ASTR 610
Theory of Galaxy Formation

Lecture 4: Newtonian Perturbation Theory
I. Linearized Fluid Equations

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Structure Formation: The Linear Regime

Thus far we have focussed on an unperturbed Universe. In this lecture we examine how small perturbations grow and evolve in a FRW metric (i.e., in a expanding space-time).

Topics that will be covered include:

- Newtonian Perturbation Theory
- Equation of State
- Jeans Criterion
- Horizons
- Linear Growth Rate
- Damping (Silk & Free Streaming)
- Meszaros Effect

The Perturbed Universe: Preliminaries

Let $\rho(\vec{x})$ be the density distribution of matter at location $\vec{x}$

It is useful to define the corresponding overdensity field

$$\boxed{\delta(\vec{x}) = \frac{\rho(\vec{x}) - \bar{\rho}}{\bar{\rho}}}$$

Note: $\delta(\vec{x})$ is believed to be the outcome of some random process in the early Universe (i.e., quantum fluctuations in inflaton)

Often it is useful to describe the matter field in Fourier space:

$$\delta(\vec{x}) = \sum_k \delta_{\vec{k}}\, e^{+i\vec{k}\cdot\vec{x}} \qquad\qquad \delta_{\vec{k}} = \frac{1}{V}\int \delta(\vec{x})\, e^{-i\vec{k}\cdot\vec{x}}$$

Here V is the volume over which the Universe is assumed to be periodic.
Note: the perturbed density field can be written as a sum of plane waves of different wave numbers k (called `modes')

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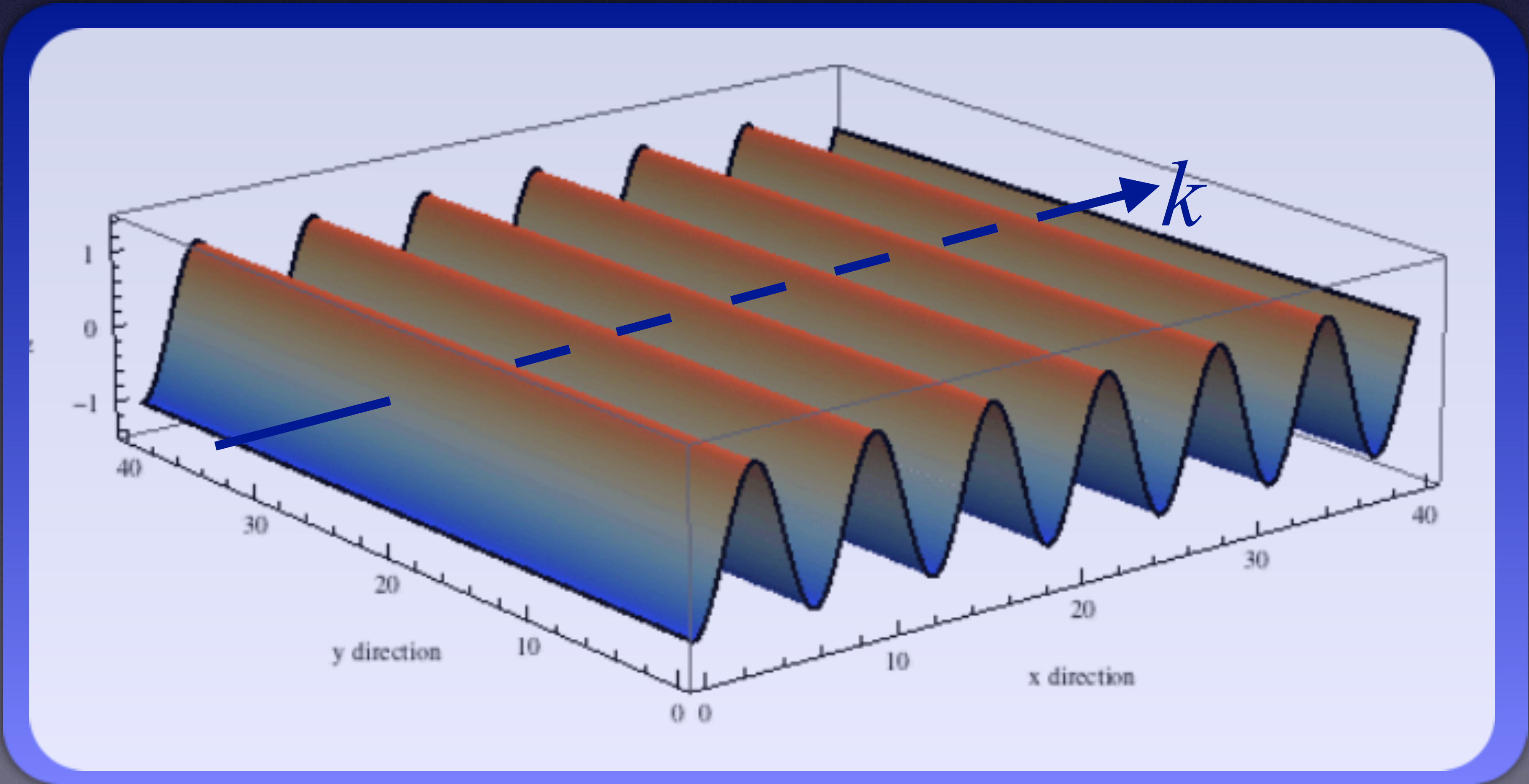


