ASTR 610: Problem Set 1

This problem set consists of 6 problems.
Due date monday Feb 5, 2024

Problem 1: The Einstein-de Sitter Cosmology
The Einstein-de Sitter (EdS) cosmology is defined as a flat, matter dominated cosmology without cosmological constant. In an EdS cosmology the universe is always matter dominated; it never experiences a phase of radiation domination.

a) `[5 points]` Use the Friedmann equation to show that in an EdS cosmology

$$\left(\frac{a}{a_0}\right) = \left(\frac{3}{2} H_0 t\right)^{2/3}$$

b) `[3 points]` Show that, in an EdS cosmology, $\bar{\rho}(t) = (6\pi G t^2)^{-1}$

c) `[3 points]` Show that, in an EdS cosmology, $H(t)\, t = 2/3$.

d) `[3 points]` Show that, in an EdS cosmology, the proper particle horizon is $\lambda_{\rm H} = 3\, c\, t$.

Problem 2: Matter-Radiation Equality `[5 points]`
In a hypothetical universe, $\Omega_\Lambda = 0$, the CMB has a temperature of 10.0 K, and the energy density of neutrinos, which are still relativistic at the present, is 1.5 times higher than that of the photons. What is the redshift of matter-radiation equality in units of $\Omega_{\rm m,0}\, h^2$?

Problem 3: Tired Light Cosmology `[6 points]`
The "tired light hypothesis" is an old hypothesis that was once used to explain the Hubble relation without having to resort to an expanding space-time. It postulates that photons 'simply' lose energy as they move through space (by some unexplained means), with the energy loss per unit distance being given by the law

$$\frac{\mathrm{d}E}{\mathrm{d}r} = -K\,E$$

where K is a constant. Show that this hypothesis gives a distance-redshift relation that is linear in the limit $z << 1$, and derive the value for K that yields a Hubble constant of $H_0 = 70\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$.

Problem 4: Curved Space: Suppose you are a two-dimensional being living on the surface of a sphere of radius R.

a) `[5 points]` Using the metric, derive an expression for the circumference of a circle of radius r [NOTE: this radius is measured within the curved space itself]

b) `[4 points]` Idealize the Earth as a perfect sphere of radius $R = 6400\,\mathrm{km}$. If you could measure distances with an accuracy of ± 1 meter, how large a circle would you have to draw/construct on the Earth's surface to convincingly demonstrate that the Earth is curved, rather than flat?

Problem 5: The Gravitational Potential `[6 points]`
Using that, in a homogeneous and isotropic expanding universe, the equation of motion of a shell of matter is given by

$$\ddot{R} = -\frac{GM}{R^2}$$

with $M = M(< R)$ the mass enclosed by that shell. Show that the Laplacian of $\Phi \equiv \phi + a\ddot{a}x^2/2$ with respect to the comoving coordinate $\vec{x}$ is equal to $4\pi G\bar{\rho}a^2\delta$, indicating that Φ is only 'sourced' by the density contrast $\bar{\rho}\delta = \rho - \bar{\rho}$. Here ϕ is the Newtonian gravitational potential, and $a = a(t)$ is the scale factor.

Problem 6: Einstein's Static Universe: Einstein, in an attempt to construct a static model for the Universe, introduced the cosmological constant Λ.

a) `[4 points]` What is the relation between Λ and the matter density ρ for a static Universe in which the energy density of relativistic matter is negligible?

b) `[3 points]` Show that Einstein's static universe has to be positively curved (i.e., $K = +1$), and demonstrate that the present-day value for the scale-factor, a_0 (also called the radius of curvature) has to be equal to $\Lambda^{-1/2}$.

c) `[4 points]` Consider once more Einstein's static universe. Suppose that some matter is converted to radiation (for example, due to stars and/or AGN). What happens to the scale factor of the universe? Explain your answer.

d) `[4 points]` Suppose the energy density associated with the cosmological constant is equal to the present-day critical density. What is the total energy associated with Λ within 1 AU, in units of the rest-mass energy of the Sun? What does this imply for the impact of Λ on the the dynamics of the Solar system?