The Evolution of Baryonic Perturbations

Master Equation:
\[
\frac{d^2 \delta_k}{dt^2} + 2\frac{\dot{a}}{a} \frac{d \delta_k}{dt} = \left[ 4\pi G \bar{\rho} - \frac{k^2 c_s^2}{a^2} \right] \delta_k - \frac{2}{3} \frac{T}{a^2} k^2 S_k
\]

Describes evolution of individual modes in Fourier space, sourced by gravity & pressure (i.e., by isentropic and isocurvature perturbations).

- Perturbations smaller than Jeans scale undergo acoustic oscillations.
- Silk damping arises from photon-diffusion close to recombination.
- After recombination, perturbations larger than Jeans length grow according to linear growth rate, \( D(a) \).
This was the picture for structure formation developed by Zel’dovich and his colleagues during the 1960’s in Moscow. However, it soon became clear that this picture was doomed.

To allow sufficient time for fragmentation, the large-scale perturbations need large amplitudes in order to collapse sufficiently early. At recombination one requires $|\delta_m| > 10^{-3}$

Using that $\delta_T = 1/4\delta_r = 1/3\delta_m$, which follows from fact that perturbations are isentropic and from $\rho_r \propto T^4$, this model implies CMB fluctuations $\Delta T/T > 10^{-3}$

Such large fluctuations were already ruled out in early 1980s (e.g., Uson & Wilkinson 1984)
While Zel’dovich was working on his adiabatic model, Peebles and his colleagues at Princeton developed an alternative model for structure formation, in which the perturbations were assumed to be isothermal (i.e., $\delta_T = 0$), which is a good approximation for isocurvature perturbations prior to the matter era.

In this model, sound speed prior to recombination is much lower, resulting in much lower Jeans mass. Also, since there are no radiation perturbations, there is NO Silk damping. All perturbations with $M > M_J \sim 10^6 M_\odot$ survive, and structure formation proceeds hierarchical (bottom-up).

Prior to recombination, radiation drag prevents perturbations from growing (they are ‘frozen’), but at least they are not damped...

Similar to adiabatic model, this isothermal baryonic model requires large temperature fluctuations in CMB to explain observed structure

In addition, isothermal perturbations are fairly “unnatural”. All these problems dissapear when considering a separate matter component: dark matter
We now turn our attention to Collisionless Dark Matter. Here `Dark' indicates that this matter has no EM interaction (no interaction with photons).

In what follows we use $X$ to indicate some generic particle species produced in the early Universe. We call such particles `cosmic relics'.

### Cosmic Relics

- **Non-thermal Relics**
  - Not produced in thermal equilibrium (TE) with rest of Universe.
  - e.g., axions, monopoles, cosmic strings

- **Thermal Relics**
  - Are held in TE with other components of Universe until they `decouple', which happens when interaction rate $\Gamma = n\langle\sigma v\rangle$ drops below expansion rate $H(a)$

### Dark Matter

- **Hot Dark Matter**
  - Particles are still relativistic at decoupling, i.e., $3k_B T_d > m_X c^2$
  - e.g., massive neutrinos

- **Cold Dark Matter**
  - Particles are non-relativistic at decoupling i.e., $3k_B T_d < m_X c^2$
  - e.g., WIMPS
In order to describe evolution of collisionless dark matter, in general one cannot use the fluid description. Instead, one has to resort to the Collisionless Boltzmann Equation (CBE):

Let $f(\vec{x}, \vec{v}, t) = dN/d^3\vec{x} d^3\vec{v}$ be the distribution function, which expresses the number of particles per unit volume in phase-space.

For a collisionless system we have that $df/dt = 0$.

This is the CBE, which expresses that in a collisionless system the phase-space density around each particle is conserved (i.e., there is no diffusion or scattering).