Notation & Convention: Fourier Transforms

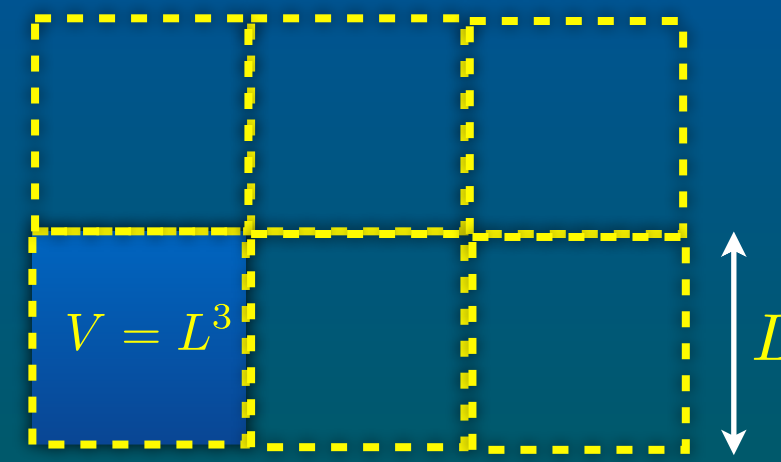
Throughout these lecture series we adopt the following Fourier Transform:

$$\delta(\vec{k}) = \int \delta(\vec{x})\, e^{-i\vec{k}\cdot\vec{x}}\, \mathrm{d}^3\vec{x} \qquad [\delta(\vec{k})] = \mathrm{cm}^3$$

$$\delta(\vec{x}) = \frac{1}{(2\pi)^3} \int \delta(\vec{k})\, e^{+i\vec{k}\cdot\vec{x}}\, \mathrm{d}^3\vec{k} \qquad [\delta(\vec{x})] = \mathrm{unitless}$$

NOTE: different authors use different conventions of where to place the factors 2π and regarding the signs of the exponents...

Rather than working with infinite space, we may also assume a finite (but large) volume in which the Universe is assumed to be periodic (periodic boundary conditions). This implies discrete modes, and the FT becomes:



