ASTR 610
Theory of Galaxy Formation

Lecture 9: Press-Schechter Theory

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Yale University, Spring 2024

Press-Schechter Theory

In this lecture we discuss Press-Schechter theory, and its extension based on upcrossing statistics of excursion sets. We show how these formalisms can be used to predict halo mass functions, but also discuss its oversimplifications and shortcomings.

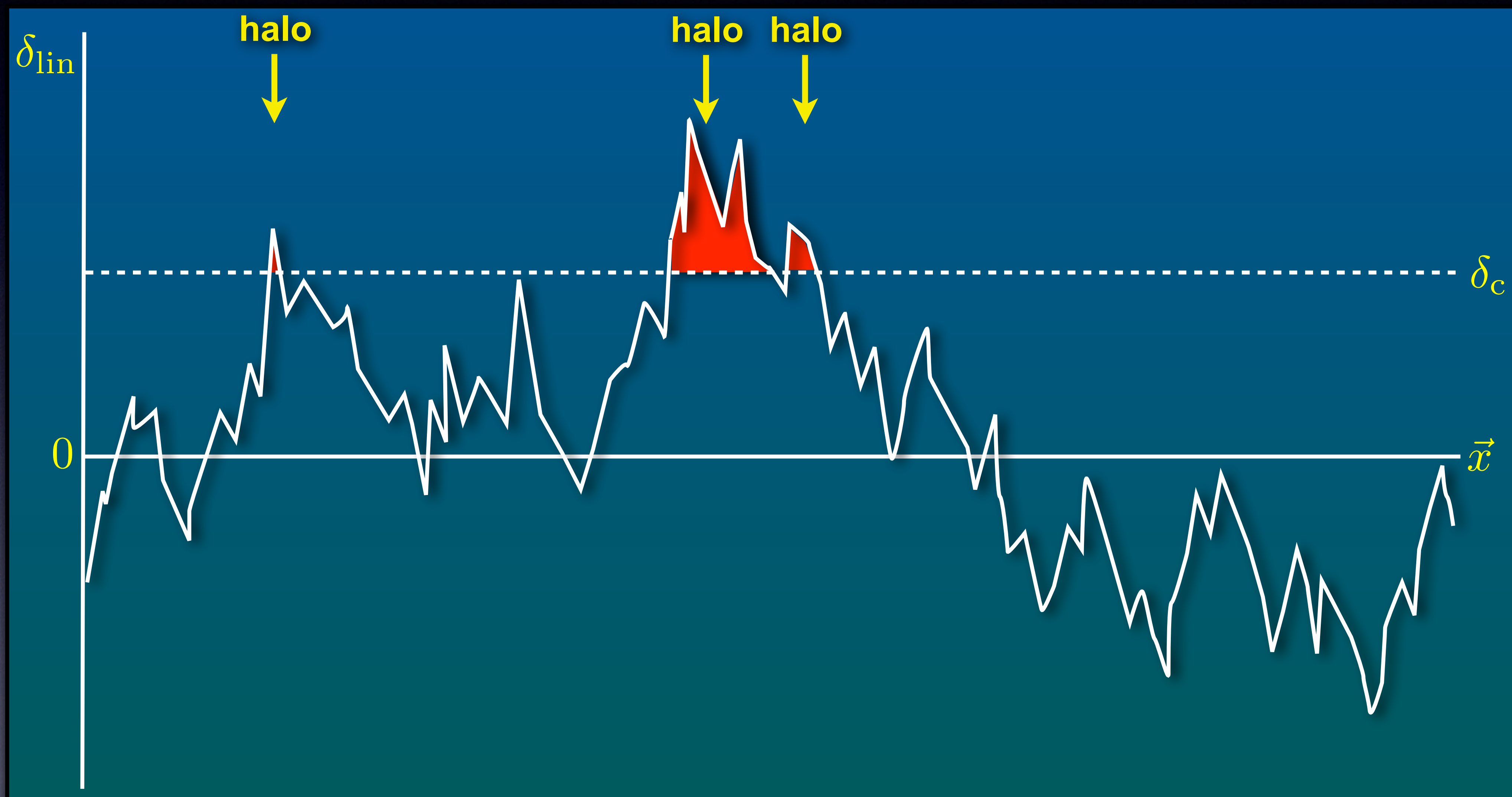
Topics that will be covered include:

- The Smoothed Density Field
- Mass Variance
- Press-Schechter Formalism
- Excursion Sets
- Extended Press-Schechter
- Halo Mass Functions
- Spherical vs. Ellipsoidal Collapse

The Linear Cosmological Density Field

According to linear theory, the density field evolves as $\delta(\vec{x},t) = D(t)\,\delta_0(\vec{x})$

Here $\delta_0(\vec{x})$ is the density field linearly extrapolated to $t = t_0$, and $D(t)$ is the linear growth rate normalized to unity at $t = t_0$



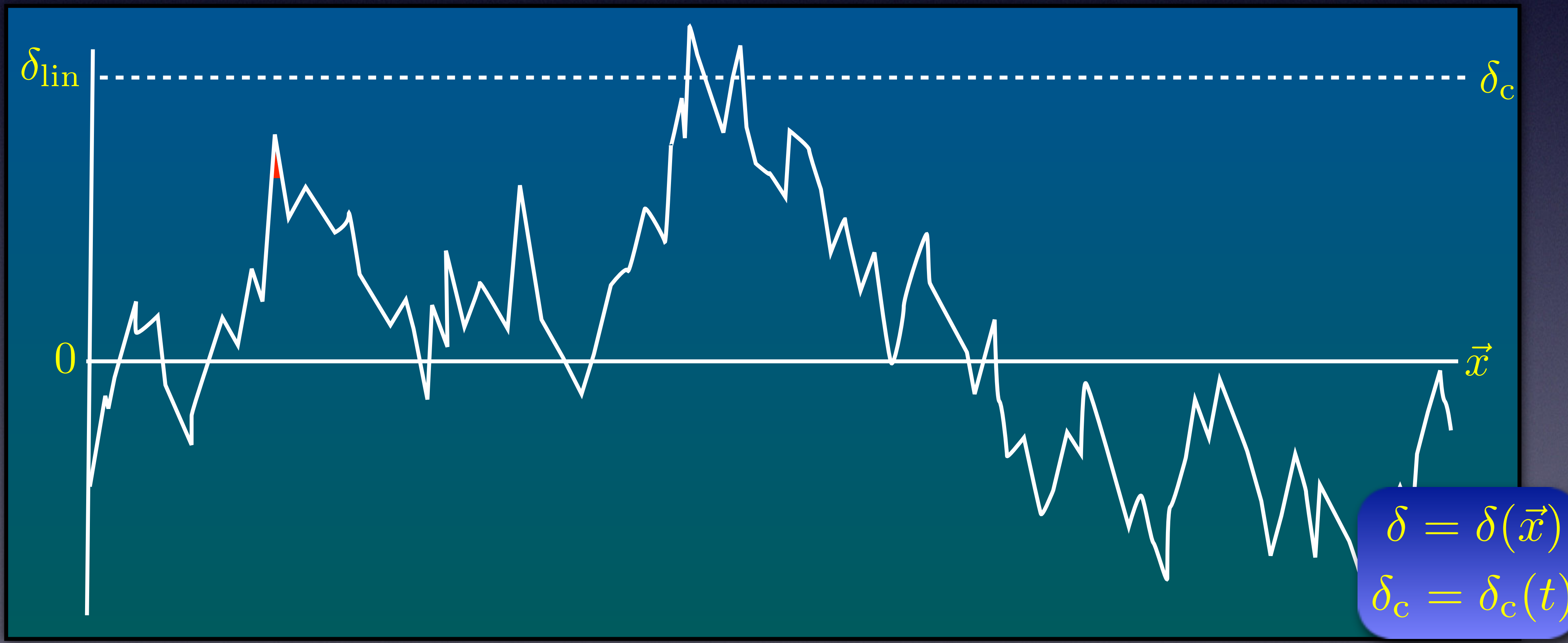
According to the spherical collapse model, regions with $\delta(\vec{x},t) > \delta_{\rm c} \simeq 1.686$ will have collapsed to produce dark matter haloes by time t. In this lecture we examine how to assign a halo mass to this structure. But first, we need to introduce some concepts...

Notation & Convention

According to the spherical collapse model, regions with $\delta(\vec{x},t) > \delta_c \simeq 1.686$ will have collapsed to produce dark matter haloes by time t

Using that $\delta(\vec{x},t) = D(t)\,\delta_0(\vec{x})$ we can also phrase this differently: regions with $\delta_0(\vec{x}) > \delta_c/D(t)$ will have collapsed to produce dark matter haloes by time t

In this latter case, we consider the density field to be static (at the one linearly extrapolated to our reference time), while the `collapse barier' evolves with time.



In the Press-Schechter formalism, the latter will be our preferred `view'.

The Variance of the Density Field

Recall that the assumption of ergodicity implies that $\langle\delta\rangle = \frac{1}{V}\int \delta(\vec{x})\,\mathrm{d}^3\vec{x}$

where V is the volume of the Universe over which we assume it to be periodic.

Similarly, we have that the variance of the density field can be written as

$$\boxed{\sigma^2 = \langle\delta^2\rangle = \frac{1}{V}\int \delta^2(\vec{x})\,\mathrm{d}^3\vec{x}}$$

Recall that $\xi(r) = \langle\delta(\vec{x})\delta(\vec{x}+\vec{r})\rangle = \frac{1}{(2\pi)^3}\int P(k)e^{+i\vec{k}\cdot\vec{r}}\mathrm{d}^3\vec{k}$, from which it is clear that

$$\boxed{\sigma^2 = \xi(0) = \frac{1}{(2\pi)^3}\int P(k)\mathrm{d}^3\vec{k} = \frac{1}{2\pi^2}\int P(k)\,k^2\,\mathrm{d}k = \int \Delta^2(k)\frac{\mathrm{d}k}{k}}$$

where $\Delta^2(k) = \frac{k^3}{2\pi^2}P(k)$ is the unitless power spectrum.

Smoothing

Given a density field $\delta(\vec{x})$, one can filter it using some window function (or "filter") $W(\vec{x};R)$ which is properly normalized such that $\int W(\vec{x};R)\,\mathrm{d}^3\vec{x} = 1$, to get a smoothed field

$$\boxed{\delta(\vec{x};R) \equiv \int \delta(\vec{x}')\, W(\vec{x}-\vec{x}';R)\,\mathrm{d}^3\vec{x}'}$$

For each filter, one can define a mass $M = \gamma_{\rm f}\,\bar{\rho}\,R^3$, where $\gamma_{\rm f}$ is some constant that depends on the shape of the filter. In what follows, we will characterize a filter intermittendly by its size R or its mass M.

The above equation for the smoothed density field is a convolution integral (the density field is convolved with the window function). Since convolution in real-space is equal to multiplication in Fourier space, we have that

$$\boxed{\delta(\vec{k};R) = \int \delta(\vec{x};R)\, e^{-i\vec{k}\cdot\vec{x}}\,\mathrm{d}^3\vec{x} = \delta(\vec{k})\,\widetilde{W}(kR)}$$

where $\widetilde{W}(kR) = \int W(\vec{x};R)\, e^{-i\vec{k}\cdot\vec{x}}\,\mathrm{d}^3\vec{x}$ is the Fourier Transform of the window function for which we have made it explicit that k and R only enter in the combination kR.

Window Functions

Throughout we will use either one of the following three window functions:

Top Hat Filter: $\gamma_{\rm f} = 4\pi/3$

$$W(\vec{x};R) = \begin{cases} \frac{3}{4\pi R^3} & r \leq R \\ 0 & r > R \end{cases} \qquad \widetilde{W}(kR) = \frac{3}{(kR)^3}\left[\sin(kR) - (kR)\,\cos(kR)\right]$$

Gaussian Filter: $\gamma_{\rm f} = (2\pi)^{3/2}$

$$W(\vec{x};R) = \frac{1}{(2\pi)^{3/2}\,R^3}\exp\left(-\frac{r^2}{2R^2}\right) \qquad \widetilde{W}(kR) = \exp\left(-\frac{(kR)^2}{2}\right)$$

Sharp k-space Filter: $\gamma_{\rm f} = 6\,\pi^2$

$$W(\vec{x};R) = \frac{1}{2\pi^2\,r^3}\left[\sin(r/R) - (r/R)\,\cos(r/R)\right] \qquad \widetilde{W}(kR) = \begin{cases} 1 & k \leq 1/R \\ 0 & k > 1/R \end{cases}$$

The Smoothed Density Field

Similar to case without smoothing, we define the variance of the smoothed density field as

$$\sigma^2(R) = \langle \delta^2(\vec{x};R) \rangle = \frac{1}{2\pi^2} \int P(k)\, \widetilde{W}^2(kR)\, k^2\, \mathrm{d}k$$

Note that $\lim_{R\to 0} \widetilde{W}(kR) = 1$ (normalization condition), from which it is clear that $\lim_{R\to 0} \sigma^2(R) = \sigma^2$ as required.

