Thus far we have focussed on an unperturbed Universe. In this lecture we examine how small perturbations grow and evolve in a FRW metric (i.e., in a expanding space-time).

Topics that will be covered include:

- Newtonian Perturbation Theory
- Equation of State
- Jeans Criterion
- Horizons
- Linear Growth Rate
- Damping (Silk & Free Streaming)
- Meszaros Effect
In Lecture 4 we derived the following fluid equations in comoving coordinates:

**continuity equation**
\[
\frac{\partial \delta}{\partial t} + \frac{1}{a} \nabla \cdot [(1 + \delta) \vec{v}] = 0
\]

**Euler equations**
\[
\frac{\partial \vec{v}}{\partial t} + \frac{\dot{a}}{a} \vec{v} + \frac{1}{a} (\vec{v} \cdot \nabla) \vec{v} = -\frac{\nabla \Phi}{a} - \frac{c_s^2}{a(1 + \delta)} - \frac{2T}{3a} \nabla S
\]

**Poisson equation**
\[
\nabla^2 \Phi = 4\pi G \bar{\rho} a^2 \delta
\]
The Linearized Fluid Equations

After linearization:

continuity equation

$$\frac{\partial \delta}{\partial t} + \frac{1}{a} \nabla \vec{v} = 0$$

Euler equations

$$\frac{\partial \vec{v}}{\partial t} + \frac{\dot{a}}{a} \vec{v} = -\frac{\nabla \Phi}{a} - \frac{c_s^2}{a} \nabla \delta - \frac{2T}{3a} \nabla S$$

Poisson equation

$$\nabla^2 \Phi = 4\pi G \rho a^2 \delta$$
Differentiating continuity equation, and substituting Euler & Poisson Equations:

\[ \frac{\partial^2 \delta}{\partial t^2} + \frac{2 \dot{a}}{a} \frac{\partial \delta}{\partial t} = 4\pi G \bar{\rho} \delta + \frac{c_s^2}{a^2} \nabla^2 \delta + \frac{2}{3} \frac{T}{a^2} \nabla^2 S \]

- 'Hubble drag' term, expresses how expansion suppresses perturbation growth.
- Gravitational term, expresses how gravity promotes perturbation growth.
- Pressure terms, expressing how pressure gradients due to spatial gradients in density and/or entropy influence perturbation growth.
The Master Equation in Fourier Space

After Fourier Transformation:

**Master Equation:**

\[
\frac{d^2 \delta_k}{dt^2} + 2\frac{\dot{a}}{a}\frac{d\delta_k}{dt} = \left[ 4\pi G \rho - \frac{k^2 c_s^2}{a^2} \right]\delta_k - \frac{2}{3} \frac{T}{a^2} k^2 S_k
\]

\[
\delta(\vec{x}, t) = \sum_k \delta_k(t)e^{+i\vec{k} \cdot \vec{x}}
\]

\[
\delta_S(\vec{x}, t) \equiv \frac{S(\vec{x}, t) - \bar{S}(t)}{\bar{S}(t)} = \sum_k S_k(t)e^{+i\vec{k} \cdot \vec{x}}
\]

**NOTE:** in linear theory, all modes evolve independently.
**Types of Perturbations**

- **Isentropic Perturbations**
  \[ \delta_S = 0 \] (pure density perturbations)

- **Isocurvature Perturbations**
  \[ \delta = 0 \] (pure entropy perturbations)

\[
\delta_S = \frac{\partial S}{S} = \frac{1}{S} \left[ \frac{\partial S}{\partial \rho_r} \partial \rho_r + \frac{\partial S}{\partial \rho_m} \partial \rho_m \right] = \frac{3}{4} \delta_r - \delta_m
\]

**Isentropic Perturbations:** \[ \frac{\delta_r}{\delta_m} = \frac{4}{3} \]

**Isocurvature Perturbations:** \[ \frac{\delta_r}{\delta_m} = -\left( \frac{\bar{\rho}_m}{\bar{\rho}_r} \right) = -\left( \frac{a}{a_{eq}} \right) \]

- **Isentropic perturbations** are often called **adiabatic perturbations**. However, it is better practice to reserve `adiabatic` to refer to an evolutionary process.

- **If evolution is adiabatic**, isentropic perturbations remain **isentropic**. If not, the **non-adiabatic** processes create non-zero \( \nabla S \).

- **At early times**, isocurvature perturbations obey approximately \[ \delta_r = 0 \]. This is why they are sometimes called **isothermal perturbations**.
The matter perturbations that we are describing consist of both baryons and dark matter (assumed to be collisionless).