$$\lambda_{\rm max} = L \qquad k_{\rm min} = \frac{2\pi}{L}$$

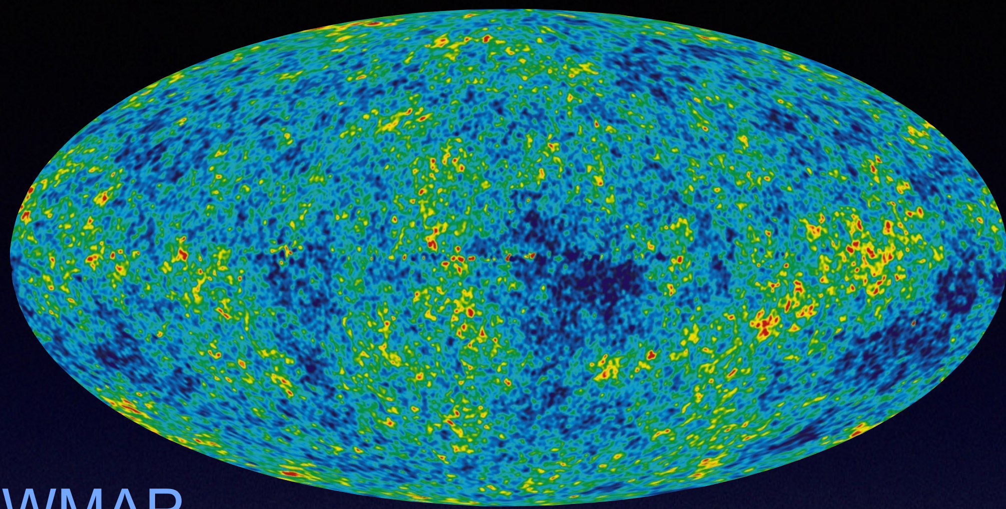
$$\delta_{\vec{k}} = \frac{1}{V} \int \delta(\vec{x})\, e^{-i\vec{k}\cdot\vec{x}}\, \mathrm{d}^3\vec{x} \qquad [\delta_{\vec{k}}] = \mathrm{unitless}$$

$$\delta(\vec{x}) = \sum \delta_{\vec{k}}\, e^{+i\vec{k}\cdot\vec{x}} \qquad [\delta(\vec{x})] = \mathrm{unitless}$$

In the limit where the number of modes goes to infinity one has that $\delta(\vec{k}) = V\,\delta_{\vec{k}}$

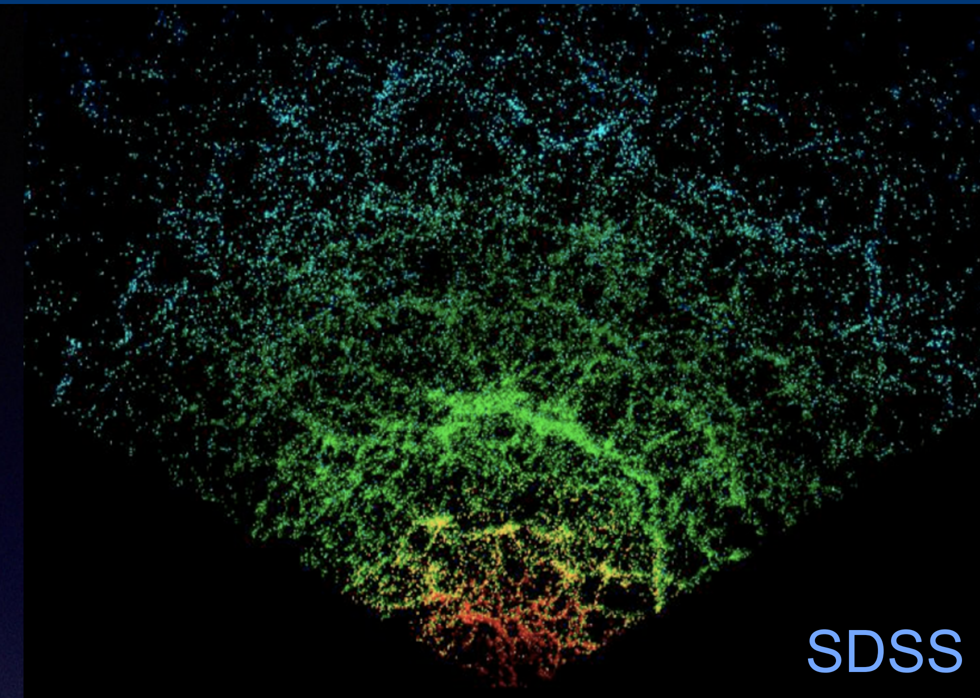
NOTE: $\delta_{\vec{k}}$ is not the same as $\delta(\vec{k})$!!!

Structure Formation: The linear regime



WMAP

the linear regime
CMB, z~1100



the non-linear regime
galaxy distribution, z~0.1

In this lecture we focus on the evolution of the density field $\delta(t)$ in the linear regime, which means that $|\delta| \ll 1$.

Observations of the CMB show that the perturbations at recombination are still very much in the linear regime $|\delta| < 10^{-5}$. However, at the present-day the Universe has clearly entered the non-linear regime, at least on scales larger than a few Mpc. On largest scales, Universe is still in linear regime, in agreement with the Cosmological Principle.