The cosmological parameter σ_8 is defined as the variance of the density field, linearly extrapolated to $z=0$, when smoothed with top-hat filter of size $R = 8h^{-1}\mathrm{Mpc}$

$$\sigma_8 = \langle \delta^2_{\mathrm{lin}}(\vec{x};R) \rangle^{1/2} = \left[\frac{1}{2\pi^2} \int P_{\mathrm{lin}}(k)\, \widetilde{W}^2_{\mathrm{TH}}(kR)\, k^2\, \mathrm{d}k \right]^{1/2}$$

This parameter is used to characterize the normalization of the power spectrum. It's currently favored value is of the order of $\sigma_8 \simeq 0.8 \pm 0.1$. A larger value of σ_8 implies larger fluctuations, and therefore earlier structure formation...

Mass Variance

Since we can equally label a filter by its size R or its mass M, we can write $\sigma^2(R) = \sigma^2(M)$.
The latter is called the mass variance, and plays an important role in what follows.

It is straightforward to show that

$$\boxed{\sigma^2(M) = \left\langle \left(\frac{M(\vec{x};R) - \bar{M}(R)}{\bar{M}(R)} \right)^2 \right\rangle}$$

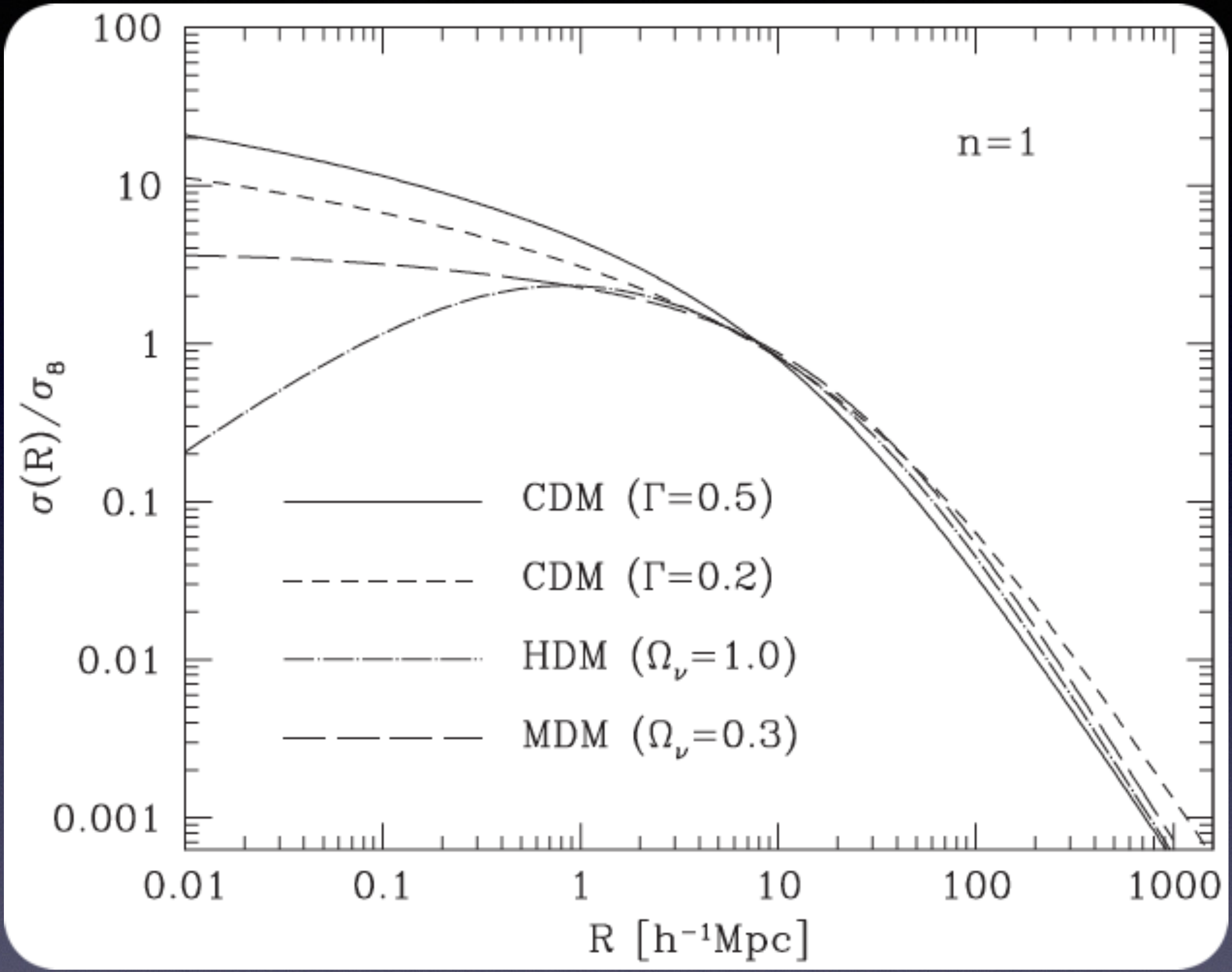
where $M(\vec{x};R) = V_R \int \rho(\vec{x}')\, W(\vec{x}-\vec{x}';R)\, \mathrm{d}^3\vec{x}'$, with V_R the volume of the filter, and $\bar{M}(R) = \langle M(\vec{x};R) \rangle$, which exemplifies the nomenclature \`mass variance'.

NOTE: If $\delta(\vec{x})$ is a Gaussian random field, then so is $\delta(\vec{x};R)$. In particular

$$\boxed{\mathcal{P}(\delta_M)\, \mathrm{d}\delta_M = \frac{1}{\sqrt{2\pi}\, \sigma_M} \exp\left[-\frac{\delta_M^2}{2\sigma_M^2} \right] \mathrm{d}\delta_M}$$

where we have used the shorthand notation $\delta_M = \delta(\vec{x};M)$ and $\sigma_M = \sigma(M)$.

Mass Variance



The variance of the smoothed, linear density field as a function of the size R of the top-hat filter. Results are shown for four different cosmogonies. The variance is normalized such that $\sigma_8 = 1$.

(see MWB §6.1.3)

Note: the shape parameter $\Gamma = \Omega_{\rm m,0}\, h$ characterizes the horizon scale at matter-radiation equality.

In hierarchical models, such as CDM-based cosmologies, the variance is a monotonically decreasing function of the filter size R (or M). In top-down cosmogonies, such as HDM, however, the lack of small scale structure introduces a characteristic scale where the variance is maximum.

Assigning Halo Mass to Collapsed Regions

We now return to our main question of interest:

According to SC model, regions in the linear density field with $\delta > \delta_c$ have collapsed to produce virialized dark matter haloes. How can we associate a mass to those haloes, and how can we use the statistics of the linear density field to infer the halo mass function, i.e., the (comoving) number density of haloes as a function of halo mass?

Idea:

Let δ_M be the linear density field smoothed on a mass scale M, i.e., $\delta_M = \delta(\vec{x};R)$ where $M = \gamma_{\rm f}\,\bar{\rho}\,R^3$, then those locations where $\delta_M = \delta_{\rm c}(t)$ are the locations where, at time t, a halo of mass M condenses out of the evolving density field....

In this case, the halo mass function simply follows from calculating the number density of peaks in the smoothed density field, i.e.,

$$n(>M) = n_{\rm pk}(\delta_M)$$

number density of haloes with mass >M

number density of peaks above $\delta_{\rm c}$ in density field smoothed on mass scale M

Peak Formalism & Cloud-in-Cloud Problem

This idea was explored in a seminal paper by Bardeen et al. (1986), known as "BBKS".

THE STATISTICS OF PEAKS OF GAUSSIAN RANDOM FIELDS

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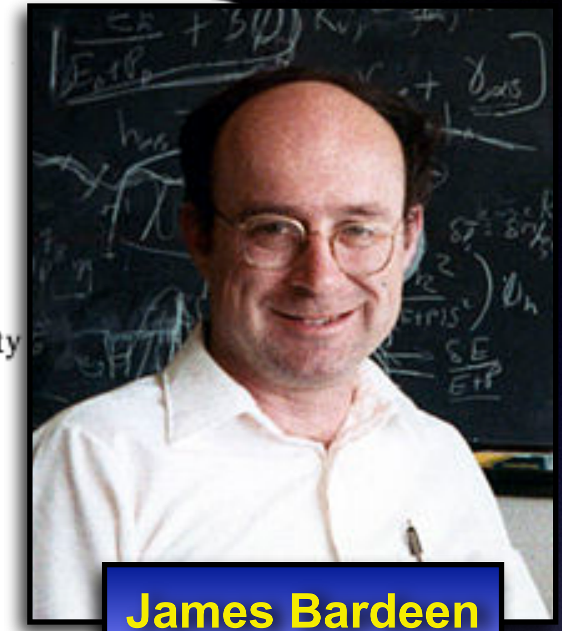

James Bardeen

ABSTRACT

Cosmological density fluctuations are often assumed to be Gaussian random fields. The local maxima of such fields are obvious sites for the formation of nonlinear structures. The statistical properties of the peaks can be used to predict the abundances and clustering properties of objects of various types. In this paper, we derive (1) the number density of peaks of various heights $\nu\sigma_0$ above the rms σ_0; (2) the factor by which the peak density is enhanced in large-scale overdense regions; (3) the n-point peak-peak correlation function in the limit that the peaks are well separated, with special emphasis on the two- and three-point correlations; and (4) the density profiles centered on peaks. To illustrate the predictive power of this semianalytic approach, we apply our formulae to structure formation in the adiabatic and isocurvature $\Omega = 1$ cold dark matter (CDM) models. We assume bright galaxies form only at those peaks in the density field (smoothed on a galactic scale) that are above some global threshold height $\nu_t \approx 3$ fixed by normalizing to the galaxy number density. We find, for example, that the shapes of the peak-peak two- and three-point correlation functions for the adiabatic CDM model agree well with observations before any dynamical evolution, just due to the propensity of the peaks to be clustered in the initial conditions. Only moderate dynamical evolution is required to bring the amplitude of the correlations up to the observed level. The corresponding redshift of galaxy formation z_g in the isocurvature model is too recent ($z_g \approx 0$) for this model to be viable. Even for the adiabatic models $z_g \approx$ 3–4 is predicted. We show that the mass-per-peak ratio in clusters, and thus presumably the cluster mass-to-light ratio, is substantially lower than in the ambient medium, alleviating the Ω problem. We also confirm that the smoothed density profiles of collapsing structures of height $\sim\nu_t$ are inherently triaxial.

Peak Formalism & Cloud-in-Cloud Problem

This idea was explored in a seminal paper by Bardeen et al. (1986), known as “BBKS”.

Using elegant, clever mathematics they were able to compute the number density, clustering properties, shapes and density profiles of peaks in a smoothed Gaussian random field (which itself is also a Gaussian random field), all as function of the peak height $\nu_{\rm pk} = \frac{\delta_{\rm pk}}{\langle \delta_M^2 \rangle^{1/2}} = \frac{\delta_{\rm pk}}{\sigma_M}$

(see MBW §7.1 for details)

Unfortunately, it soon became clear that the identification

$$\boxed{\text{peak in } \delta_M \Longleftrightarrow \text{halo with mass} > M}$$

faces a very serious problem:

Consider the same density field, but smoothed on two different mass scales, M_1 and M_2, where $M_2 > M_1$. Let δm be a mass element associated with a peak of $\delta_1 = \delta(\vec{x}; M_1)$ but also with a peak of $\delta_2 = \delta(\vec{x}; M_2)$. Is δm part of a halo of mass M_1 or M_2?

- If $\delta_2 < \delta_1$ the obvious interpretation is that δm is part of M_1 at some early time t_1, and part of $M_2 > M_1$ at some later time $t_2 > t_1$.

- If $\delta_2 > \delta_1$ then δm can never be part of a halo with mass M_1; apparently, contrary to the \`ansatz', not every peak in δ_1 can be associated with a halo...

Peak Formalism & Cloud-in-Cloud Problem

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Using elegant, clever mathematics they were able to compute the number density, clustering properties, shapes and density profiles of peaks in a smoothed Gaussian random field (which itself is also a Gaussian random field), all as function of the peak height $\nu_{\rm pk} = \frac{\delta_{\rm pk}}{\langle \delta_M^2 \rangle^{1/2}} = \frac{\delta_{\rm pk}}{\sigma_M}$

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Unfortunately, it soon became clear that the identification

peak in δ_M ⟷ halo with mass $> M$

faces a very serious problem:

Apparently, some peaks (those that are part of a higher peak when smoothed with a larger filter) have to be excluded when identifying peaks with haloes...

This is called the cloud-in-cloud problem.

The Press-Schechter Mass Function

Because of the cloud-in-cloud problem, the peak formalism of BBKS has largely been abandoned in favor of the less rigorous, but more succesfull, Press-Schechter formalism

Bill Press

FORMATION OF GALAXIES AND CLUSTERS OF GALAXIES BY SELF-SIMILAR GRAVITATIONAL CONDENSATION*

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Received 1973 August 1

ABSTRACT

We consider an expanding Friedmann cosmology containing a "gas" of self-gravitating masse
The masses condense into aggregates which (when sufficiently bound) we identify as single particl
of a larger mass. We propose that after this process has proceeded through several scales, the mas spectrum of condensations becomes "self-similar" and independent of the spectrum initially assumed. Some details of the self-similar distribution, and its evolution in time, can be calculated with the linear perturbation theory. Unlike other authors, we make no ad hoc assumptions about the spectrum of long-wavelength initial perturbations: the nonlinear N-body interactions of the mass points randomize their positions and generate a perturbation to all larger scales; this should fix the self-similar distribution almost uniquely. The results of numerical experiments on 1000 bodies are presented; these appear to show new nonlinear effects: condensations can "bootstrap" their way up in size faster than the linear theory predicts. Our self-similar model predicts relations between the masses and radii of galaxies and clusters of galaxies, as well as their mass spectra. We compare the predictions with available data, and find some rather striking agreements. If the model is to explain galaxies, then isothermal "seed" masses of $\sim 3 \times 10^7\, M_\odot$ must have existed at recombination. To explain clusters of galaxies, the only necessary seeds are the galaxies themselves. The size of clusters determines, in principle, the deceleration parameter q_0; presently available data give only very broad limits, unfortunately.