In what follows we will first treat these separately.

- We start by considering a Universe without dark matter (only baryons + radiation).
- Next we considering a Universe without baryons (only dark matter + radiation).
- We end with discussing a more realistic Universe (radiation + baryons + dark matter).
The special case \( \omega = 0 \) defines a characteristic mode, \( k_J \), which translates into a characteristic scale

\[
\lambda_J^{\text{prop}} = a(t) \lambda_J^{\text{com}} = a(t) \frac{2\pi}{k_J} = c_s \sqrt{\frac{\pi}{G \bar{\rho}}}
\]

Hence, we have the following Jeans criterion:

- \( \lambda < \lambda_J \) \quad \omega^2 > 0 \quad \delta_k(t) \propto e^{\pm i\omega t} \quad \text{sound wave, propagating w. sound speed}
- \( \lambda > \lambda_J \) \quad \omega^2 < 0 \quad \delta_k(t) \propto e^{\pm i\omega t} \quad \text{static mode, growing or decaying exponentially with time}
**Prior to recombination:** photon-baryon fluid 

\[ c_s = \frac{c}{\sqrt{3}} \left[ \frac{3}{4} \frac{\rho_b(t)}{\rho_r(t)} + 1 \right]^{-1/2} \]

(see problem set 2)

\[ \rho_b \propto a^{-3} \]
\[ \rho_r \propto a^{-4} \] \( t < t_{eq} \) \( \Rightarrow \) \( c_s = c/\sqrt{3} \propto a^0 \)
\( t > t_{eq} \) \( \Rightarrow \) \( c_s \propto a^{-1/2} \)

**WARNING:** the fluid equations that we used above are only valid for a non-relativistic fluid; our Newtonian treatment is not valid for the tightly coupled photon-baryon fluid. However, we may still use the Newtonian Jeans criterion for an order-of-magnitude analysis, which is what we will do here and in what follows. A proper treatment requires solving the Boltzmann equation in a perturbed space-time metric, which is beyond the scope of these lectures. (see MBW §4.2 for details if interested)
The Jeans Mass

Prior to recombination: photon-baryon fluid
\[ c_s = \frac{c}{\sqrt{3}} \left[ \frac{3 \rho_b(t)}{4 \rho_r(t)} + 1 \right]^{-1/2} \]
\[
\begin{align*}
\rho_b &\propto a^{-3} & t < t_{eq} &\Rightarrow c_s = c/\sqrt{3} \propto a^0 \\
\rho_r &\propto a^{-4} & t > t_{eq} &\Rightarrow c_s \propto a^{-1/2}
\end{align*}
\]

After recombination: baryon fluid is `ideal gas'
\[ c_s = (\partial P/\partial \rho)^{1/2} \propto T^{1/2} \]
\[ T \propto a^{-2} \Rightarrow c_s \propto a^{-1} \]

Using Jeans length, we can also define Jeans mass:
\[
M_J = \frac{4\pi}{3} \bar{\rho} \left( \frac{\lambda_J}{2} \right)^3 = \frac{\pi}{6} \bar{\rho} \lambda_J^3
\]

- Immediately after recombination,
  \[ M_J = 1.5 \times 10^5 (\Omega_{b,0} h^2)^{-1/2} M_\odot \]
  while at matter-radiation equality,
  \[ M_J = 1.5 \times 10^{16} (\Omega_{b,0} h^2)^{-2} M_\odot \]

- At recombination, photons decouple from baryons, which dramatically reduces the pressure, causing a huge drop in the Jeans mass...
Using how the sound speed and density scale with the scale factor we obtain the following evolution of the Jeans length.

**Baryon** perturbations with $\lambda > \lambda_J$ grow, while those with $\lambda < \lambda_J$ become acoustic waves.
According to the Jeans criterion discussed above, which ignored the expansion of the Universe, baryonic perturbations with \( \lambda > \lambda_J \) will grow exponentially.

In the case of expansion, there are two modifications:
- Hubble drag
- Horizons

As is evident from linearized fluid equation, Hubble drag resists perturbation growth:
If \( \lambda > \lambda_J \) perturbations do not grow exponentially, but as a power-law \( \delta_k(t) \propto t^a \)
The index \( a \) depends on cosmology, EoS and epoch, as discussed at a later stage.

In addition, in an expanding space-time, one needs to account for the presence of horizons, and distinguish sub-horizon perturbations (those with \( \lambda < \lambda_H \)) from super-horizon perturbation (\( \lambda > \lambda_H \)).
In an expanding space-time, one needs to account for the presence of horizons.