The Fluid Description

QUESTION: what equations describe the evolution of $\delta(t)$?

Before recombination: photon & baryons are a tightly coupled fluid

After recombination: photons are decoupled from baryons
baryons behave as an `ideal gas' (i.e. a fluid)

Throughout: dark matter is (assumed to be) a collisionless fluid

ANSWER: it seems we can describe $\delta(t)$ using fluid equations...

The Fluid Description

QUESTION: when is a fluid description adequate?

ANSWER:

- when frequent collisions can establish local thermal equilibrium, i.e., when the mean free path is much smaller than scales of interest (baryons).

- for a collisionless system, as long as the velocity dispersion of the particles is sufficiently small that particle diffusion can be neglected on the scale of interest....

NOTE: As we will see, Cold Dark Matter (CDM) can be described as fluid with zero pressure (on all scales of interest to us). However, in the case of Hot Dark Matter (HDM) or Warm Dark Matter (WDM), the non-zero peculiar velocities of the dark matter particles causes free-streaming damping on small scales. On those scales the fluid description brakes down, and one has to resort to the (collisionless) Boltzmann equation.

The Fluid Equations

Let $\rho(\vec{x},t)$, $\vec{u}(\vec{x},t)$, and $P(\vec{x},t)$ describe the density, velocity, and pressure of a fluid at location $\vec{x}$ at time t , with $\phi(\vec{x},t)$ the corresponding gravitational potential.

The time evolution of this fluid is given by the continuity equation (describing mass conservation), the Euler equations (the eqs. of motion), and the Poisson equation (describing the gravitational field:

continuity equation	$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\,\nabla_r \cdot \vec{u} = 0$
Euler equations	$\frac{\mathrm{D}\vec{u}}{\mathrm{D}t} = -\frac{\nabla_r P}{\rho} - \nabla_r \phi$
Poisson equation	$\nabla_r^2 \phi = 4\pi G \rho$

NOTE: $\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \vec{u} \cdot \nabla_r$ is the Lagrangian (or \`convective') derivative, which means the derivative wrt a moving fluid element (as opposed to the Eulerian derivatives wrt some fixed grid point...).

NOTE: The continuity & Euler eqs are moment eqs of the Boltzmann eq.

The Fluid Equations

continuity equation $\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho \nabla_r \cdot \vec{u} = 0$

Euler equations $\frac{\mathrm{D}\vec{u}}{\mathrm{D}t} = -\frac{\nabla_r P}{\rho} - \nabla_r \phi$

Poisson equation $\nabla_r^2 \phi = 4\pi G \rho$

This is a set of 5 equations (3 Euler eqs + continuity + Poisson) for 6 unknowns $(\rho, u_x, u_y, u_z, P, \phi)$

In order to close the set, we require an additional equation, which is the equation of state (EoS)

Before we address this EoS, we first rewrite these fluid equations in a different form, more suited to describe perturbations in an expanding space....

Because of expansion, it is best to use comoving coordinates, $\vec{x}$, where $\boxed{\vec{r} = a(t)\,\vec{x}}$

$\vec{u} = \dot{r} = \dot{a}\vec{x} + a\dot{\vec{x}} \equiv \dot{a}\vec{x} + \vec{v}$ $\vec{v}$ = peculiar velocity

$\nabla_r \rightarrow \frac{1}{a}\nabla_x = \frac{1}{a}\nabla$ it is understood that gradient is wrt $\vec{x}$

$\frac{\partial}{\partial t} \rightarrow \frac{\partial}{\partial t} - \frac{\dot{a}}{a}\vec{x} \cdot \nabla$ Eulerian time-derivative depends on H(t)

at fixed r

at fixed x

The Fluid Equations

what follows is only valid for a non-relativistic fluid

If we now also write $\rho(\vec{x}) = \bar{\rho}[1+\delta(\vec{x})]$, and use that $\bar{\rho} \propto a^{-3}$, the fluid eqs. become

continuity equation
$$\frac{\partial \delta}{\partial t} + \frac{1}{a}\nabla \cdot [(1+\delta)\,\vec{v}] = 0$$