Subject headings: cosmology — galaxies — galaxies, clusters of

Paul Schechter

The Press-Schechter Mass Function

Because of the cloud-in-cloud problem, the peak formalism of BBKS has largely been abandoned in favor of the less rigorous, but more succesfull, Press-Schechter formalism

Press & Schechter (1974) postulated that:

> "the probability that $\delta_M > \delta_{\rm c}(t)$ is the same as the mass fraction that at time t is contained in halos with mass greater than M"

For a Gaussian random field, one has that

$$\boxed{\mathcal{P}(\delta_M > \delta_{\rm c}) = \frac{1}{\sqrt{2\pi}\,\sigma_M} \int_{\delta_{\rm c}}^{\infty} \exp\left[-\frac{\delta_M^2}{2\sigma_M^2}\right] {\rm d}\delta_M = \frac{1}{2}{\rm erfc}\left[\frac{\delta_{\rm c}}{2\sigma_M}\right]}$$

Here ${\rm erfc}(x) = 1 - {\rm erf}(x)$ is the complimentary error function, and we consider it understood that $\delta_{\rm c} = \delta_{\rm c}(t)$. According to the PS postulate, we thus have that

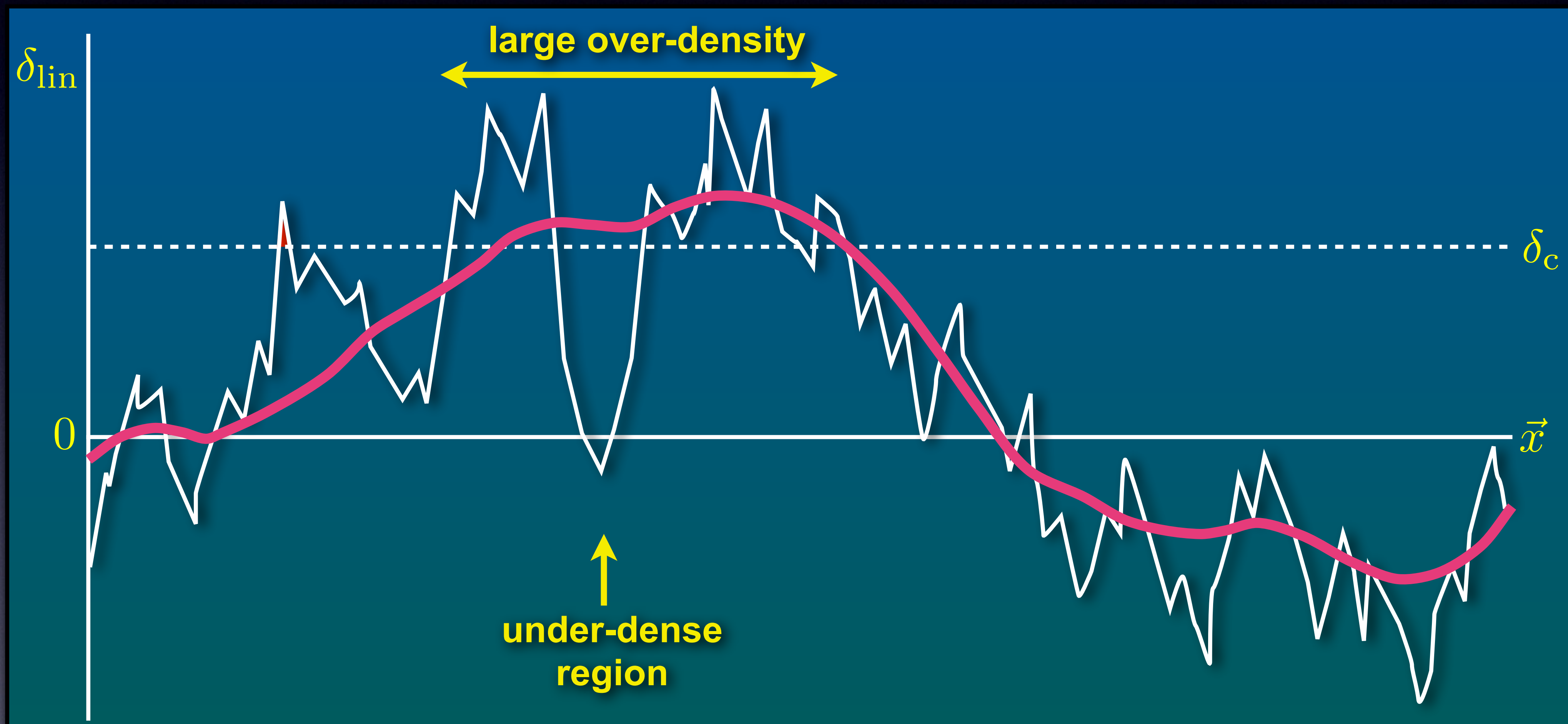
$$F(>M,t) = \frac{1}{2}\,{\rm erfc}\left[\frac{\delta_c}{2\,\sigma_M}\right]$$

Note: since $\lim_{M\to 0}\sigma_M = \infty$ and ${\rm erfc}(0) = 1$ we see that the PS postulate predicts that only 1/2 of all matter in the Universe is locked-up in collapsed haloes...

The Press-Schechter Mass Function

This may seem logical from the fact that $\mathcal{P}(\delta < 0) = \frac{1}{2}$; i.e., only regions that are initially overdense end up in collapsed objects...

However, underdense regions can be enclosed within larger overdense regions, giving them a finite probability of being included in some larger collapsed object (see illustration)



Press & Schechter `solved' this problem by simply introducing a fudge factor two:

$$F(>M,t) = 2\mathcal{P}\left[\delta_M > \delta_{\rm c}(t)\right]$$

The Press-Schechter Mass Function

We are now ready to write down the PS halo mass function:

We define the mass function as $n(M,t)\,\mathrm{d}M$, which is the number of haloes with masses in the range $[M, M+\mathrm{d}M]$ per volume. Hence, $n(M,t) = \frac{\mathrm{d}n}{\mathrm{d}M} = \frac{1}{M}\frac{\mathrm{d}n}{\mathrm{d}\ln M}$.

Beware of units and different notations!!!!

We have that $\frac{\partial F(>M)}{\partial M}\,\mathrm{d}M$ is equal to the fraction of mass that is locked up in haloes with masses in the range $[M, M+\mathrm{d}M]$.

Multiplying by $\bar{\rho}$ yields the total mass per unit volume that is locked up in those haloes.

Hence, the halo mass function is simply given by $n(M,t)\,\mathrm{d}M = \frac{\bar{\rho}}{M}\frac{\partial F(>M)}{\partial M}\,\mathrm{d}M$

Using the Press-Schechter ansatz plus fudge factor we thus obtain:

$$\boxed{n(M,t)\,\mathrm{d}M = 2\,\frac{\bar{\rho}}{M}\frac{\partial \mathcal{P}(>\delta_{\rm c})}{\partial M}\,\mathrm{d}M = \sqrt{\frac{2}{\pi}}\,\frac{\bar{\rho}}{M^2}\,\frac{\delta_{\rm c}}{\sigma_M}\exp\left(-\frac{\delta_{\rm c}^2}{2\sigma_M^2}\right)\left|\frac{\mathrm{d}\ln\sigma_M}{\mathrm{d}\ln M}\right|\,\mathrm{d}M}$$

where we have used that $\partial\mathcal{P}/\partial M = \partial\mathcal{P}/\partial\sigma_M \times |\mathrm{d}\sigma_M/\mathrm{d}M|$

The Press-Schechter Mass Function

Upon defining the variable $\nu \equiv \delta_{\rm c}(t)/\sigma(M)$ the PS mass function can be written in a more compact form:

$$n(M,t)\,{\rm d}M = \frac{\bar{\rho}}{M^2}\, f_{\rm PS}(\nu) \left|\frac{{\rm d}\ln\nu}{{\rm d}\ln M}\right| {\rm d}M \quad \text{where} \quad f_{\rm PS}(\nu) = \sqrt{\frac{2}{\pi}}\,\nu\, e^{-\nu^2/2}$$

$f_{\rm PS}(\nu)$ is called the multiplicity function and gives the mass fraction associated with haloes in a unit range of $\ln\nu$. Note that time enters only through $\delta_{\rm c}(t) \simeq 1.686/D(t)$

WARNING: some authors define $\nu = \delta_{\rm c}^2(t)/\sigma^2(M)$ which results in a somewhat modified multiplicity function.....always check how ν is defined!!

If we define a characteristic mass, M^*, by $\sigma(M^*) = \delta_{\rm c}(t)$ (i.e., by $\nu(M^*) = 1$) then:

- For $M \ll M^*$ we have that $n(M,t) \propto M^{\alpha-2}$, where $\alpha = {\rm d}\ln\sigma/{\rm d}\ln M$.

  For a CDM cosmology $\alpha \rightarrow 0$ at low mass end so that $n(M) \propto M^{-2}$

- For $M \gg M^*$ the abundance of haloes is exponentially suppressed.

- Since $\delta_{\rm c}(t)$ decreases with time, the characteristic halo mass grows as function of time; as time passes more and more massive haloes will start to form...

The Excursion Set Formalism

Bond et al. (1991) came up with an alternative derivation of the halo mass function that does not suffer from a `fudge-factor problem'

Dick Bond

EXCURSION SET MASS FUNCTIONS FOR HIERARCHICAL GAUSSIAN FLUCTUATIONS

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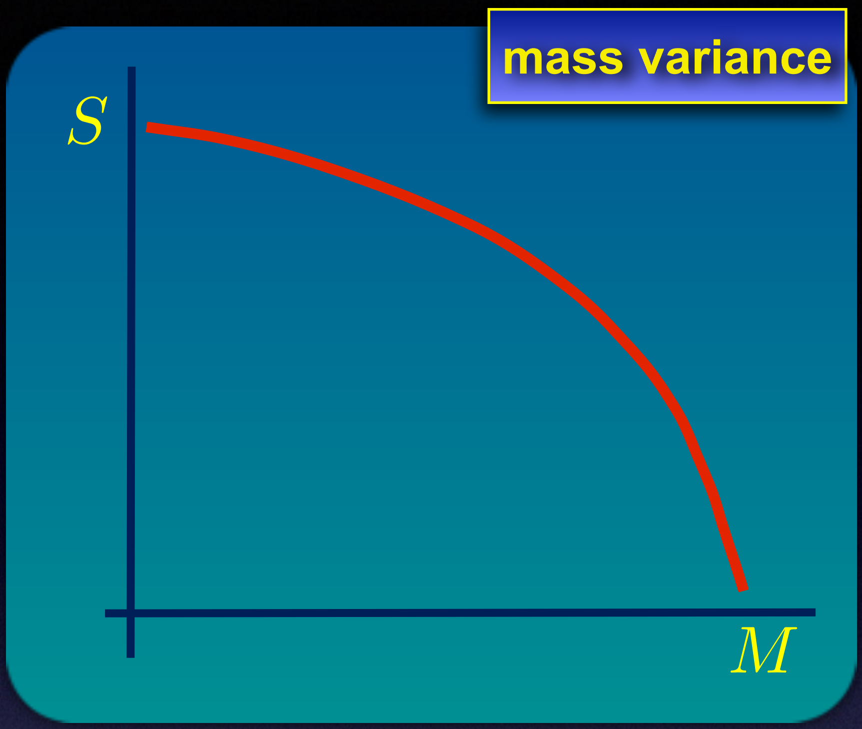
ABSTRACT

Most schemes for determining the mass function of virialized objects from the statistics of the initial density perturbation field suffer from the "cloud-in-cloud" problem of miscounting the number of low-mass clumps, many of which would have been subsumed into larger objects. We propose a solution based on the theory of the excursion sets of $F(\boldsymbol{r}, R_f)$, the four-dimensional initial density perturbation field smoothed with a continuous hierarchy of filters of radii R_f. We identify the mass fraction of matter in virialized objects with mass greater than M with the fraction of space in which the initial density contrast lies above a critical overdensity when smoothed on some filter of radius greater than or equal to $R_f(M)$. The differential mass function is then given by the rate of first upcrossings of the critical overdensity level as one decreases R_f at constant position $\boldsymbol{r}$. The shape of the mass function depends on the choice of filter function. The simplest case is "sharp k-space" filtering, in which the field performs a Brownian random walk as the resolution changes. The first upcrossing rate can be calculated analytically and results in a mass function identical to the formula of Press and Schechter—complete with their normalizing "fudge factor" of 2. For general filters (e.g., Gaussian or "top hat") no analogous analytical result seems possible, though we derive useful analytical upper and lower bounds. For these cases, the mass function can be calculated by generating an ensemble of field trajectories numerically. We compare the results of these calculations with group catalogs found from N-body simulations. Compared to the sharp k-space result, less spatially extended filter functions give fewer large-mass and more small-mass objects. Over the limited mass range probed by the N-body simulations, these differences in the predicted abundances are less than a factor of 2 and span the values found in the simulations. Thus the mass functions for sharp k-space and more general filtering all fit the N-body results reasonably well. None of the filter functions is particularly successful in identifying the particles which form low-mass groups in the N-body simulations, illustrating the limitations of the excursion set approach. We have extended these calculations to compute the evolution of the mass function in regions that are constrained to lie within clusters or underdensities at the present epoch. These predictions agree well with N-body results, although the sharp k-space result is slightly preferred over the Gaussian or top hat results.