Using that $df = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x_i} dx_i + \frac{\partial f}{\partial v_i} dv_i$ (Einstein summation convention) and that $dv_i/dt = -\partial \Phi / \partial x_i$ we can write the CBE as:

$$\frac{\partial f}{\partial t} + v_i \frac{\partial f}{\partial x_i} - \frac{\partial \Phi}{\partial x_i} \frac{\partial f}{\partial v_i} = 0$$
Collisionless Fluids

Rather than solving CBE itself, one normally focuses on the moment equations. These are obtained by multiplying all terms by $v^k$ and integrating over velocity space. When using comoving coordinates in an expanding space with scale factor $a(t)$, one obtains, for $k = 0$

$$\frac{\partial \delta}{\partial t} + \frac{1}{a} \sum_j \frac{\partial}{\partial x_j} [(1 + \delta) \langle v_j \rangle] = 0$$

These are called the Jeans equations, and are the collisionless analog of the Euler equations for a fluid. The only difference is that the stress tensor $\rho \sigma_{ij}^2$ now plays the role of the pressure...

Similarly, for $k = 1$ one obtains

$$\frac{\partial \langle v_i \rangle}{\partial t} + \frac{\dot{a}}{a} \langle v_i \rangle + \frac{1}{a} \sum_j \langle v_i \rangle \frac{\partial \langle v_j \rangle}{\partial x_j} = -\frac{1}{a} \frac{\partial \Phi}{\partial x_i} - \frac{1}{\rho a (1 + \delta)} \sum_j \frac{\partial \rho \sigma_{ij}^2}{\partial x_j}$$

Here $\rho_{\text{com}}(\vec{x}) = \int f d^3 \vec{v} = \bar{\rho} a^3 [1 + \delta(\vec{x})]$, and $\langle v_i \rangle$ is the mean streaming motion in the $i$-direction.

NOTE: summation signs have been added for clarity, but can be removed according to the Einstein summation convention.
The velocity dispersion tensor is defined as

\[ \sigma_{ij}^2 \equiv \langle v_i v_j \rangle - \langle v_i \rangle \langle v_j \rangle \]

Since this tensor is manifest symmetric, i.e., \( \sigma_{ij} = \sigma_{ji} \) it is characterized by 6 independent numbers. Thus, whereas fluid pressure is characterized by a single scalar quantity, the stress tensor has 6 unknowns.

This also means that a simple `Equation-of-State' will NOT suffice to close the set of equations in general the Jeans equations cannot be solved.

One typically proceeds by making a number of simplifying assumptions (see MBW §5.4). In what follows, we will do the same and assume that the velocity dispersion is isotropic \( \sigma_{ij} = \sigma \delta_{ij} \)

If we further assume that \( \sigma \) is independent of location, then \( \frac{\nabla \rho \sigma^2}{\rho} = \sigma^2 \nabla \delta \)

Substituting this in the Jeans equations, and comparing the result to the Euler equations, it is clear that the velocity dispersion \( \sigma = \langle v_i^2 \rangle^{1/2} \) now plays the same role as the sound speed, \( c_s \), in the fluid equations.
Collisional vs. Collisionless Fluids

Moments of Collisionless Boltzmann Equation

\[
\frac{\partial \delta}{\partial t} + \frac{1}{a} \sum_j \frac{\partial}{\partial x_j} [(1 + \delta) \langle v_j \rangle] = 0
\]

\[
\frac{\partial \langle v_i \rangle}{\partial t} + \frac{\dot{a}}{a} \langle v_i \rangle + \frac{1}{a} \sum_j \langle v_j \rangle \frac{\partial \langle v_i \rangle}{\partial x_j} = -\frac{1}{a} \frac{\partial \Phi}{\partial x_i} - \frac{1}{\rho a(1 + \delta)} \sum_j \frac{\partial \rho \sigma_{ij}^2}{\partial x_j}
\]

**vector notation**

\[
\frac{\partial \delta}{\partial t} + \frac{1}{a} \nabla \cdot [(1 + \delta) \vec{v}] = 0
\]

\[
\frac{\partial \vec{v}}{\partial t} + \frac{\dot{a}}{a} \vec{v} + \frac{1}{a} (\vec{v} \cdot \nabla) \vec{v} = -\frac{\nabla \Phi}{a} - \frac{1}{a(1 + \delta)} \frac{\nabla \rho \sigma^2}{\rho}
\]

\[
\frac{\partial \delta}{\partial t} + \frac{1}{a} \nabla \cdot [(1 + \delta) \vec{v}] = 0
\]

\[
\frac{\partial \vec{v}}{\partial t} + \frac{\dot{a}}{a} \vec{v} + \frac{1}{a} (\vec{v} \cdot \nabla) \vec{v} = -\frac{\nabla \Phi}{a} - \frac{\sigma^2}{a} \frac{\nabla \delta}{1 + \delta}
\]

