sqrt{\frac{\pi}{G\bar{\rho}}}$$

Using that the sound speed during the radiation dominated era is $c_{\rm s} = c/\sqrt{3}$, and that $\bar{\rho} = \bar{\rho}_{\rm r} = \bar{\rho}_{\rm r,0}a^{-4}$, we have

$$\lambda_{\rm J}^{\rm prop} = c\sqrt{\frac{\pi}{3G\bar{\rho}_{\rm r,0}}}a^2 = \sqrt{\frac{32\pi^2}{9}}\;c\,t$$

In the last step we have used that $a = \beta t^{1/2}$. Thus, we have that

$$\lambda_{\rm J}^{\rm prop} = \frac{2\sqrt{2}\pi}{3}\lambda_{\rm H}^{\rm prop} \simeq 2.96\lambda_{\rm H}^{\rm prop}$$

Problem 5: Primordial Matter
Primordial gas has zero metallicity and a Helium mass fraction of $Y = 0.25$. What is the mean particle mass μ, in units of the proton mass $m_{\rm p}$, for such a primordial gas when it is fully ionized? And what are the number densities of Hydrogen and Helium nuclei relative to that of electrons?

ANSWER: Using that a Helium nucleus weighs approximately 4 proton masses, (mass of proton is very similar to that of a neutron), a Hydrogen atom 1 proton mass, while the masses of electrons are negligble. Using that the mean mass per particle is equal to the total mass per volume divided by the total number of particles per volume, it is easy to see that

$$\mu = \frac{n_{\rm H} \cdot 1 + n_{\rm He} \cdot 4}{n_{\rm H} \cdot 2 + n_{\rm He} \cdot 3} = \frac{1 + 4(n_{\rm He}/n_{\rm H})}{2 + 3(n_{\rm He}/n_{\rm H})}$$

where we have used that there is 1 electron per Hydrogren nucleus, and 2 electrons per Helium nucleus. Using that $n_{\rm He} = \rho_{\rm He}/m_{\rm He} = \rho_{\rm He}/(4m_{\rm p})$ and $n_{\rm H} = \rho_{\rm H}/m_{\rm p}$, while

$$Y = \frac{\rho_{\rm He}}{\rho_{\rm H} + \rho_{\rm He}}$$

we have that

$$\frac{n_{\rm He}}{n_{\rm H}} = \frac{\rho_{\rm He}}{4\,\rho_{\rm H}} = \frac{1}{4}\left(\frac{1}{Y} - 1\right)^{-1} = \frac{1}{12}$$

Substituting in the equation for μ we obtain that $\mu = 16/27 \simeq 0.59$.
In order to compute the number densities of Hydrogen and Helium nuclei relative to that of electrons, we use that the number density of electrons is equal to that of protons, and thus $n_{\rm e} = n_{\rm p} = n_{\rm H} + 2n_{\rm He}$. Dividing by $n_{\rm H}$ and using that $n_{\rm He}/n_{\rm H} = 1/12$, one trivially obtains that $n_{\rm H}/n_{\rm e} = 6/7$ and $n_{\rm He}/n_{\rm e} = 1/14$.

Problem 6:
The Einstein-de Sitter (EdS) cosmology is defined as a flat, matter dominated cosmology without cosmological constant. In an EdS cosmology the universe is always matter dominated; it never experiences a phase of radiation domination.

a) Use the Friedmann equation to show that in an EdS cosmology

$$\left(\frac{a}{a_0}\right) = \left(\frac{3}{2}H_0 t\right)^{2/3}$$

ANSWER: The Friedmann equation can be written as

$$\left(\frac{\dot{a}}{a}\right) = H_0\, E(z) = H_0 \left[\Omega_{\Lambda,0} + (1 - \Omega_0)(1+z)^2 + \Omega_{\rm m,0}(1+z)^3 + \Omega_{\rm r,0}(1+z)^4\right]^{1/2}$$

Hence, for a flat, matter dominated EdS cosmology ($\Omega_{\mathrm{m},0} = \Omega_0 = 1.0$, and $\Omega_{\Lambda,0} = \Omega_{\mathrm{r},0} = 0$), we have that

$$\left(\frac{\dot{a}}{a}\right) = H_0 \left(\frac{a}{a_0}\right)^{-3/2}$$

which we rewrite as

$$\frac{\mathrm{d}(a/a_0)}{\mathrm{d}t} = H_0 \left(\frac{a}{a_0}\right)^{-1/2}$$