Euler equations
$$\frac{\partial \vec{v}}{\partial t} + \frac{\dot{a}}{a}\vec{v} + \frac{1}{a}(\vec{v}\cdot\nabla)\,\vec{v} = -\frac{\nabla P}{a\bar{\rho}(1+\delta)} - \frac{\nabla \Phi}{a}$$

Poisson equation
$$\nabla^2 \Phi = 4\pi G \bar{\rho} a^2 \delta$$

where we defined the modified gravitational potential, $\Phi = \phi + a\ddot{a}x^2/2$, which assures that gravity is only sourced by density contrast $\rho - \bar{\rho}$ (see problem set 1)

This set of equations can in principle be solved once

(i) we pick a particular cosmology $\Rightarrow$ $a(t)$

(ii) we pick an equation of state $\Rightarrow$ $P = P(\rho, S)$

where, for reasons that will become clear soon, we elect to describe the EoS as a function of density, ρ, and specific entropy, S (i.e. entropy per unit mass)

Ideal Gas and the Equation of State

Throughout, we shall assume that after recombination our baryonic fluid is ideal:,

ideal gas law $$\boxed{P = \frac{k_{\rm B}T}{\mu m_{\rm p}}\rho}$$

The specific internal energy (i.e., internal energy per unit mass) of an ideal gas depends only on its temperature:

$$\boxed{\varepsilon = \frac{1}{\gamma - 1}\frac{k_{\rm B}T}{\mu m_{\rm p}}}$$

where γ is the adiabatic index (see MBW App B1.1). For an (idealized) mono-atomic gas of point particles $\gamma = 5/3$, which is what we shall adopt in what follows.

Although the general EoS of an ideal gas has the form $P = P(\rho, T)$, one often considers special cases in which the EoS is barotropic, meaning $P = P(\rho)$. If $P \propto \rho^{\Gamma}$ the EoS is called polytropic, with Γ the polytropic index.

Examples:

isothermal evolution $\mathrm{d}T/\mathrm{d}t = 0$ $\Rightarrow$ $P \propto \rho$ $(\Gamma = 1)$

isentropic evolution $\mathrm{d}S/\mathrm{d}t = 0$ $\Rightarrow$ $P \propto \rho^{\gamma}$ $(\Gamma = \gamma)$

Note: an isentropic flow is a flow that is adiabatic and reversible.

Closing the set of Fluid Equations

Recall that the EoS is needed to close the set of equations (unless the fluid is pressureless...).

In the case of a barotropic EoS, the set of fluid equations is closed. However, in the more general case where $P = P(\rho, T)$, or equivalently, $P = P(\rho, S)$, we have introduced a new variable (T or S), and thus also require a new equation...

This is provided by the 2nd law of thermodynamics: $\mathrm{d}S = \mathrm{d}Q/T$
where $\mathrm{d}Q$ is the amount of energy added to the fluid per unit mass.

$$\boxed{T\frac{\mathrm{d}S}{\mathrm{d}t} = \frac{\mathrm{d}Q}{\mathrm{d}t} = \frac{\mathcal{H} - \mathcal{C}}{\rho}}$$

Here $\mathcal{H}$ and $\mathcal{C}$ are the heating and cooling rates per unit volume, respectively.

As we shall see throughout this course, $\mathcal{H}$ and $\mathcal{C}$ are determined by a variety of physical processes (see chapter 8 of MBW) and can be calculated...

For most of what follows, we assume that the evolution is adiabatic, which implies that $\mathrm{d}S/\mathrm{d}t = 0$.

The Equation of State

We now proceed to use the ideal gas law to close our set of fluid equations:

To start, we use the first two laws of thermodynamics to write

$$\boxed{\mathrm{d}U = T\mathrm{d}S - P\mathrm{d}V \quad \Rightarrow \quad \mathrm{d}\varepsilon \equiv \mathrm{d}U/M = T\mathrm{d}S - P\mathrm{d}(1/\rho)}$$ ‡

Assuming an ideal, mono-atomic gas $\varepsilon = \frac{3}{2}\frac{k_\mathrm{B}\,T}{\mu\,m_\mathrm{p}} = \frac{3}{2}\frac{P}{\rho}$ $\Rightarrow$ $\mathrm{d}\left(\frac{P}{\rho}\right) = \frac{2}{3}\mathrm{d}\varepsilon = \frac{2}{3}T\mathrm{d}S - \frac{2}{3}P\,\mathrm{d}\left(\frac{1}{\rho}\right)$