**continuity equation**

**Jeans equations**

**Euler equations**

Fluid Equations (see Lecture 4)
Thus, a collisionless fluid with isotropic and homogeneous velocity dispersion can be described by exactly the same continuity and momentum equations as a collisional fluid, but with the sound speed \( c_s \) replaced by \( \sigma = \left\langle v_i^2 \right\rangle^{1/2} \).

Hence we can define the collisionless analog of the Jeans length:

\[
\lambda_J^{\text{prop}} = a(t) \lambda_J^{\text{com}} = a(t) \frac{2\pi}{k_J} = \sigma \sqrt{\frac{\pi}{G \rho}}
\]

However, there is one very important difference wrt a collisional fluid:

**Collisional Fluid**
- \( \lambda > \lambda_J \) → gravitational collapse
- \( \lambda < \lambda_J \) → acoustic oscillations

**Collisionless Fluid**
- \( \lambda > \lambda_J \) → gravitational collapse
- \( \lambda < \lambda_J \) → free streaming damping

Free streaming damping arises because particles disperse in random directions due to their non-zero velocity dispersion. This `erases' perturbations with \( \lambda < \lambda_J \).

Free streaming (or collisionless) damping is basically the same as Landau damping for a collisionless plasma...
The fact that perturbations with $\lambda < \lambda_J$ damp due to `free streaming’ is easy to understand:

The time for collisionless particles to disperse over a distance $\lambda$ is $\tau_{\text{disp}} = \frac{\lambda}{v}$. Hence, the criterion $\lambda < \lambda_J$ implies that $\tau_{\text{disp}} < \sqrt{\frac{\pi}{G \bar{\rho}}}$, where we have used the definition of the Jeans length.

The Friedmann equation for a flat Universe (without cosmological constant) is:

$$H(a) = \sqrt{\frac{8\pi G \bar{\rho}}{3}}$$

from which we see that (roughly) $\tau_{\text{disp}} < \tau_H = 1/H(a)$

Hence for perturbations with $\lambda < \lambda_J$ the dispersion time is shorter than the Hubble time, and those perturbations will thus have dispersed (=damped).
We now proceed to estimate how far these neutrinos could have free-streamed up to the time of equality. We define the \textit{free-streaming length} as the comoving distance travelled, i.e.,

\[ \lambda_{fs}^{\text{com}} = \int_{0}^{t_{eq}} \frac{v(t')}{a(t')} \, dt' \]

As an example, consider massive neutrinos with a mass \( m_\nu c^2 \sim 30 \text{eV} \)

This was actually a popular dark matter candidate during early 1980s for two reasons:

- There was a (false) experimental claim that neutrinos indeed were this massive.
- One can show that \( \Omega_\nu \sim 0.32 h^2 (m_\nu c^2 / 30 \text{eV}) \) (of order unity, as required).

These neutrinos become \textit{non-relativistic} when \( 3 k_B T_\gamma \sim m_\nu c^2 \)

Using that \( T_\gamma = 2.7 \, \text{K}(1 + z) \) this yields \( (1 + z_{NR}) \simeq 4.3 \times 10^4 \)

A more accurate calculation that accounts for fact that after decoupling \( T_\nu \neq T_\gamma \) yields \( (1 + z_{NR}) \simeq 6.0 \times 10^4 \)

For comparison, neutrinos \textit{decouple} (’freeze-out’) at \( (1 + z_{\text{dec}}) \sim 4.3 \times 10^9 \)

so that we clearly have \( z_{NR} \ll z_{\text{dec}} \Rightarrow \) massive neutrinos are HDM

We now proceed to estimate how far these neutrinos could have free-streamed up to the time of equality. We define the \textit{free-streaming length} as the comoving distance travelled, i.e.,
Using that the peculiar velocity of neutrinos is $v \sim c$ prior to $t_{NR}$ and $v \propto a^{-1}$ thereafter, one obtains that at the time of equality

$$\lambda_{fs}^{\text{com}} = \frac{2c t_{NR}}{a_{NR}} \left[1 + \ln\left(\frac{a_{eq}}{a_{NR}}\right)\right]$$

Where we have used that $a(t) \propto t^{1/2}$ during the radiation dominated era.