Inflation creates perturbations on all scales, including on super-horizon scales.

A proper treatment of super-horizon perturbations requires general relativistic perturbation theory, which is beyond the scope of this course (see MBW §4.2)

Crudely speaking, a super-horizon perturbation (\(\lambda > \lambda_H\)) doesn’t “know” it is a perturbation as it has no causal knowledge of the metric on scales larger than itself. → metric perturbations with \(\lambda > \lambda_H\) don’t evolve (‘frozen’).
Horizons

Recall: a photon moves along a geodesic (ds=0). Since we can always choose our coordinate system such that the photon moves along $d\theta = d\phi = 0$, the FRW metric implies that a photon path is characterized by $d\chi = d\tau$.

Using that the conformal time interval $d\tau = c\, dt/a(t)$, and that $dt = da/\dot{a}$

$$\chi(r_e) = \int_{t_e}^{t_0} \frac{c\, dt}{a} = \int_{a_e}^{a_0} \frac{c\, da}{a\, \dot{a}}$$

where the subscripts $e$ and $o$ refer to `emitted’ and `observed’.

If $\chi(r_e)$ converges to a finite value $\chi_H$ when $a_e \rightarrow 0$, then there are events for which $\chi > \chi_H$ and from which no communication can have reached the observer.

$$\chi_H = \int_0^{t_0} \frac{c\, dt}{a}$$

is called the comoving particle horizon

If $\lim_{t_0 \rightarrow \infty} \chi_H$ converges to a finite value, it means that there are fundamental observers with whom the observer can never communicate. Such fundamental observers are said to lie outside the comoving event horizon.

In what follows we are mainly concerned with particle horizons only....
Using the Friedmann equation

\[
\left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \rho - \frac{K c^2}{a^2}
\]

(see Lecture 3)

we can write the dependence of the comoving (particle) horizon on scale factor as

\[
\chi_H(a) = \int_0^a \frac{c \, da}{a \, \dot{a}} = \int_0^a \frac{da}{a} \left[ \frac{8\pi G \rho a^2}{3 c^2} - K \right]^{-1/2}
\]

For a flat Universe (\(K=0\)), this reduces to

\[
\chi_H(a) = \left( \frac{3 c^2}{8\pi G} \right)^{1/2} \int_0^a \frac{da}{a^2 \sqrt{\rho}}
\]

\[
t \ll t_{eq} \quad \Rightarrow \quad \rho \propto a^{-4} \quad \Rightarrow \quad \chi_H(a) \propto \int_0^a da \propto a
\]

\[
t \gg t_{eq} \quad \Rightarrow \quad \rho \propto a^{-3} \quad \Rightarrow \quad \chi_H(a) \propto \int_0^a a^{-1/2} da \propto a^{1/2}
\]

\[
\Lambda \text{ dominates} \quad \Rightarrow \quad \rho \propto a^0 \quad \Rightarrow \quad \chi_H(a) \propto \int_0^a a^{-2} da \propto a^{-1}
\]
It can be shown that the proper (particle) horizon during the radiation dominated era follows
\[ \lambda_{\text{prop}} = 2 ct \]
which is the same as the Jeans length during this era, except for a factor of order unity....
(see problem set 1)

For comparison, in a matter-dominated Universe we have that
\[ \lambda_{\text{prop}} = 3 ct \]

Using that super-horizon perturbations experience growth, we can distinguish three different regions in our scale-size diagram for the evolution of baryonic perturbations in an expanding space-time...

However, this picture is not yet complete....
Before **decoupling** photons & baryons are tightly coupled via Compton scattering. However, this coupling is imperfect as the photons have a mean-free path \( \lambda = (\sigma_T n_e)^{-1} \) that is not zero.

- Photon diffusion
- Damping of perturbations in photon distribution
- Damping of acoustic oscillations (also in the baryons)

This damping mechanism is known as **Silk damping**.

The **Silk damping** scale, \( \lambda_d \), is typical distance photon can diffuse in Hubble time.

- Let photon path be a random walk with mean step length \( \lambda \)
- During a time \( t \) the photon takes on average \( N = c t / \lambda \) steps
- Kinetic Theory \[ \lambda_d \simeq (N/3)^{1/2} \lambda = \left( \frac{c t}{3\sigma_T n_e} \right)^{1/2} \]

\( \sigma_T \) is Thomson cross-section
We can now complete our picture of evolution of baryonic perturbations:

**Silk damping** will `erase’ all baryonic fluctuations on scales $\lambda < \lambda_d$

Consequently, if matter is purely baryonic, after recombination there are no perturbations left on small scales so structure formation has to proceed in top-down fashion....