Using that $\mathrm{d}(P/\rho) = \frac{\partial(P/\rho)}{\partial\rho}\,\mathrm{d}\rho + \frac{\partial(P/\rho)}{\partial P}\,\mathrm{d}P = -\frac{P}{\rho^2}\,\mathrm{d}\rho + \frac{\mathrm{d}P}{\rho}$ and that $\mathrm{d}(1/\rho) = -\frac{\mathrm{d}\rho}{\rho^2}$

when combined with the ideal gas law yields $\mathrm{d}\ln P = \frac{5}{3}\mathrm{d}\ln\rho + \frac{2}{3}\frac{\mu m_\mathrm{p}}{k_\mathrm{B}}S\,\mathrm{d}\ln S$

Hence, our EoS can be written as $$\boxed{P = P(\rho, S) \propto \rho^{5/3}\exp\left(\frac{2}{3}\frac{\mu m_\mathrm{p}}{k_\mathrm{B}}S\right)}$$

‡ **Caution**: both here and in MBW the symbol S is used to denote both entropy and specific entropy.

The Fluid Equations

We now use this EoS to rewrite the $\nabla P/\bar{\rho}$ -term in the Euler equation.

$$\boxed{\frac{\nabla P}{\bar{\rho}} = \frac{1}{\bar{\rho}}\left[\left(\frac{\partial P}{\partial \rho}\right)_S \nabla\rho + \left(\frac{\partial P}{\partial S}\right)_\rho \nabla S\right] = c_{\rm s}^2 \nabla\delta + \frac{2}{3} T(1+\delta)\nabla S}$$

Here $c_{\rm s} = (\partial P/\partial\rho)_S^{1/2}$ is the sound speed, and we have used that $\nabla\rho/\bar{\rho} = \nabla\delta$

Substituting the above in the Euler equation yields the following fluid equations:

continuity equation
$$\frac{\partial\delta}{\partial t} + \frac{1}{a}\nabla\cdot[(1+\delta)\,\vec{v}] = 0$$

Euler equations
$$\frac{\partial\vec{v}}{\partial t} + \frac{\dot{a}}{a}\vec{v} + \frac{1}{a}(\vec{v}\cdot\nabla)\,\vec{v} = -\frac{\nabla\Phi}{a} - \frac{c_{\rm s}^2}{a}\frac{\nabla\delta}{(1+\delta)} - \frac{2T}{3a}\nabla S$$

Poisson equation
$$\nabla^2\Phi = 4\pi G\bar{\rho}a^2\delta$$

Linearizing the Fluid Equations

The next step is to linearize the fluid equations: Using that both ρ and v are small, we can neglect all higher order terms (those with ρ^2, v^2, or $\rho\, v$).

If we write $T = \bar{T} + \delta T$ and also igore higher-order terms containing the small temperature perturbation δT , the fluid equations simplify to

continuity equation
$$\frac{\partial \delta}{\partial t} + \frac{1}{a} \nabla \cdot \vec{v} = 0$$

Euler equations
$$\frac{\partial \vec{v}}{\partial t} + \frac{\dot{a}}{a} \vec{v} = -\frac{\nabla \Phi}{a} - \frac{c_{\rm s}^2}{a} \nabla \delta - \frac{2\bar{T}}{3a} \nabla S$$

Differentiating the continuity eq. wrt t and using the Euler & Poisson eqs yields:

$$\boxed{\frac{\partial^2 \delta}{\partial t^2} + 2\frac{\dot{a}}{a}\frac{\partial \delta}{\partial t} = 4\pi G \bar{\rho} \delta + \frac{c_{\rm s}^2}{a^2} \nabla^2 \delta + \frac{2}{3}\frac{\bar{T}}{a^2} \nabla^2 S}$$

This `master equation' describes the evolution of the density perturbations in the linear regime ($|\delta| \ll 1$), but only for a non-relativistic fluid !!!