Using that $1/a_{NR} = (1 + z_{NR}) \approx 6 \times 10^4$ one obtains a (comoving) free-streaming length of $\lambda_{fs}^{\text{com}} \approx 30 \text{ Mpc}$

The corresponding free-streaming mass is

$$M_{fs} \equiv \frac{\pi}{6} \bar{\rho}_{m,0} \left(\lambda_{fs}^{\text{com}}\right)^3 \approx 1.3 \times 10^{15} M_\odot$$

Thus, all perturbations on scales below that of a massive cluster will have been erased by the time of equality. †

Structure formation in HDM cosmologies proceeds top-down.

† **Note:** In problem set 2 you will compute $M_{fs}$ for a CDM model, and find it to be negligible....
In order to compute the Jeans length, and thus also the Jeans mass, for dark matter, all we need is to determine the velocity dispersion as a function of time. 

- When particles are still relativistic: \( v_X \sim c \)
- Particles that are non-relativistic, and decoupled: \( v_X \propto a^{-1} \)
- Particles that are non-relativistic, but still coupled to photons: \( v_X \propto a^{-1/2} \)

The latter follows from the fact that \( 3k_B T_\gamma = m_X v_X^2 \) and that \( T_\gamma \propto a^{-1} \).
The Jeans Length of Dark Matter

**Hot Relics (HDM)**

- $t < t_{NR} \Rightarrow \lambda_{J}^{\text{com}} \propto a$
- $t_{NR} < t < t_{eq} \Rightarrow \lambda_{J}^{\text{com}} \propto a^0$
- $t > t_{eq} \Rightarrow \lambda_{J}^{\text{com}} \propto a^{-1/2}$

**Cold Relics (CDM)**

- $t < t_{NR} \Rightarrow \lambda_{J}^{\text{com}} \propto a$
- $t_{NR} < t < t_{dec} \Rightarrow \lambda_{J}^{\text{com}} \propto a^{1/2}$
- $t_{dec} < t < t_{eq} \Rightarrow \lambda_{J}^{\text{com}} \propto a^0$
- $t > t_{eq} \Rightarrow \lambda_{J}^{\text{com}} \propto a^{-1/2}$
In order to complete our description of the growth of dark matter perturbations we now focus on perturbations with $\lambda > \lambda_J$ during the matter-dominated era:

Since $\sigma_X \propto a^{-1}$ we have that the `equivalent of pressure' $P_X = \rho_X \sigma_X^2 \propto a^{-5}$

Hence, the `pressure' drops like a rock, and we may thus treat our collisionless fluid as a pressureless fluid (but only for $\lambda > \lambda_J$!!!)

As we have already seen, the (linearized) fluid equation describing perturbation growth of pressureless fluid is

$$\frac{d^2 \delta_k}{dt^2} + 2\frac{\dot{a}}{a} \frac{d\delta_k}{dt} = 4\pi G \bar{\rho}_m \delta_k$$

When discussing baryonic perturbation growth, we have already seen that the growing mode solution for such perturbations is given by $\delta \propto D(a)$

This linear growth rate applies to both baryonic matter and collisionless dark matter!

EdS cosmology $\Rightarrow$ $D(a) \propto a$

$\Lambda$CDM cosmology $\Rightarrow$ $D(a) \propto a^\gamma$ ($\gamma < 1$)

Once the cosmological constant starts to dominate, the accelerated expansion suppresses structure formation. This is simply a consequence of the increased Hubble drag.
Since whereas, one has different growth rates during the matter- and radiation-dominated periods. Introducing the new time variable this can be written as:

$$\frac{\ddot{\delta}_k}{a^2} + 2\frac{\dot{a}}{a} \frac{\dot{\delta}_k}{a} = 4\pi G (\bar{\rho}_m + \bar{\rho}_r) \delta_k$$