Integration yields

$$\int \left(\frac{a}{a_0}\right)^{1/2} \mathrm{d}\left(\frac{a}{a_0}\right) = H_0 \int \mathrm{d}t = H_0 t$$

which solves to

$$\frac{2}{3}\left(\frac{a}{a_0}\right)^{3/2} = H_0 t$$

from which it is clear that

$$\frac{a}{a_0} = \left(\frac{3}{2} H_0 t\right)^{2/3}$$

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b) Show that, in an EdS cosmology, $\bar{\rho}(t) = (6\pi G t^2)^{-1}$

ANSWER: The density of an EdS cosmology evolves as

$$\rho(t) = \rho_{\mathrm{m},0}\left(\frac{a}{a_0}\right)^{-3} = \rho_{\mathrm{crit},0}\left(\frac{a}{a_0}\right)^{-3} = \frac{3H_0^2}{8\pi G}\left(\frac{3}{2}H_0 t\right)^{-2} = \frac{1}{6\pi G t^2}$$

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c) Show that, in an EdS cosmology, $H(t)\, t = 2/3$.

ANSWER: Substituting what we learned under **a)** in the Friedmann equation yields

$$H(t) = H_0\left(\frac{a}{a_0}\right)^{-3/2} = H_0\left[\left(\frac{3}{2}H_0 t\right)^{2/3}\right]^{-3/2} = \frac{2}{3t}$$

Hence, it is clear that $H(t)\,t = 2/3$.

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d) Show that, in an EdS cosmology, the proper particle horizon is $\lambda_{\rm H} = 3\,c\,t$.

ANSWER: The proper particle horizon is given by

$$\lambda_{\rm H}^{\rm prop} = a\lambda_{\rm H}^{\rm com} = a\int_0^t \frac{c\,{\rm d}t}{a(t)}$$

Using the expression for the scale-factor derived under **a)** yields

$$\lambda_{\rm H}^{\rm prop} = ac\left(\frac{3}{2}H_0\right)^{-2/3}\int_0^t t'^{-2/3}\,{\rm d}t' = 3ct$$

Problem 7: The Sound-Speed of the Photon-Baryon fluid
Consider a Universe that consists solely of baryons and photons (no dark matter, no dark energy, no neutrinos). Show that, during the radiation era, the sound speed of the photon-baryon fluid can be written as

$$c_{\rm s} = \frac{c}{\sqrt{3}}\left[\frac{3}{4}\frac{\bar{\rho}_{\rm b}(z)}{\bar{\rho}_\gamma(z)} + 1\right]^{-1/2}$$

where $\bar{\rho}_{\rm b}(z)$ and $\bar{\rho}_\gamma(z)$ are the mean energy densities of baryons and photons at redshift z, and c is the speed of light.

ANSWER: The sound speed is defined as $c_{\rm s}^2 = (\partial P/\partial\rho)_S$. Using that $\rho = \rho_\gamma + \rho_{\rm b}$ while the pressure is dominated by that of the radiation, $P = P_\gamma = \frac{1}{3}\rho_\gamma c^2$, we have that

$$c_{\rm s} = \left(\frac{{\rm d}P_\gamma}{{\rm d}\rho_\gamma}\frac{\partial\rho_\gamma}{\partial\rho}\right)^{1/2} = \frac{c}{\sqrt{3}}\left[1 + \frac{\partial\rho_{\rm b}}{\partial\rho_\gamma}\right]^{-1/2}$$

Using that $\rho_{\rm b} = \bar{\rho}_{\rm b}(z) = \bar{\rho}_{{\rm b},0}a^{-3}$ and $\rho_\gamma = \bar{\rho}_\gamma(z) = \bar{\rho}_{\gamma,0}a^{-4}$, we have that

$$\frac{\partial\rho_{\rm b}}{\partial\rho_\gamma} = \frac{\partial\rho_{\rm b}}{\partial a}\frac{\partial a}{\partial\rho_\gamma} = \frac{-3\bar{\rho}_{{\rm b},0}a^{-4}}{-4\bar{\rho}_{\gamma,0}a^{-5}} = \frac{3}{4}\frac{\bar{\rho}_{{\rm b},0}a^{-3}}{\bar{\rho}_{\gamma,0}a^{-4}} = \frac{3}{4}\frac{\bar{\rho}_{\rm b}(z)}{\bar{\rho}_\gamma(z)}$$

Combing the above we find that, indeed,

$$c_{\mathrm{s}} = \frac{c}{\sqrt{3}} \left[\frac{3}{4} \frac{\bar{\rho}_{\mathrm{b}}(z)}{\bar{\rho}_{\gamma}(z)} + 1 \right]^{-1/2}$$