Subject headings: cosmology — galaxies: clustering — numerical methods

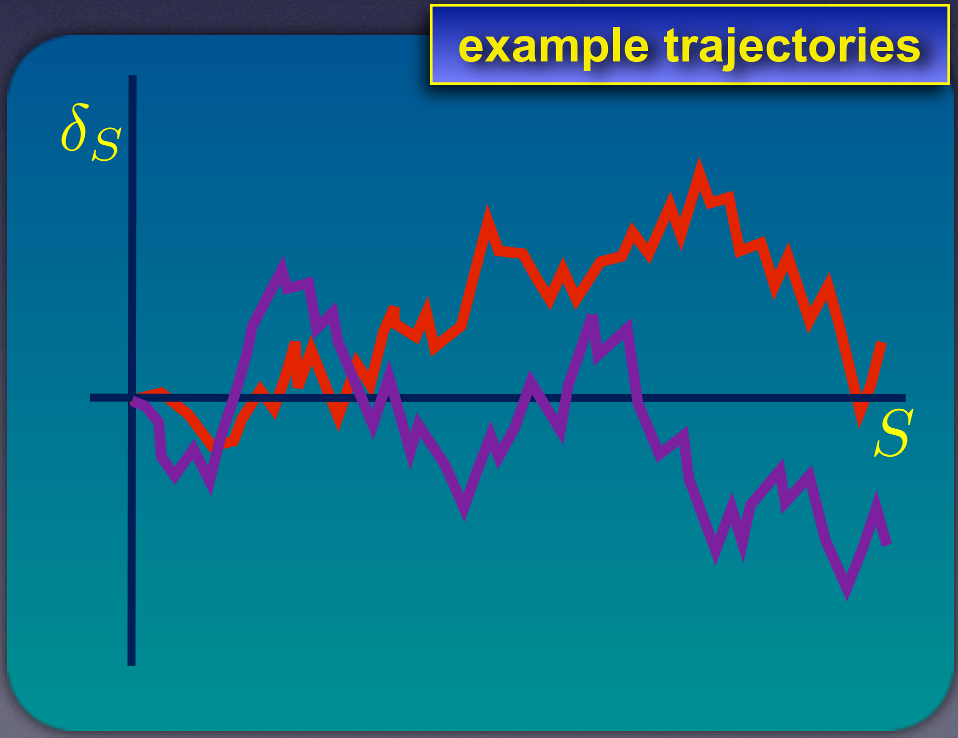
The Excursion Set Formalism

In what follows we adopt $S \equiv \sigma^2(M)$ as our mass variable. For a hierarchical cosmogony such as CDM, S is a monotonically declining function of halo mass, so that there is a clear, one-to-one relation between S and M.



Consider a point $\vec{x}$, for which the overdensity, linearly extrapolated to the present day is $\delta_0(\vec{x})$. For each value of the filtering mass M, i.e. for each value of S, the smoothed overdensity $\delta_S = \delta_M(\vec{x})$ will have a different value.

With each point $\vec{x}$ corresponds a trajectory δ_S



- For $S \rightarrow 0$ we have that $M \rightarrow \infty$, and thus $\delta_S \rightarrow 0$. Hence, each trajectory starts at $(S, \delta_S) = (0, 0)$
- If the filter is a sharp k-space filter, changing S adds new (and independent) modes. As a consequence, the trajectory is Markovian....

Markovian Random Walks

A **random walk** is a mathematical formalization of a path that consists of a succession of random steps. If the next step depends only on the current state (i.e., has no `memory' of its prior path), the random walk is called Markovian.

For a sharp k-space filter the smoothed density field is given by

$$\boxed{\delta_S(\vec{x}) = \int \mathrm{d}^3\vec{k}\, \widetilde{W}_{\rm sk}(\vec{k}R)\, \delta_{\vec{k},0}\, e^{i\vec{k}\cdot\vec{x}} = \int_{k<k_c} \mathrm{d}^3\vec{k}\, \delta_{\vec{k},0}\, e^{i\vec{k}\cdot\vec{x}}}$$

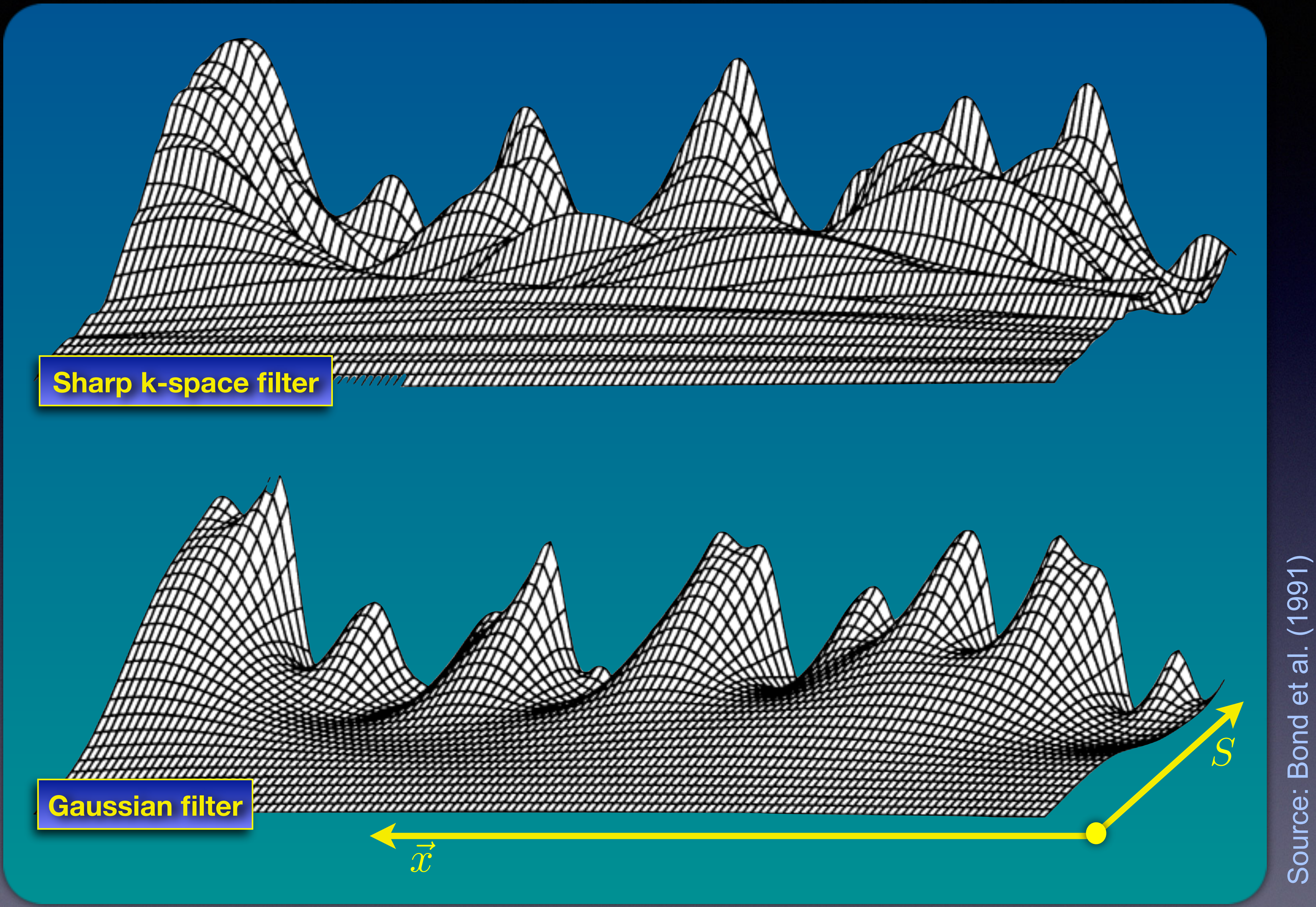
Here $k_c = 1/R$ is the size of the top-hat in k-space, and $\delta_{\vec{k},0}$ are Fourier modes of $\delta_0(\vec{x})$

When increasing S (decreasing R, and thus increasing k_c), you add new and independent modes (at least for a Gaussian random field). Since these new and independent modes have random phases, the step $\Delta(\delta_S)$ associated with the change ΔS is Markovian.

NOTE: for any other filter, the trajectories $\delta_S(S)$ will not be Markovian!!

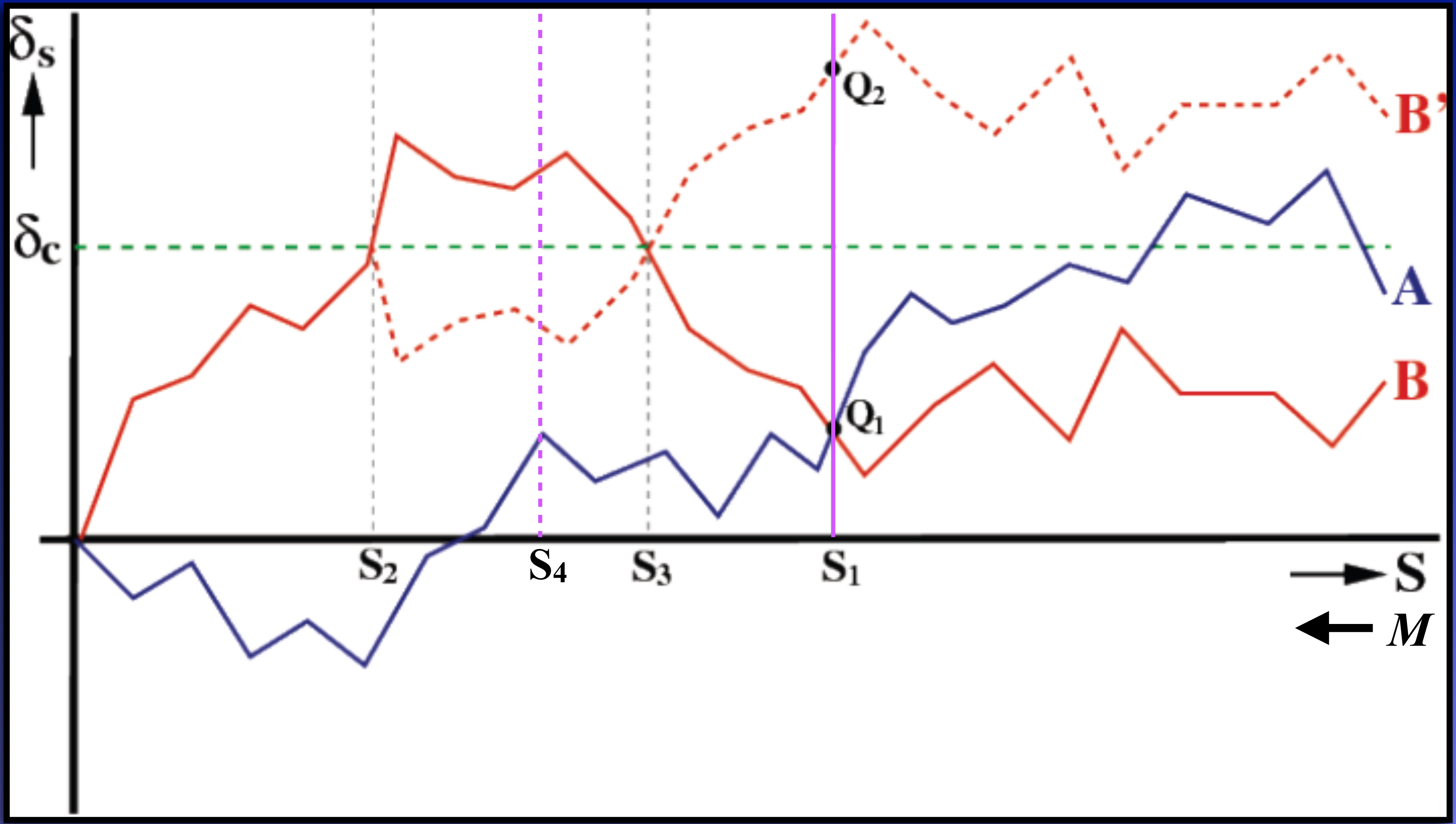
In what follows we will always assume a sharp k-space filter (unless stated otherwise), so that our trajectories can be considered Markovian.