(for $t < t_{eq}$)

new term

Since $\bar{\rho}_m \propto a^{-3}$ whereas $\bar{\rho}_r \propto a^{-4}$, one has different growth rates during the matter- and radiation-dominated periods. Introducing the new time variable $\zeta \equiv \bar{\rho}_m/\bar{\rho}_r = a/a_{eq}$ this can be written as:

$$\frac{d^2\delta_k}{d\zeta^2} + \frac{(2 + 3\zeta)}{2\zeta(1 + \zeta)} \frac{d\delta_k}{d\zeta} = \frac{3}{2} \frac{\delta_k}{\zeta(1 + \zeta)}$$

for which the growing mode solution is $\delta_+ \propto 1 + \frac{3}{2} \zeta$

$t \ll t_{eq} \quad \rightarrow \quad \zeta \ll 1 \quad \rightarrow \quad \delta_+ \text{ constant} \quad \rightarrow \quad \text{stagnation}$

$t \gg t_{eq} \quad \rightarrow \quad \zeta \gg 1 \quad \rightarrow \quad \delta_+ \propto a \quad \rightarrow \quad \text{growth}$
The stagnation of growth in pressureless matter perturbations during radiation dominated era is known as the Meszaros effect. This differs from stagnation due to radiation drag that plays role in isothermal, baryonic perturbations. After all, CDM experiences no radiation drag (i.e., has no EM interaction).

The Meszaros effect is simply a manifestation of the fact that the Hubble drag term during the radiation dominated era is larger than during the matter dominated era. Consider the following qualitative argument:

Characteristic time for growth (‘collapse’) of perturbation of pressureless material (e.g., dark matter) is the free-fall time \( \tau_{ff} \propto (G \rho_m)^{-1/2} \)

For comparison, the characteristic time for the expansion of the Universe is the Hubble time \( \tau_H = 1/H \). Using the Friedmann equation, one immediately sees that \( \tau_H \propto (G \rho_r)^{-1/2} \) during radiation era, and \( \tau_H \propto (G \rho_m)^{-1/2} \) during matter era.

Hence, \( \tau_H/\tau_{ff} \propto (\rho_m/\rho_r)^{1/2} \) during radiation era, strongly suppressing structure growth. During matter era, on the other hand, \( \tau_H/\tau_{ff} \approx 1 \)
Growth of Dark Matter Perturbations

**Hot Relics (HDM)**

- \( t < t_{NR} \) \( \Rightarrow \lambda_{J}^{\text{com}} \propto a \)
- \( t_{NR} < t < t_{eq} \) \( \Rightarrow \lambda_{J}^{\text{com}} \propto a^{0} \)
- \( t > t_{eq} \) \( \Rightarrow \lambda_{J}^{\text{com}} \propto a^{-1/2} \)

**Cold Relics (CDM)**

- \( t < t_{NR} \) \( \Rightarrow \lambda_{J}^{\text{com}} \propto a \)
- \( t_{NR} < t < t_{dec} \) \( \Rightarrow \lambda_{J}^{\text{com}} \propto a^{1/2} \)
- \( t_{dec} < t < t_{eq} \) \( \Rightarrow \lambda_{J}^{\text{com}} \propto a^{0} \)
- \( t > t_{eq} \) \( \Rightarrow \lambda_{J}^{\text{com}} \propto a^{-1/2} \)
Growth of isentropic perturbations in a mixture of radiation, baryons and CDM.

- The acoustic oscillations of the baryons during 2 - 6 are now driven by the potential perturbations in the dark matter component. (see MBW §4.1.6c)

- After recombination, baryons fall in DM potential wells; gravity `re-creates’ the baryonic perturbations, un-doing the effect of Silk damping.
Growth of isentropic perturbations in a mixture of radiation, baryons and CDM.