Detailed calculations, accounting for the evolution in ionization fraction, $X_e$, yield

**at recombination:**

$$\lambda_d^{\text{com}} \sim 5.7 (\Omega_{m,0} h^2)^{-3/4} \left( \frac{\Omega_{b,0}}{\Omega_{m,0}} \right)^{-1/2} \left( \frac{X_e}{0.1} \right)^{-1/2} \left( \frac{1 + z_{\text{dec}}}{1100} \right)^{-5/4} \text{Mpc}$$

$$M_d = \frac{4 \pi}{3} \rho \left( \frac{\lambda_d^{\text{phys}}}{2} \right)^3 \sim 2.7 \times 10^{13} (\Omega_{m,0} h^2)^{-5/4} \left( \frac{\Omega_{b,0}}{\Omega_{m,0}} \right)^{-3/2} \left( \frac{X_e}{0.1} \right)^{-3/2} \left( \frac{1 + z_{\text{dec}}}{1100} \right)^{-15/4} \text{M}_0$$
Having explored baryonic perturbations prior to recombination, we now focus on their evolution past recombination, when the baryonic matter can be treated as a pressureless fluid ($c_s = 0$). We will focus on the adiabatic evolution of isentropic perturbations ($\delta_S = 0$), and since we are now in the matter-dominated era, we can ignore radiation.

\[
\delta_S = 0, \quad c_s = 0 \quad \Rightarrow \quad \frac{d^2\delta_k}{dt^2} + 2\frac{\dot{a}}{a} \frac{d\delta_k}{dt} = 4\pi G \bar{\rho}_m \delta_k
\]

Without derivation (see MBW §4.1.6a), there are two solutions to this equation:

- **Decaying mode**: $\delta_- \propto H(t)$
- **Growing mode**: $\delta_+ \propto H(t) \int_0^t \frac{dt'}{a^2(t') H^2(t')}$

**Question**: what, physically, do these different modes represent?
In general, the solution is a linear combination of the growing and decaying modes

\[ \delta(t) = C_1 \delta_+ + C_2 \delta_- \]

Since, by definition, the decaying mode disappears with time, we shall only be concerned with the growing mode, and simply write \( \delta(t) \propto \delta_+ \) in what follows.

In what follows, we write that \( \delta_+ \propto D(z) \propto \frac{g(z)}{1+z} \) with \( D(z) \) the linear growth rate.

An accurate approximation for the linear growth rate (Carroll et al. 1992) has

\[ g(z) \simeq \frac{5}{2} \Omega_m(z) \left\{ \Omega_m^{4/7}(z) - \Omega_\Lambda(z) + \left[ 1 + \frac{\Omega_m(z)}{2} \right] \left[ 1 + \frac{\Omega_\Lambda(z)}{70} \right] \right\}^{-1} \]

For an Einstein-de Sitter (EdS) cosmology \((\Omega_{m,0}, \Omega_{\Lambda,0}) = (1, 0)\) the solutions are particularly simple: \( \delta_+ \propto a \propto t^{2/3} \) and \( \delta_- \propto t^{-1} \)

Note that, as already mentioned before, because of the expansion of the Universe, the growth-rate is not exponential (as in static case), but a power-law.
The linear growth rate, \( D(z) \), is normalized such that \( D(z=0) = 1.0 \).

Physically, this is due to the fact that in an open universe, or in one with a non-zero cosmological constant, the expansion rate is larger than in an EdS universe, causing a reduction of perturbation growth due to enhanced Hubble drag.

**NOTE:** Perturbations grow faster in an EdS universe than in one with \( \Omega_{m,0} < 1 \).
We can now complete our picture of evolution of baryonic perturbations:

**Silk damping** will `erase’ all baryonic fluctuations on scales $\lambda < \lambda_d$

Consequently, if matter is purely baryonic, after recombination there are no perturbations left on small scales, structure formation has to proceed in **top-down** fashion.