The Excursion Set Formalism



NOTE: for any filter other than sharp k-space filter, the random walks are NOT Markovian

The Excursion Set Formalism



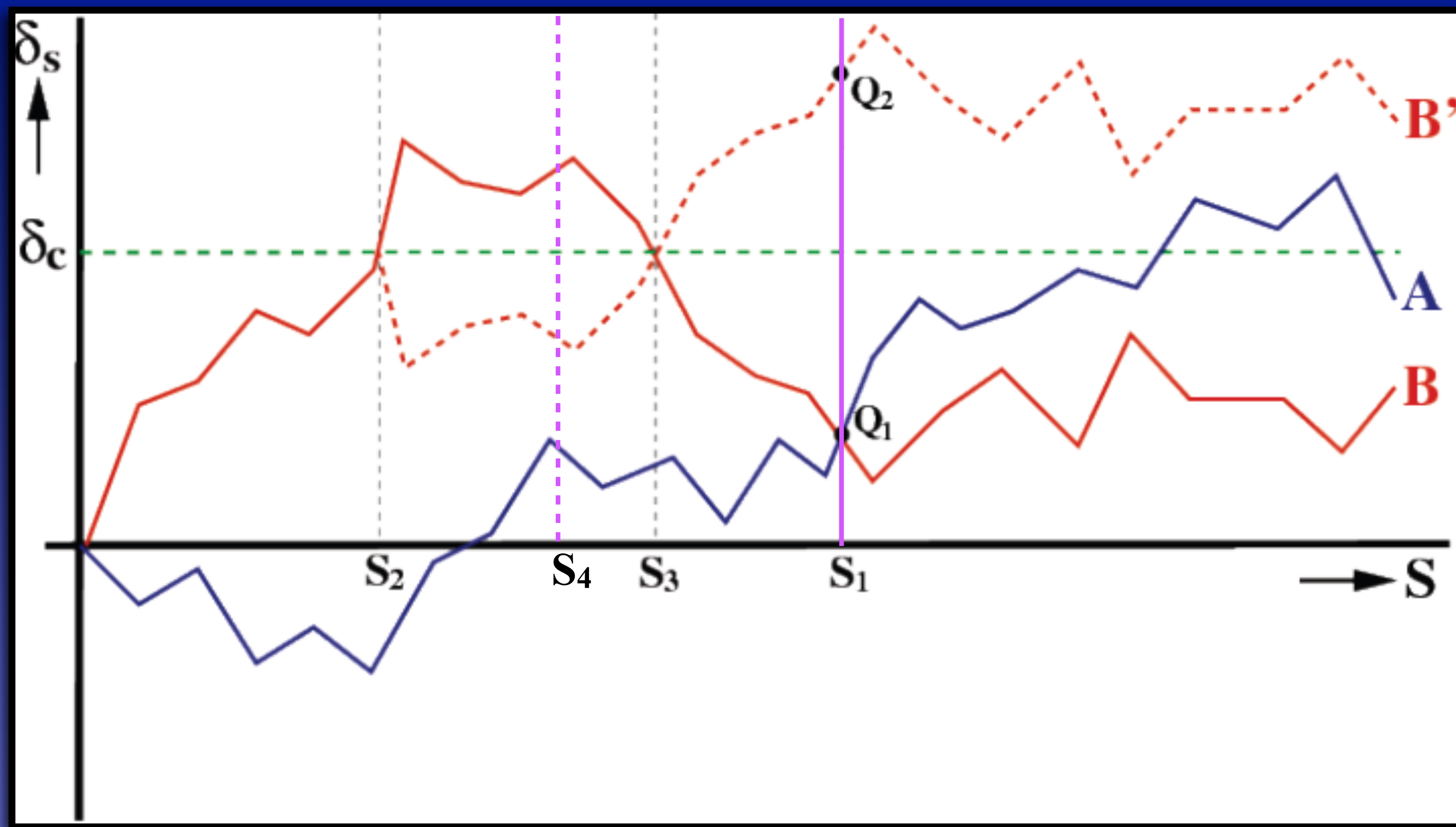
Three trajectories corresponding to three different mass elements in a Gaussian random field. Note that B' is obtained mirroring trajectory B in the line $\delta_S = \delta_c$ for $S \geq S_2$. Since the trajectories are Markovian B and B' are equally likely!

Consider $\delta_0(\vec{x})$ smoothed on a mass scale M_1 corresponding to $S_1 = \sigma^2(M_1)$

According to PS ansatz, mass elements whose trajectory $\delta_S > \delta_c$ at S_1 reside in dark matter haloes with mass $M > M_1$ ➡ neither A or B are in halo with $M > M_1$

BUT: according to same PS ansatz, mass element associated with trajectory B resides in a halo with $M > M_4 > M_1$: PS ansatz is not self-consistent!!!

The Excursion Set Formalism



Three trajectories corresponding to three different mass elements in a Gaussian random field. Note that B' is obtained mirroring trajectory B in the line $\delta_S = \delta_c$ for $S \geq S_2$. Since the trajectories are Markovian B and B' are equally likely!

The problem with the PS ansatz is that it fails to account for trajectories such as B when counting mass elements in haloes with mass $M > M_1$.

Correcting for this is easy though, by realizing that each trajectory B has a mirror version, B', that is equally likely (as a result of the Markovian nature of the trajectories).

Double-counting trajectories with $\delta_S > \delta_c$ at S_1 corrects for `missed trajectories'.....

A natural explanation for the fudge-factor two in PS formalism!

The Excursion Set Formalism

In the excursion set formalism , also called the Extended Press-Schechter (EPS) formalism, one uses the (statistics of) Markovian random walks (the trajectories of mass elements in (S, δ_S)-space) to infer the halo mass function (and more).

PS ansatz: fraction of mass elements with $\delta_S > \delta_c(t)$ is equal to the mass fraction that at time t resides in haloes with masses $> M$, where S and M are related according to $S = \sigma^2(M)$

EPS ansatz: fraction of trajectories with a first upcrossing (FU) of the barrier $\delta_S = \delta_c(t)$ at $S > S_1 = \sigma^2(M_1)$ is equal to the mass fraction that at time t resides in haloes with masses $M < M_1$

Since, each trajectory is guaranteed to upcross the barrier $\delta_S = \delta_c(t)$ at some (arbitrarily large) S, the EPS ansatz predicts that every mass element is in a halo of some (arbitrarily low) mass

$$\boxed{F(< M_1) = 1 - F(> M_1)}$$

The EPS Mass Function

Based on the EPS ansatz, we can write the EPS mass function as:

$$\begin{aligned} n(M,t)\,\mathrm{d}M &= \frac{\bar\rho}{M}\,\frac{\partial F(>M)}{\partial M}\,\mathrm{d}M = -\frac{\bar\rho}{M}\,\frac{\partial F(<M)}{\partial M}\,\mathrm{d}M \\ &= -\frac{\bar\rho}{M}\,\frac{\partial F_{\rm FU}(>S)}{\partial S}\,\frac{\mathrm{d}S}{\mathrm{d}M}\,\mathrm{d}M = \frac{\bar\rho}{M}\,f_{\rm FU}(S,\delta_{\rm c})\,\left|\frac{\mathrm{d}S}{\mathrm{d}M}\right|\,\mathrm{d}M \end{aligned}$$

Here $f_{\rm FU}(S,\delta_{\rm c})\,\mathrm{d}S$ is the fraction of trajectories that have their first upcrossing of barrier $\delta_{\rm c}(t)$ between S and $S+\mathrm{d}S$.

Without proof:

$$f_{\rm FU}(\nu) = \frac{1}{\sqrt{2\pi}}\,\frac{\delta_{\rm c}}{S^{3/2}}\,\exp\left[-\frac{\delta_{\rm c}^2}{2S}\right] = \frac{1}{2\,S}\,f_{\rm PS}(\nu)$$

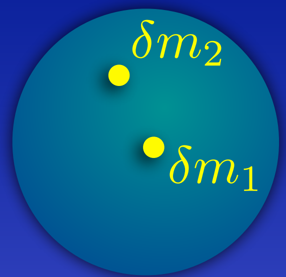
(see MBW §7.2.2 for derivation)

where, as before, we defined $\nu = \delta_{\rm c}(t)/\sigma(M) = \delta_{\rm c}/\sqrt{S}$ and we expressed the result in terms of the PS multiplicity function $f_{\rm PS}(\nu) = \sqrt{2/\pi}\,\nu\,\exp(-\nu^2/2)$

It is straightforward to show that this yields exactly the same halo mass function as before, but this time there has been no need for a fudge factor....

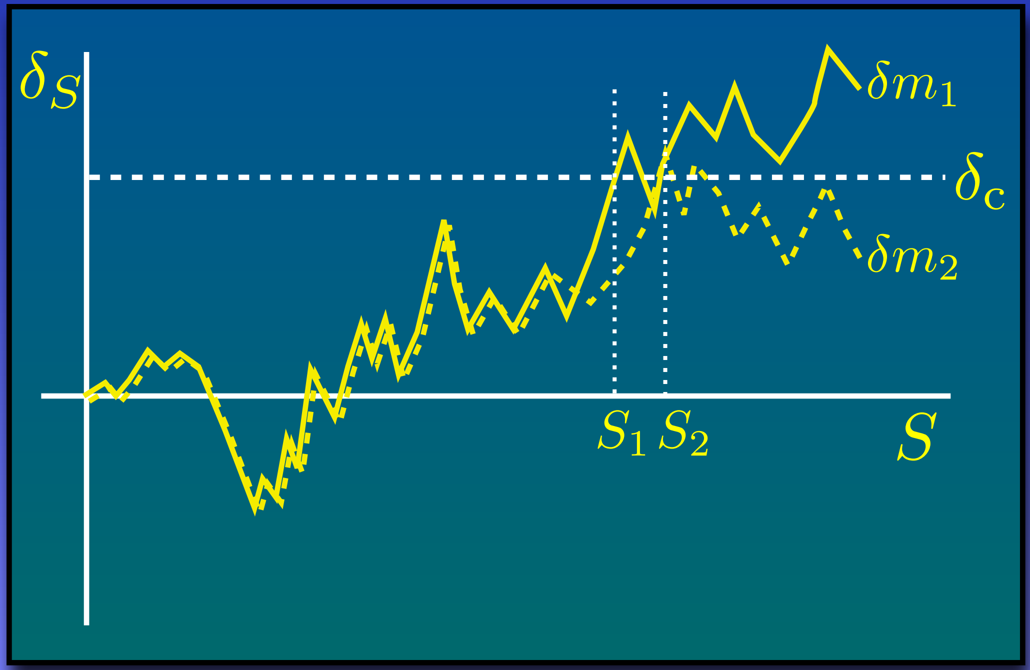
The EPS Mass Function

Although the EPS mass function is used very frequently in modern astronomy, it is important to be aware of its assumptions, shortcomings and pitfalls:



Consider two mass elements (yellow `dots') in the same dark matter halo: one near the center, the other near the outskirts.

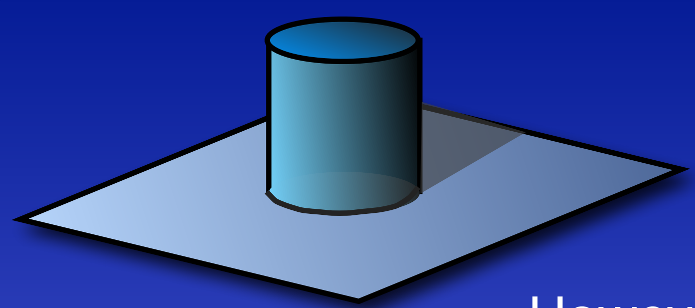
Since both particles have very similar large-scale environments (on scales larger than halo itself), their trajectories are very similar for small S:



Although both particles reside in same halo, their trajectories have first upcrossings at different S: according to EPS formalism, δm_2 resides in a less massive halo than δm_1: excursion set formalism only predicts how much mass ends up in haloes of different mass in a statistical sense....

The EPS Mass Function

Although the EPS mass function is used very frequently in modern astronomy, it is important to be aware of its assumptions, shortcomings and pitfalls:

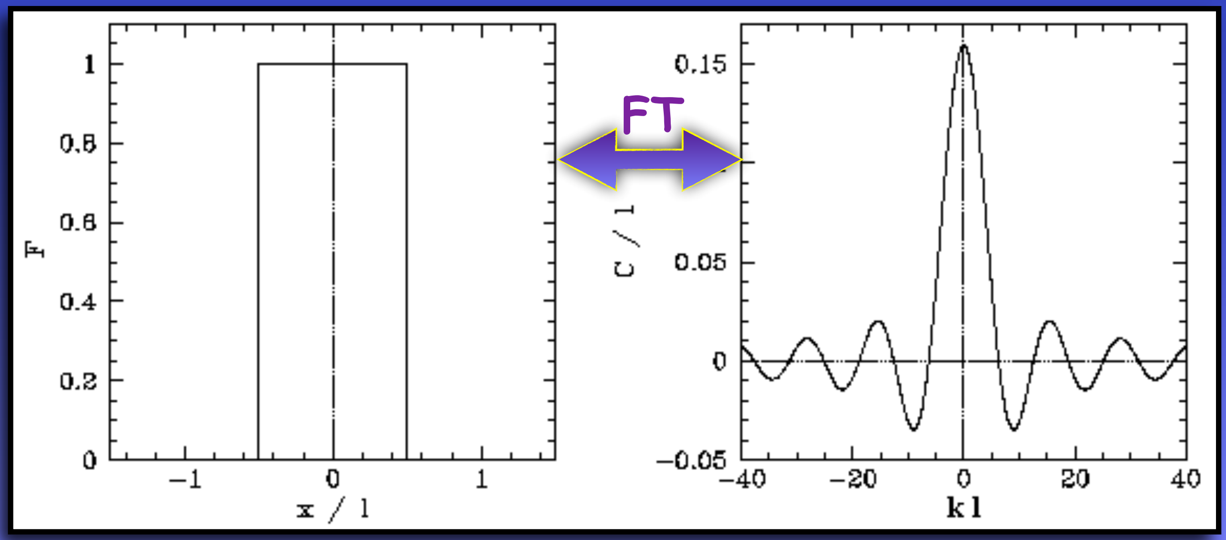


Trajectories have to be constructed with sharp k-space filter in order to guarantee Markovian nature of the random walks.