The Linearized Fluid Equations

$$\frac{\partial^2 \delta}{\partial t^2} + 2\frac{\dot{a}}{a}\frac{\partial \delta}{\partial t} = 4\pi G \bar{\rho} \delta + \frac{c_s^2}{a^2}\nabla^2 \delta + \frac{2}{3}\frac{\bar{T}}{a^2}\nabla^2 S$$

\`Hubble drag' term, expresses how expansion suppresses perturbation growth

gravitational term, expresses how gravity promotes perturbation growth

pressure terms, expressing how pressure gradients due to spatial gradients in density and/or entropy influence perturbation growth

The Linearized Fluid Equations

$$\boxed{\frac{\partial^2 \delta}{\partial t^2} + 2\frac{\dot{a}}{a}\frac{\partial \delta}{\partial t} = 4\pi G \bar{\rho}\delta + \frac{c_{\rm s}^2}{a^2}\nabla^2\delta + \frac{2}{3}\frac{\bar{T}}{a^2}\nabla^2 S}$$

As already mentioned, it is often advantageous to work in Fourier Space:

$$\delta(\vec{x},t) = \sum_k \delta_{\vec{k}}(t)e^{+i\vec{k}\cdot\vec{x}} \qquad \delta_S(\vec{x},t) \equiv \frac{S(\vec{x},t) - \bar{S}(t)}{\bar{S}(t)} = \sum_k S_{\vec{k}}(t)e^{+i\vec{k}\cdot\vec{x}}$$

Since the above equation is linear, we obtain, for each independent mode, that

$$\boxed{\frac{{\rm d}^2\delta_{\vec{k}}}{{\rm d}t^2} + 2\frac{\dot{a}}{a}\frac{{\rm d}\delta_{\vec{k}}}{{\rm d}t} = \left[4\pi G\bar{\rho} - \frac{k^2 c_{\rm s}^2}{a^2}\right]\delta_{\vec{k}} - \frac{2}{3}\frac{\bar{T}}{a^2}k^2\, S_{\vec{k}}}$$

where we used that when going to Fourier space $\nabla \rightarrow i\vec{k}$ and thus $\nabla^2 \rightarrow -k^2$

This expresses something very important: as long as linearized equation suffices to describe the evolution of the density field, all modes evolve independently

Types of Perturbations

$$\frac{\mathrm{d}^2\delta_{\vec{k}}}{\mathrm{d}t^2} + 2\frac{\dot{a}}{a}\frac{\mathrm{d}\delta_{\vec{k}}}{\mathrm{d}t} = \left[4\pi G\bar{\rho} - \frac{k^2 c_\mathrm{s}^2}{a^2}\right]\delta_{\vec{k}} - \frac{2}{3}\frac{\bar{T}}{a^2}k^2\,S_{\vec{k}}$$

As one can see from this equation, both density perturbation and entropy perturbations can seed structure formation! Hence, one distinguishes

Isentropic Perturbations	$\delta_S = 0$	(pure density perturbations)
Isocurvature Perturbations	$\delta = 0$	(pure entropy perturbations)

- Isocurvature perturbations do not perturb the (FRW) metric; rather, they are perturbations in the EoS (baryons per photon ratio)
- If evolution is adiabatic, isentropic perturbations remain isentropic. If not, the non-adiabatic processes create non-zero ∇S
- Isentropic perturbations are often called adiabatic perturbations. However, it is better practice to reserve \`adiabatic' to refer to an evolutionary process
- Isentropic and isocurvature perturbations are orthogonal, and any perturbation can be written as a linear combination of both.

Types of Perturbations

The entropy density s (=entropy per unit volume) is dominated by relativistic particles, and scales with photon temperature as $s \propto T_\gamma^3$ (see MBW §3.3.3).

Hence, the entropy per unit mass is simply given by $S = \frac{s}{\rho_{\rm m}} \propto \frac{\rho_{\rm r}^{3/4}}{\rho_{\rm m}}$
where we have used that $\rho_{\rm r} = (4\sigma_{\rm SB}/c^3)\,T^4$

$$\boxed{\delta_S = \frac{\partial S}{S} = \frac{1}{S}\left[\frac{\partial S}{\partial \rho_{\rm r}}\,\partial\rho_{\rm r} + \frac{\partial S}{\partial \rho_{\rm m}}\,\partial\rho_{\rm m}\right] = \frac{3}{4}\delta_{\rm r} - \delta_{\rm m}}$$