- The acoustic oscillations of the baryons during 2 - 6 are now driven by the potential perturbations in the dark matter component. (see MBW §4.1.6c)

- After recombination, baryons fall in DM potential wells; gravity `re-creates’ the baryonic perturbations, un-doing the effect of Silk damping.
The Poisson equation in Fourier space reads:

\[ -k^2 \Phi_k = 4\pi G \bar{\rho} a^2 \delta_k \]

In a matter dominated Universe \( \bar{\rho} \propto a^{-3} \), so that

\[ \Phi_k \propto D(a)/a \]

EdS cosmology: \( D(a) \propto a \) \( \Rightarrow \) \( \Phi_k \propto \text{const} \)

\( \Lambda \)CDM cosmology: \( D(a) \propto a^\gamma \) \( \Rightarrow \) \( \Phi_k \propto a^{\gamma-1} \)

Since \( \gamma < 1 \) for a typical \( \Lambda \)CDM cosmology, potential perturbations (in the linear regime), will decay with time.

A photon moving through (linear) perturbation will fall into deeper potential well than what it climbs out of: it gains energy. This implies that large-scale (=linear) structure between last-scattering surface and us produce temperature fluctuations in CMB. This is called the Integrated Sachs Wolfe (ISW) effect.

**NOTE:** In an EdS cosmology the ISW effect is absent!
It is customary to distinguish

“early-time” ISW effect: arises immediately after recombination, when radiation-contribution is still significant

“late-time” ISW effect: arises at late times (close to $z=0$) due to impact of cosmological constant

ISW effect reveals itself in the form of cross-correlation between matter distribution (e.g., as probed by galaxies) and CMB temperature fluctuations.

Tens of studies have tried to detect (late-time) ISW effect. Most detections are mildly significant at best ($<5\sigma$). One of the most intriguing detections is shown here, based on SDSS-LRG vs WMAP5.

Granett, Neyrinck & Szapudi (2008)
### Key words

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- A collisionless fluid with isotropic and homogeneous velocity dispersion is described by the same continuity and momentum equations as a collisional fluid, but with the sound speed \( c_s \) replaced by \( \sigma = \langle v_i^2 \rangle^{1/2} \).

- A collisionless fluid does **not** have an EoS; moment equations are not a closed set.

- Collisionless dark matter and baryonic matter have the same linear growth rate.

- **Collisional fluid**: perturbations below Jeans mass undergo acoustic oscillations
  **Collisionless fluid**: perturbations below Jeans mass undergo free streaming.

- After recombination, baryons fall in DM potential wells, thereby un-doing Silk damping.

- The integrated Sachs-Wolfe effect probes (linear) growth rate of structure. In an EdS cosmology \( D(a) \propto a \) and the ISW effect is absent.
Collisionless Boltzmann Equation (CBE)
\[ \frac{df}{dt} = \frac{\partial f}{\partial t} + v_i \frac{\partial f}{\partial x_i} - \frac{\partial \Phi}{\partial x_i} \frac{\partial f}{\partial v_i} = 0 \]

Moment equations: multiply all terms by \( v_i^k \) and integrate over all of velocity space

\( k = 0 \) → Continuity equation
\[ \frac{\partial \delta}{\partial t} + \frac{1}{a} \sum_j \frac{\partial}{\partial x_j} [(1 + \delta) \langle v_j \rangle] = 0 \]

\( k = 1 \) → Jeans equations
\[ \frac{\partial \langle v_i \rangle}{\partial t} + \frac{\dot{a}}{a} \langle v_i \rangle + \frac{1}{a} \sum_j \langle v_j \rangle \frac{\partial \langle v_i \rangle}{\partial x_j} = -\frac{1}{a} \frac{\partial \Phi}{\partial x_i} - \frac{1}{\bar{\rho}a(1 + \delta)} \sum_j \frac{\partial \rho \sigma_{ij}^2}{\partial x_j} \]

Free-streaming scale
\[ \chi_{fs}^{\text{com}} = \int_{0}^{t_{\text{eq}}} \frac{v(t')}{a(t')} dt' \]

Linear growth rate
- EdS cosmology \( \Rightarrow D(a) \propto a \)
- \( \Lambda \)CDM cosmology \( \Rightarrow D(a) \propto a^{\gamma} \) \((\gamma < 1)\)

Poisson equation in Fourier space:
\[ -k^2 \Phi_k = 4\pi G \bar{\rho} a^2 \delta_k \]

In matter dominated Universe:
\[ \bar{\rho} \propto a^{-3} \]
\[ \Phi_k \propto D(a)/a \]