However, the corresponding real-space filter has complicated (sinc-like) form; difficult to interpret....

In particular, the real-space filter is not spatially localized; it has oscillating wings that extent out to large distances...

Yet, according to EPS formalism, this structure corresponds to a collapsed dark matter halo, which *is* spatially localized...



The EPS Mass Function

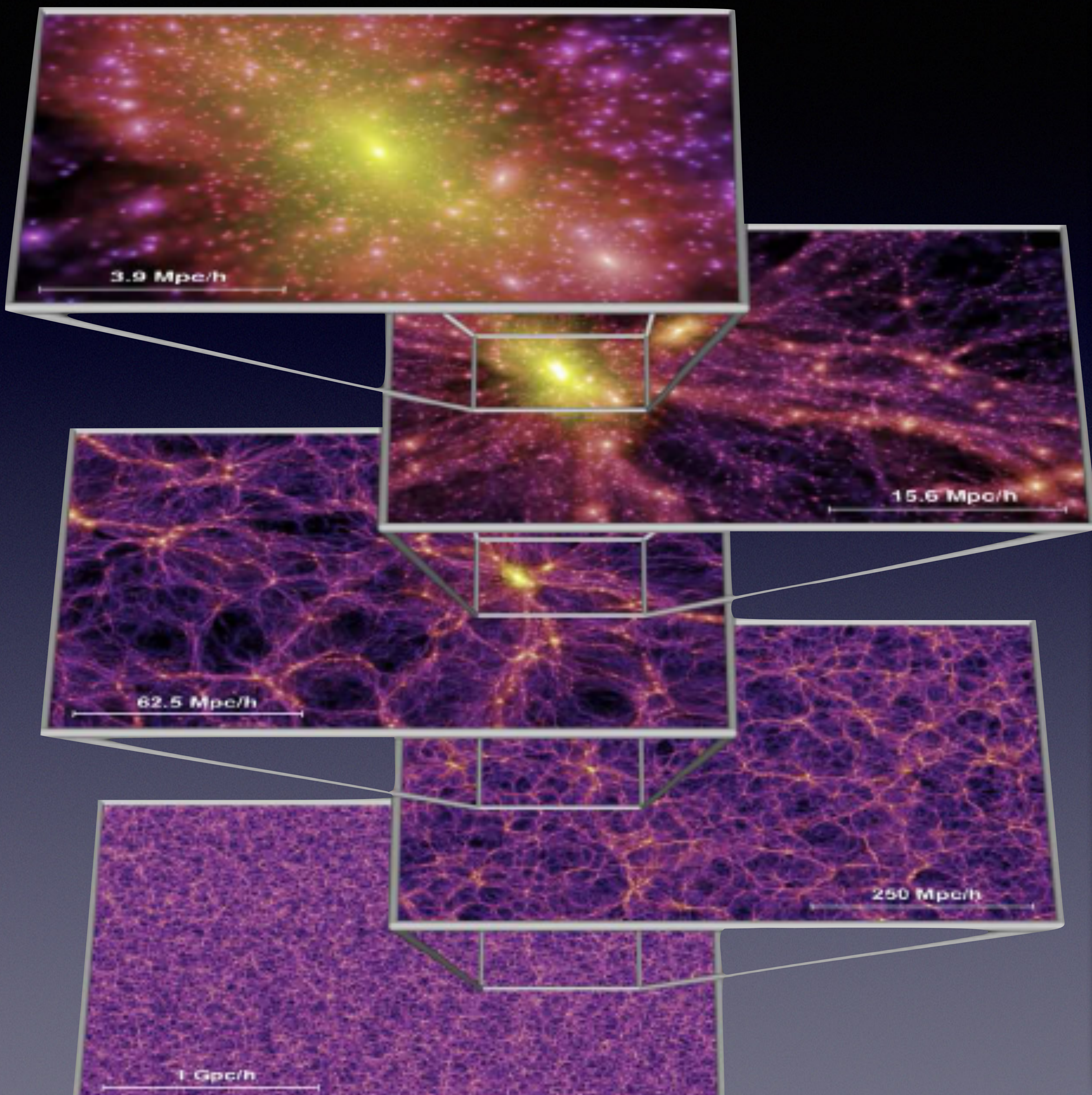
Although the EPS mass function is used very frequently in modern astronomy, it is important to be aware of its assumptions, shortcomings and pitfalls:

The Spherical Cow: The upcrossing barrier used is based on the spherical collapse model; as we have seen collapse is believed to be ellipsoidal instead...

As we will see, though, this can be taken into account...

Finally, the mere idea that one can use the linear density field to identify collapsed structures in the non-linear field constitutes a leap of faith...

Comparison with Numerical Simulations



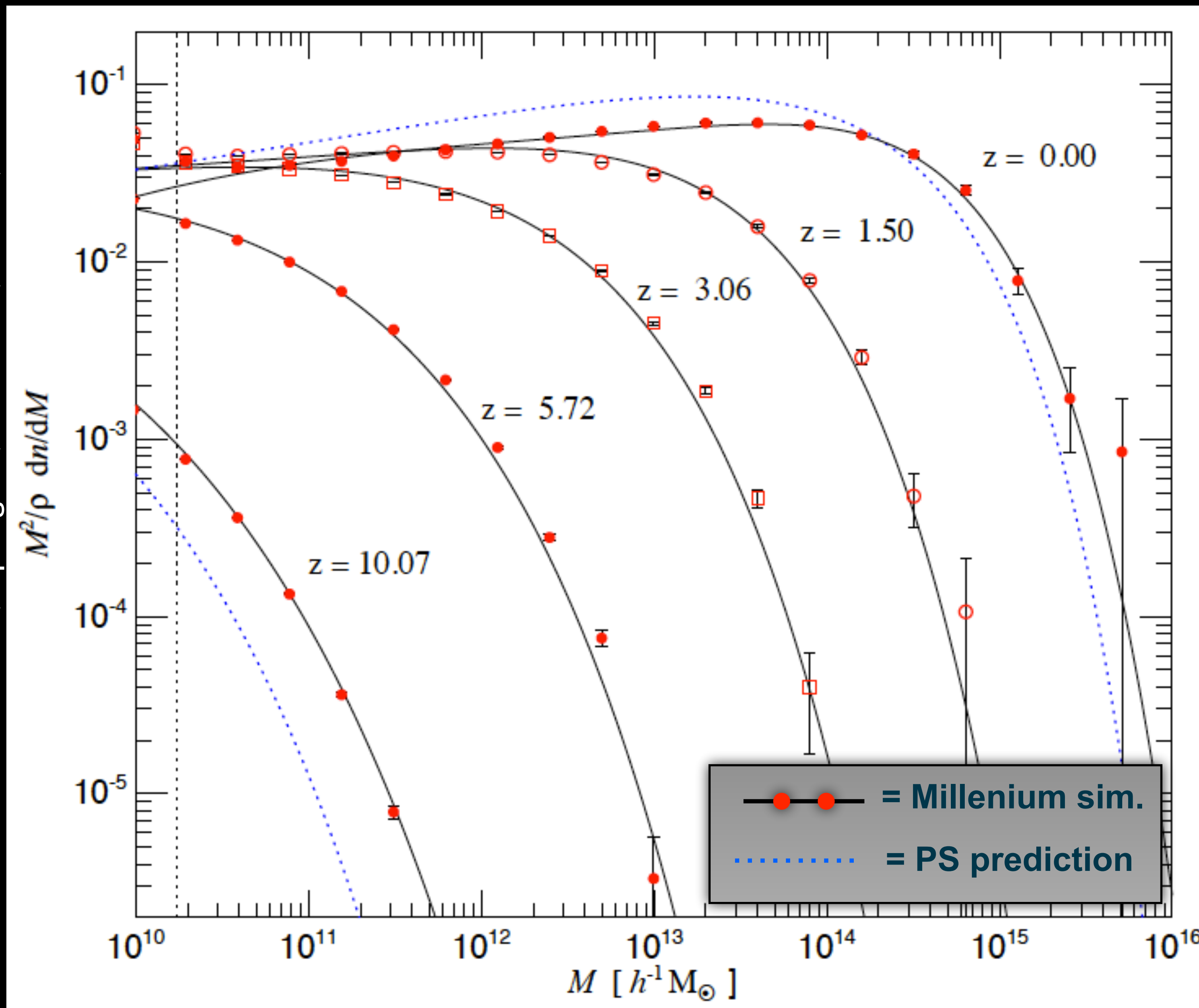
Given the various crude assumptions underlying the PS & EPS formalisms, it is important to test their predictions for halo mass function against numerical simulations...

These follow the growth & collapse of structures directly by solving the equations of motion for dark matter particles. However, as will be discussed later, identifying haloes in simulations is a non-trivial task.....

Until end of 1990s, most simulations yielded results in fair agreement with PS predictions....

However, when larger and more accurate simulations became available, it became clear that there where some problems....

Comparison with Numerical Simulations



Source: Millenium Simulation; Springel V., et al. 2005, Nature, 435. 629

The Millenium Simulation followed the evolution of 2160^3 (~10 billion) particles in a periodic box 500 Mpc/h on a side in a ΛCDM cosmology.

At the time it was run (2005) it was one of the biggest simulations to date. Because of its superb statistics, it is ideally suited to test the PS mass functions...

At low redshift, the PS mass function under- (over)-predicts the abundance of massive (low mass) haloes. These problems become more pronounced at higher redshifts...

WARNING: this statement is sensitive to how haloes are identified in the simulation box. Here a Friends-Of-Friends (FOF) algorithm has been used (see lecture 11)

EPS with Ellipsoidal Collapse

Ellipsoidal collapse and an improved model for the number and spatial distribution of dark matter haloes

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ABSTRACT

The Press–Schechter, excursion set approach allows one to make predictions about the shape and evolution of the mass function of bound objects. The approach combines the assumption that objects collapse spherically with the assumption that the initial density fluctuations were Gaussian and small. The predicted mass function is reasonably accurate, although it has fewer high-mass and more low-mass objects than are seen in simulations of hierarchical clustering. We show that the discrepancy between theory and simulation can be reduced substantially if bound structures are assumed to form from an ellipsoidal, rather than a spherical, collapse. In the original, standard, spherical model, a region collapses if the initial density within it exceeds a threshold value, δ_{sc}. This value is independent of the initial size of the region, and since the mass of the collapsed object is related to its initial size, this means that δ_{sc} is independent of final mass. In the ellipsoidal model, the collapse of a region depends on the surrounding shear field, as well as on its initial overdensity. In Gaussian random fields, the distribution of these quantities depends on the size of the region considered. Since the mass of a region is related to its initial size, there is a relation between the density threshold value required for collapse and the mass of the final object. We provide a fitting function to this $\delta_{ec}(m)$ relation which simplifies the inclusion of ellipsoidal dynamics in the excursion set approach. We discuss the relation between the excursion set predictions and the halo distribution in high-resolution N-body simulations, and use our new formulation of the approach to show that our simple parametrization of the ellipsoidal collapse model represents an improvement on the spherical model on an object-by-object basis. Finally, we show that the associated statistical predictions, the mass function and the large-scale halo-to-mass bias relation, are also more accurate than the standard predictions.

Key words: galaxies: clusters: general – cosmology: theory – dark matter.


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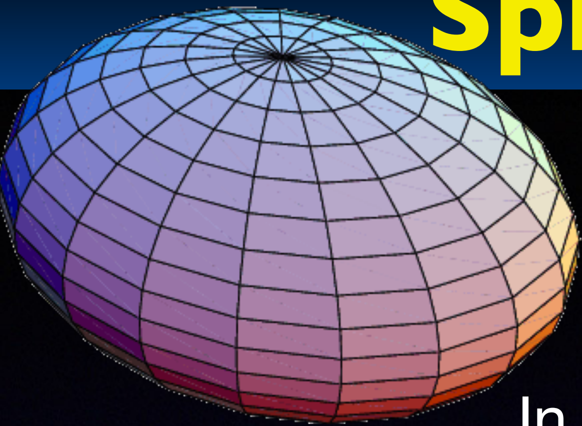
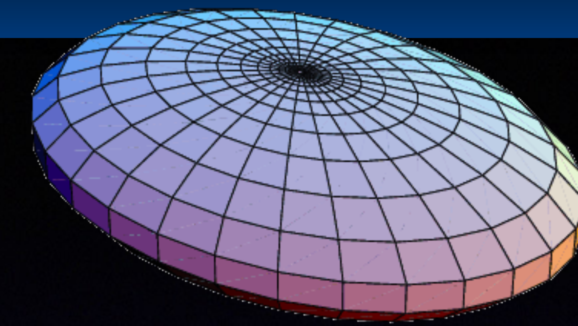


Spherical vs. Ellipsoidal Collapse



As we have seen, because of the non-zero tidal field, collapse will not be spherical, but ellipsoidal.