Detailed calculations, accounting for the evolution in ionization fraction, $X_e$, yield

**at recombination:**

$$\lambda_d^{\text{com}} \sim 5.7 (\Omega_{m,0} h^2)^{-3/4} \left( \frac{\Omega_{b,0}}{\Omega_{m,0}} \right)^{-1/2} \left( \frac{X_e}{0.1} \right)^{-1/2} \left( \frac{1 + z_{\text{dec}}}{1100} \right)^{-5/4} \text{Mpc}$$

$$M_d = \frac{4\pi}{3} \tilde{\rho} \left( \frac{\lambda_d^{\text{phys}}}{2} \right)^3 \sim 2.7 \times 10^{13} (\Omega_{m,0} h^2)^{-5/4} \left( \frac{\Omega_{b,0}}{\Omega_{m,0}} \right)^{-3/2} \left( \frac{X_e}{0.1} \right)^{-3/2} \left( \frac{1 + z_{\text{dec}}}{1100} \right)^{-15/4} M_\odot$$
This was the picture for structure formation developed by Zel’dovich and his colleagues during the 1960’s in Moscow. However, it soon became clear that this picture was doomed.

To allow sufficient time for fragmentation, the large-scale perturbations need large amplitudes in order to collapse sufficiently early. At recombination, one requires $|\delta_m| > 10^{-3}$

Using that $\delta_T = 1/4\delta_r = 1/3\delta_m$, which follows from fact that perturbations are isentropic and from $\rho_r \propto T^4$, this model implies CMB fluctuations

$$\frac{\Delta T}{T} > 10^{-3}$$

Such large fluctuations were already ruled out in early 1980s (e.g., Uson & Wilkinson 1984)
While Zel’dovich was working on his adiabatic model, Peebles and his colleagues at Princeton developed an alternative model for structure formation, in which the perturbations were assumed to be isothermal (i.e., $\delta_t = 0$), which is a good approximation for isocurvature perturbations prior to the matter era.

In this model, sound speed prior to recombination is much lower, resulting in much lower Jeans mass. Also, since there are no radiation perturbations, there is NO Silk damping. All perturbations with $M > M_J \sim 10^6 M_\odot$ survive, and structure formation proceeds hierarchical (bottom-up).

Prior to recombination, radiation drag prevents perturbations from growing (they are ‘frozen’), but at least they are not damped...

Similar to adiabatic model, this isothermal baryonic model requires large temperature fluctuations in CMB to explain observed structure.

In addition, isothermal perturbations are fairly “unnatural”.

All these problems disappear when considering a separate matter component: dark matter.
## Summary: key words & important facts

<table>
<thead>
<tr>
<th>Key words</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jeans criterion</td>
</tr>
<tr>
<td>Jeans length</td>
</tr>
<tr>
<td>Horizons (particle vs. event)</td>
</tr>
<tr>
<td>Linear growth rate</td>
</tr>
<tr>
<td>Silk damping</td>
</tr>
<tr>
<td>Radiation drag</td>
</tr>
</tbody>
</table>

- Perturbations below the Jeans mass do not grow, but cause acoustic oscillations.
- At recombination photons decouple from baryons, huge drop in the Jeans mass.
- Hubble drag resists perturbation growth, perturbations above the Jeans mass do not grow exponentially, but as a power-law: $\delta_k(t) \propto t^\alpha$
  The index $\alpha$ depends on cosmology and EoS, as characterized by linear growth rate.
- Growth of super-horizon density-perturbations is governed by conservation of the associated perturbations in the metric.
- If matter is purely baryonic, at recombination Silk damping has erased all perturbations on relevant scales ($M_d \sim 10^{15} M_\odot$), structure formation proceeds in top-down fashion.
prior to recombination: relativistic photon-baryon fluid

\[ c_s = \frac{c}{\sqrt{3}} \left[ \frac{3}{4} \frac{\rho_b(t)}{\rho_r(t)} + 1 \right]^{-1/2} \]

after recombination: baryon fluid is ‘ideal gas’

\[ c_s = \left( \frac{\partial P}{\partial \rho} \right)^{1/2} \propto T^{1/2} \]

Jeans length & mass:

\[ \lambda_J^{\text{prop}} = c_s \sqrt{\frac{\pi}{G \bar{\rho}}} \quad M_J = \frac{4\pi}{3} \bar{\rho} \left( \frac{\lambda_J}{2} \right)^3 = \frac{\pi}{6} \bar{\rho} \lambda_J^3 \]

comoving particle horizon:

\[ \chi_H(a) = \int_0^t \frac{c \, dt}{a} = \int_0^a \frac{c \, da}{a \, \dot{a}} \quad \Rightarrow \quad \lambda_H^{\text{prop}} = \frac{2 \, c \, t}{3 \, c \, t} \]

Poisson equation (Fourier space)

\[ -k^2 \Phi_k = 4\pi G a^2 \bar{\rho} \delta_k \]

\[ \Phi_k \text{ is constant implies that } \delta_k \propto (\bar{\rho} a^2)^{-1} \]