Thus, for isentropic perturbations we have that $\delta_{\rm r} = (4/3)\delta_{\rm m}$

For isocurvature perturbations $\delta_\rho = \rho - \bar{\rho} = 0$, which implies that
$\rho_{\rm r} + \rho_{\rm m} - \bar{\rho}_{\rm r} - \bar{\rho}_{\rm m} = \bar{\rho}_{\rm r}\,\delta_{\rm r} + \bar{\rho}_{\rm m}\,\delta_{\rm m} = 0$, and thus $\delta_{\rm r}/\delta_{\rm m} = -(\bar{\rho}_{\rm m}/\bar{\rho}_{\rm r}) = -(a/a_{\rm eq})$

At early times, during the radiation dominated era, we have that $\rho_{\rm m} \ll \rho_{\rm r}$
so that isocurvature perturbations obey approximately $\delta_{\rm r} = 0$.
For this reason, isocurvature perturbations are also sometimes called isothermal perturbations (especially in older literature).

Note though that isocurvature is only ~isothermal for $t \ll t_{\rm eq}$

Different Matter Components

The matter perturbations that we are describing consist of both baryons and dark matter (assumed to be collisionless).

In what follows we will first treat these separately.

- We start by considering a Universe without dark matter (only baryons + radiation).
- Next we considering a Universe without baryons (only dark matter + radiation).
- We end with discussing a more realistic Universe (radiation + baryons + dark matter)

Lecture 4

SUMMARY

Summary: key words & important facts

Key words	
Euler equations	Hubble drag
Equation of State	Perturbation analysis
Ideal Gas	Isentropic perturbations
Sound Speed	Isocurvature perturbations

- Dark matter can be described as a collisionless fluid as long as the velocity dispersion of the particles is sufficiently small that particle diffusion can be neglected on the scale of interest. This is true on scales larger than the free-streaming scale.
- In the linear regime, all modes evolve independently (there is no `mode-coupling')
- If evolution is adiabatic, isentropic perturbations remain isentropic. If not, the non-adiabatic processes create non-zero ∇S
- Isentropic and isocurvature perturbations are orthogonal; any perturbation can be written as a linear combination of both.

Summary: key equations & expressions

continuity equation $\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\,\nabla_r \cdot \vec{u} = 0$

Euler equations $\frac{\mathrm{D}\vec{u}}{\mathrm{D}t} = -\frac{\nabla_r P}{\rho} - \nabla_r \phi$

Poisson equation $\nabla_r^2 \phi = 4\pi G \rho$

ideal gas $P = \frac{k_{\mathrm{B}} T}{\mu m_{\mathrm{p}}} \rho \qquad \varepsilon = \frac{1}{\gamma - 1} \frac{k_{\mathrm{B}} T}{\mu m_{\mathrm{p}}}$

sound speed $c_{\mathrm{s}} = (\partial P / \partial \rho)_S^{1/2}$

Perturbation analysis in expanding space-time

$$\frac{\partial^2 \delta}{\partial t^2} + 2\frac{\dot{a}}{a}\frac{\partial \delta}{\partial t} = 4\pi G \bar{\rho} \delta + \frac{c_{\mathrm{s}}^2}{a^2} \nabla^2 \delta + \frac{2}{3}\frac{\bar{T}}{a^2} \nabla^2 S$$

Fourier Transform

"Master equation"

$$\frac{\mathrm{d}^2 \delta_{\vec{k}}}{\mathrm{d}t^2} + 2\frac{\dot{a}}{a}\frac{\mathrm{d}\delta_{\vec{k}}}{\mathrm{d}t} = \left[4\pi G \bar{\rho} - \frac{k^2 c_{\mathrm{s}}^2}{a^2}\right] \delta_{\vec{k}} - \frac{2}{3}\frac{\bar{T}}{a^2} k^2\, S_{\vec{k}}$$

$$\delta_S = \frac{3}{4}\delta_{\mathrm{r}} - \delta_{\mathrm{m}}$$

isentropic perturbations $\delta_{\mathrm{r}} = (4/3)\delta_{\mathrm{m}}$

isocurvature perturbations $\delta_{\mathrm{r}} / \delta_{\mathrm{m}} = -(a/a_{\mathrm{eq}})$