In that case, the critical (linear) over density for collapse is given by

$$\boxed{\frac{\delta_{\rm ec}}{\delta_{\rm sc}} \approx 1 + 0.47 \left[5(e^2 \pm p^2) \frac{\delta_{\rm ec}^2}{\delta_{\rm sc}^2} \right]^{0.615}}$$

Ellipsoidal collapse

Here $\delta_{\rm ec} = \delta_{\rm ec}(e,p)$ is the critical overdensity for ellipsoidal collapse, $\delta_{\rm sc} = \delta_{\rm c} \simeq 1.686$ is the critical overdensity for spherical collapse, and the parameters e and p characterize the asymmetry of the initial tidal field. (see lecture 8)

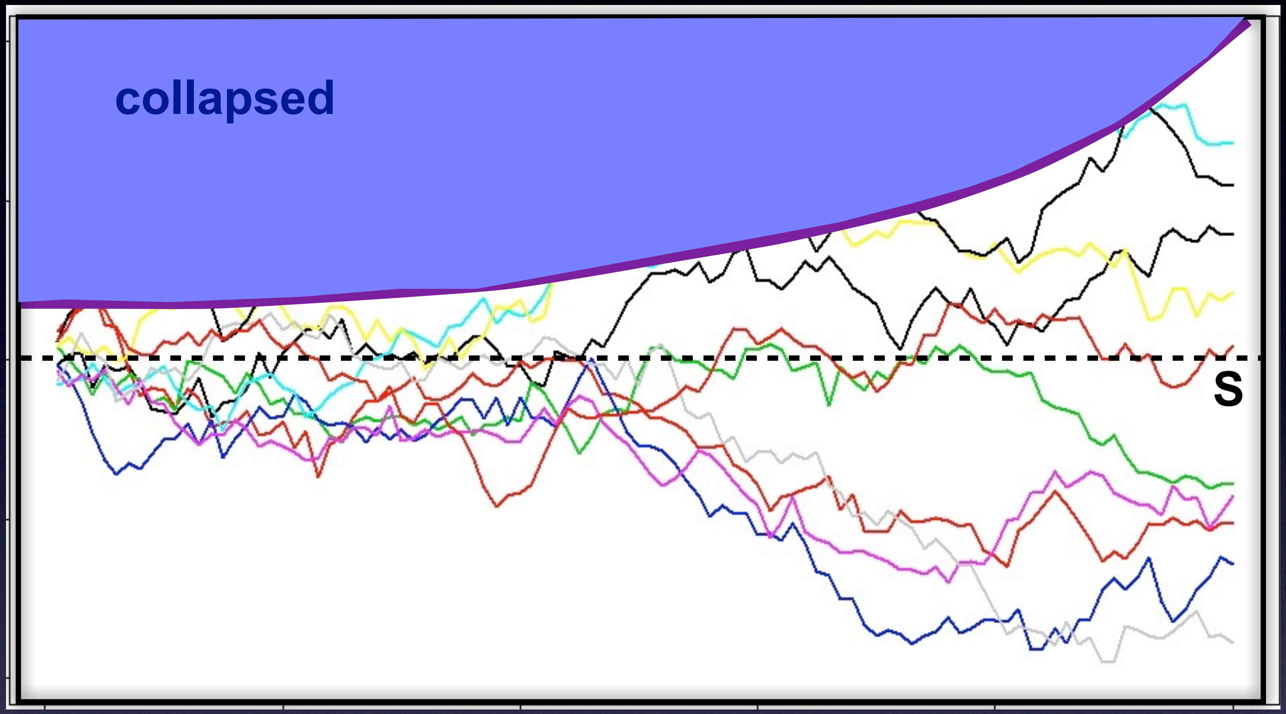
Adopting the most probable values for e and p, Sheth, Mo & Tormen (2001; SMT) showed that the upcrossing boundary for ellipsoidal collapse can be written as:

$$\boxed{\delta_{\rm ec} \simeq \delta_{\rm ec}(S,t) = \delta_{\rm c}(t) \left[1 + 0.47 \left(\frac{S}{\delta_{\rm c}^2(t)} \right)^{0.615} \right]}$$

Contrary for spherical collapse, for which the boundary is constant, the boundary for ellipsoidal collapse increases with S (less massive structures need higher overdensity for collapse). Because of this S-dependence, $\delta_{\rm ec}$, is called a "moving barrier".

EPS Mass Function for Ellipsoidal Collapse

Knowing the critical overdensity for ellipsoidal collapse, we can compute the corresponding PS mass function: all we need to do is to work out the first-upcrossing statistics....



Unfortunately, for a moving barrier one cannot compute this analytically. Rather, one has to resort to Monte Carlo simulations of independent random walks, and register their first upcrossings.

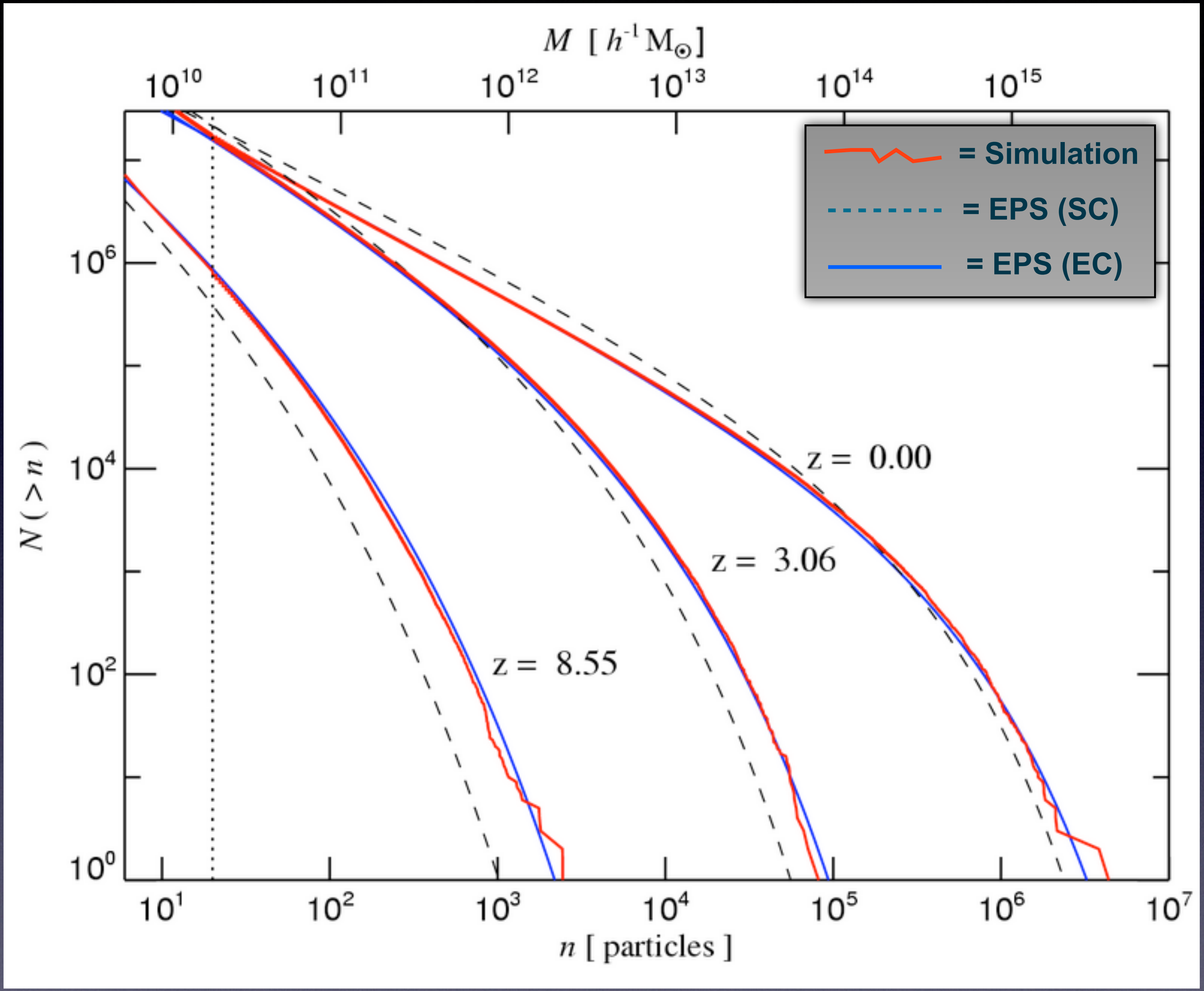
This was done by SMT, who found that the resulting multiplicity function is well approximated by

$$f_{\rm EC}(\nu) = 0.322\left[1 + \frac{1}{\tilde{\nu}^{0.6}}\right] f_{\rm PS}(\tilde{\nu}) \qquad \text{where} \qquad \tilde{\nu} = 0.84\,\nu$$

The normalization 0.322 is set by requiring that $\int_0^\infty n(M)\,M\,{\rm d}M = \bar{\rho}_{\rm m}$, which implies that <u>all</u> matter is in collapsed objects. The PS mass function for ellipsoidal collapse simply follows from replacing $f_{\rm PS}(\nu)$ with $f_{\rm EC}(\nu)$; i.e.,

$$n(M,t)\,{\rm d}M = \frac{\bar{\rho}}{M^2}\, f_{\rm EC}(\nu) \left|\frac{{\rm d}\ln\nu}{{\rm d}\ln M}\right| {\rm d}M$$

Spherical vs. Ellipsoidal Collapse



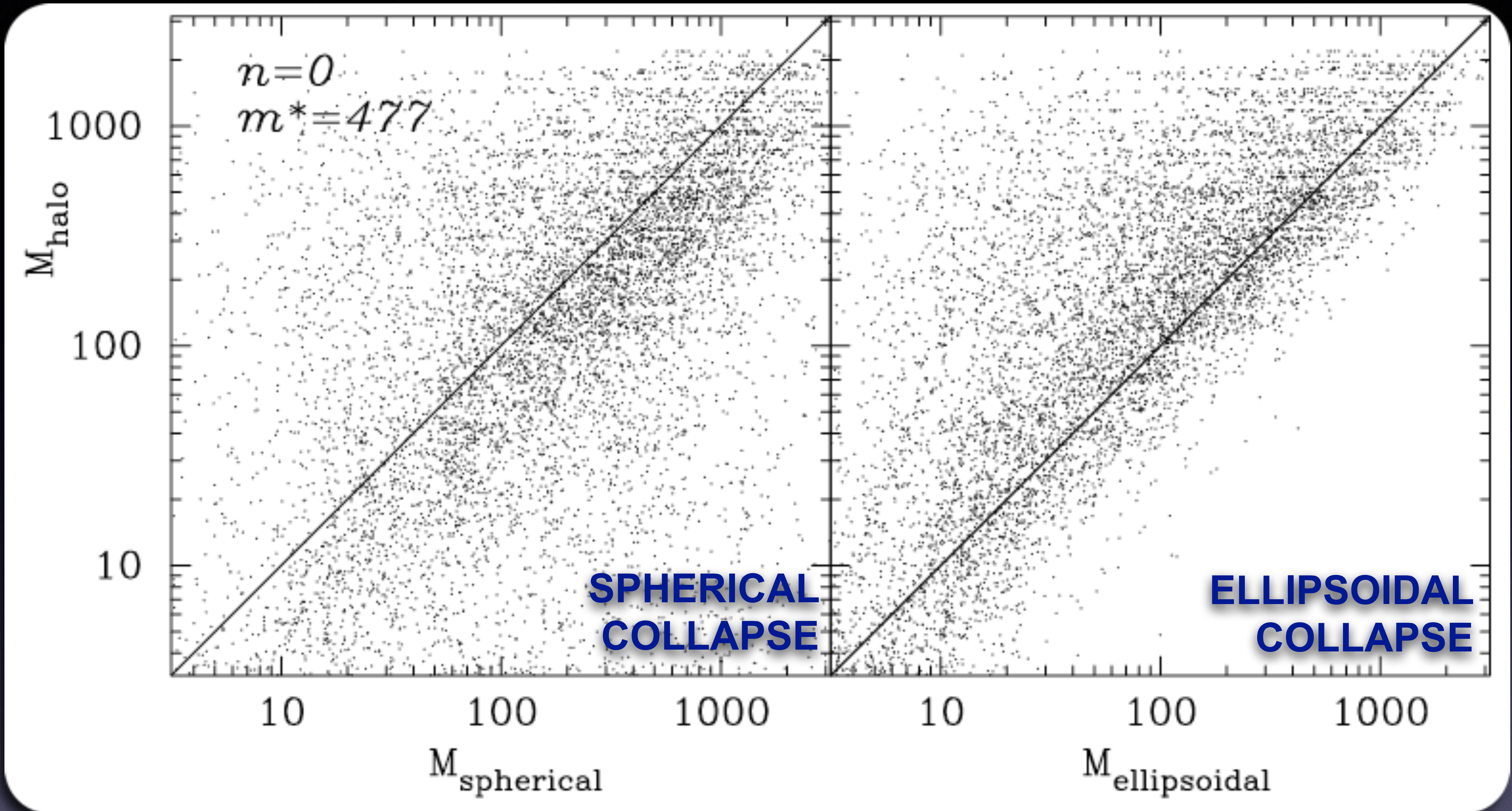
Source: Millenium Simulation; Springel V., 2004 (MPA research highlight)

The Millenium Simulation followed the evolution of 2160^3 (~10 billion) particles in a periodic box 500 Mpc/h on a side in a ΛCDM cosmology.

Clearly, the EPS mass function based on ellipsoidal collapse is in much better agreement with numerical simulations than the spherical collapse-based model prediction...

WARNING: this statement is sensitive to how haloes are identified in the simulation box. Here a Friends-Of-Friends (FOF) algorithm has been used (see lecture 11)

Spherical vs. Ellipsoidal Collapse

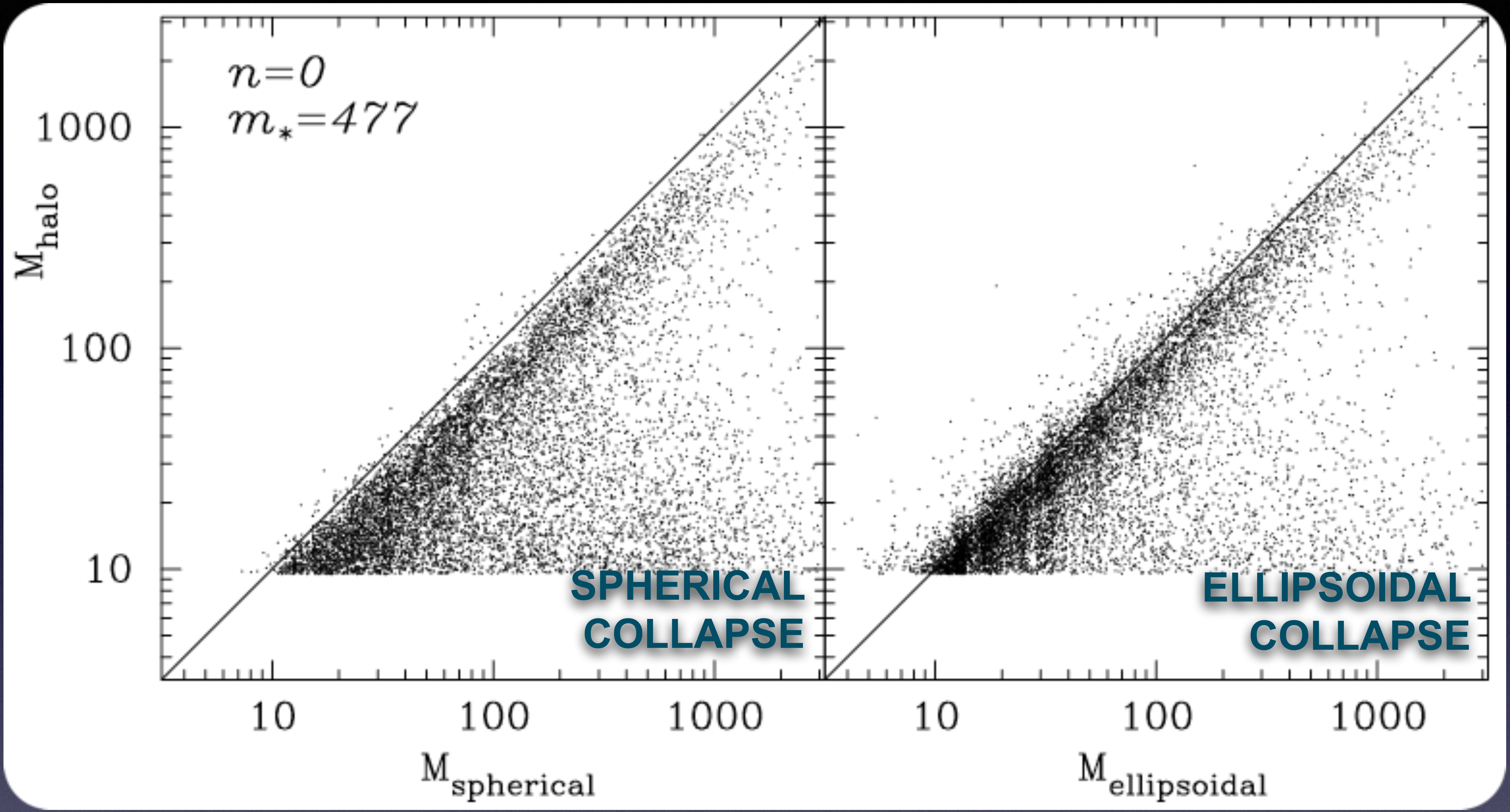


Source: Sheth, Mo & Tormen, 2001, MNRAS, 323, 1

Comparison of the mass of the halo of particles in a N-body simulation vs. the halo mass predicted by EPS based on the particle's location in the initial (linear) density field. Ellipsoidal collapse clearly performs much better than spherical collapse, but neither are very impressive... EPS performs poorly for individual particles (=mass elements), but nevertheless yields impressive results statistically....

Spherical vs. Ellipsoidal Collapse

Source: Sheth, Mo & Tormen, 2001, MNRAS, 323, 1



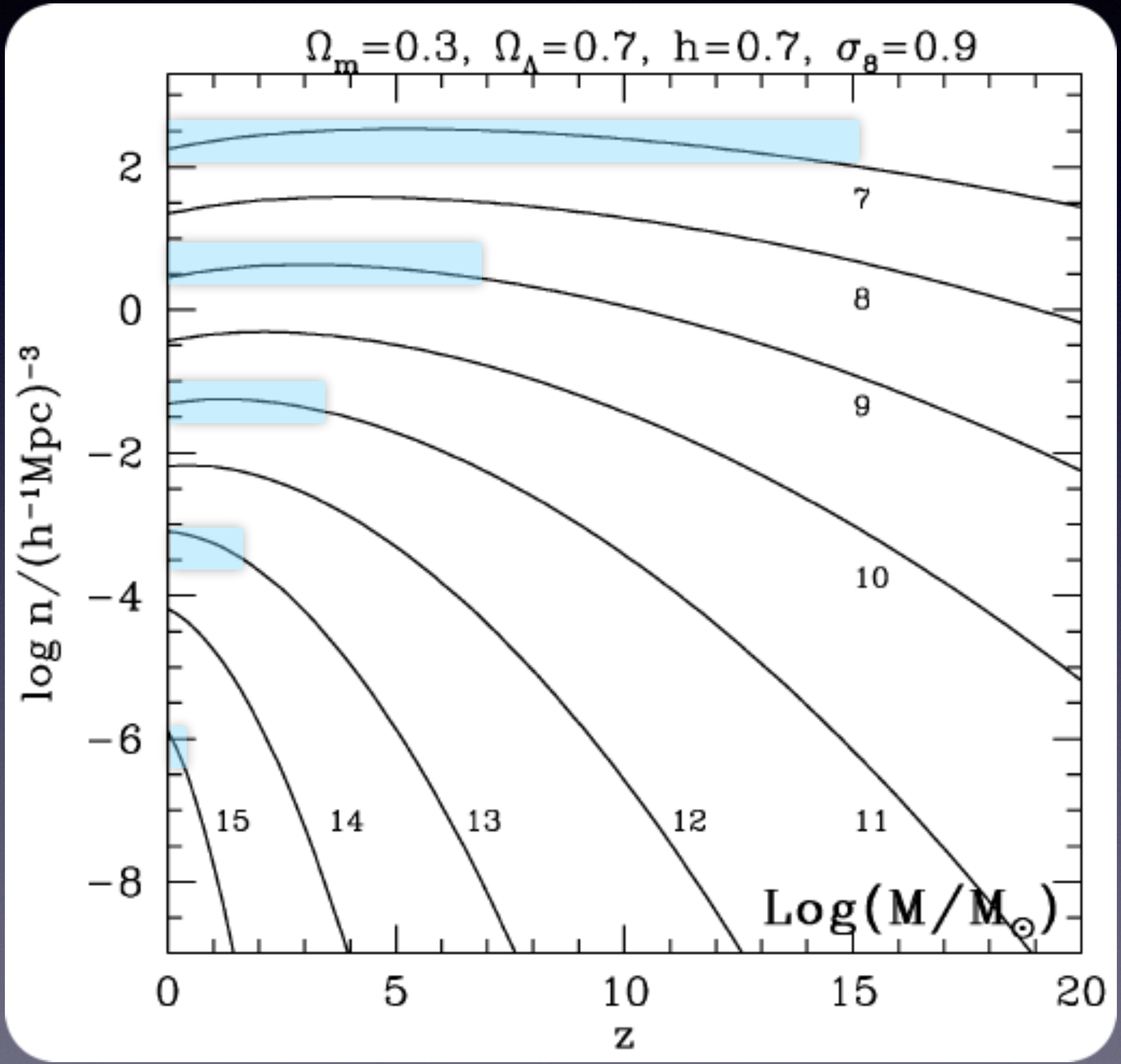
Same as on previous page, but this time only showing the results for the N-body particles located at the centers of their dark matter haloes. This removes the swath of points in the upper-left corner...for ellipsoidal collapse EPS is able to make object-by-object predictions that are not too far off...

Evolution of the Halo Mass Function

One can get some useful insight into how structure forms, by studying how the halo mass function (computed using EPS under ellipsoidal collapse conditions) evolves as function of time...

The figure to the right shows how the comoving number density of dark matter haloes of different mass evolve as function of redshift in a ΛCDM cosmology.

Note how the abundance of low mass haloes has hardly evolved at all since z=20, while the abundance of massive haloes is a very strong function of redshift. This is a manifestation of hierarchical structure formation.



Source: Mo & White, 2002, MNRAS, 336, 112

Lecture 9

SUMMARY

Summary: key words & important facts

Key words

(Extended) Press-Schechter	Mass Variance
Excursion Set Formalism	Halo Mass Function
Moving Collapse Barrier	Multiplicity Function
Markovian random walk	Characteristic Halo Mass

- Locations in linearly extrapolated density field where $\delta > \delta_c \simeq 1.686$ correspond to collapsed objects (halos)
- If $\delta(x)$ is Gaussian, then so is the smoothed density field $\delta(x;R)$
- Excursion sets are Markovian if, and only if, the density field is smoothed with a sharp-k space filter
- The cosmological parameter σ_8 is defined as the mass variance of the linearly extrapolated density field at $z=0$, smoothed with a Top-Hat filter of size $R=8\,h^{-1}$Mpc
- The ellipsoidal collapse model gives rise to a moving barrier in excursion set formalism

Summary: key equations & expressions

Mass Smoothing

$$\delta(\vec{x};R) \equiv \int \delta(\vec{x}')\,W(\vec{x}-\vec{x}';R)\,\mathrm{d}^3\vec{x}' \qquad \delta(\vec{k};R) = \delta(\vec{k})\,\widetilde{W}(kR)$$

Mass Variance

$$\sigma^2(M) = \langle\delta^2(\vec{x};R)\rangle = \frac{1}{2\pi^2}\int P(k)\,\widetilde{W}^2(kR)\,k^2\,\mathrm{d}k \qquad M = \gamma_{\rm f}\bar{\rho}R^3$$

(E)PS ansatz

PS $\quad F(>M,t) = 2\,\mathcal{P}[\delta_M > \delta_{\rm c}(t)]$

EPS $\quad F(>M,t) = 1 - F_{\rm FU}(>S) \qquad S = \sigma^2(M)$

Halo Mass Function

$$n(M,t) \equiv \frac{\mathrm{d}n}{\mathrm{d}M} = \frac{1}{M}\frac{\mathrm{d}n}{\mathrm{d}\ln M} = \frac{\bar{\rho}}{M}\frac{\partial F(>M,t)}{\partial M}$$

EPS + *Gaussian*

$$n(M,t) = \sqrt{\frac{2}{\pi}}\,\frac{\bar{\rho}}{M^2}\,\frac{\delta_{\rm c}}{\sigma_M}\,\exp\left(-\frac{\delta_{\rm c}^2}{2\sigma_{\rm M}^2}\right)\left|\frac{\mathrm{d}\ln\sigma_M}{\mathrm{d}\ln M}\right| = \frac{\bar{\rho}}{M^2}\,f_{\rm PS}(\nu)\left|\frac{\mathrm{d}\ln\nu}{\mathrm{d}\ln M}\right|$$

shorthand $\quad \nu \equiv \delta_{\rm c}(t)/\sigma(M) \qquad f_{\rm PS}(\nu) = \sqrt{2/\pi}\,\nu\,\mathrm{e}^{-\nu^2/2}$

Ellipsoidal Collapse Model

$$\delta_{\rm c}(t) \rightarrow \delta_{\rm c}(t)\left[1 + 0.47\left(\frac{\sigma^2(M)}{\delta_{\rm c}^2(t)}\right)^{0.615}\right]$$

$$f_{\rm PS}(\nu) \rightarrow f_{EC}(\nu) = 0.322\left[1 + \frac{1}{(0.84\nu)^{0.6}}\right] f_{\rm PS}(0.84\nu)$$

Characteristic Mass

$$\sigma^2(M^*) = \delta_{\rm c}(t)$$