ASTR 610
Theory of Galaxy Formation

Lecture 3: Overview of Cosmology II
(General Relativity & Friedmann Eqs)

Frank van den Bosch
Yale University, Spring 2024

The Friedmann Equations

In this second part of our brief review of cosmology, we follow Einstein's thought-process that resulted in his derivation of the GR Field Equations. Next we derive the all-important Friedmann equation by substituting the FWR metric in the Field Equation and briefly discuss some implications.

Topics that will be covered include:

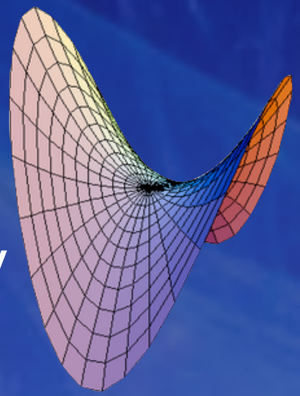
- General Relativity (conceptual)
- Equivalence Principles
- More Riemannanian Geometry
- Einstein Field Equations
- Friedmann Equations
- Critical Density

Cosmology in a Nutshell

Lecture 2

Cosmological Principle

Universe is homogeneous & Isotropic

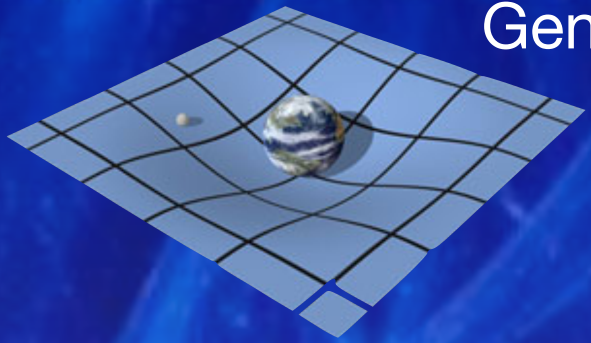


Riemannian Geometry

Friedmann-Robertson-Walker Metric

$$\mathrm{d}s^2 = a^2(\tau)\left[\mathrm{d}\tau^2 - \mathrm{d}\chi^2 - f_K^2(\chi)\ \left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right]$$

Lecture 3



General Relativity

Einstein's Field Equation

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - g_{\mu\nu}\Lambda = \frac{8\pi G}{c^4}T_{\mu\nu}$$

Friedmann Equations

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{Kc^2}{a^2} + \frac{\Lambda c^2}{3}$$

Inertial Frames, Invariance & Covariance

A frame of reference is a standard relative to which motion and rest may be measured. Any set of points or objects that are at rest with respect to each other can serve as a frame of reference (i.e., coordinate system, Earth).

An inertial frame is a frame of reference that has a constant velocity with respect to the `distant stars' (CMB), i.e., it is moving in a straight line at a constant speed, or it is standing still. It is a non-accelerating frame, in which the laws of physics take on their simplest forms, because there are no fictitious forces.

A non-inertial frame is a frame of reference that is accelerating. In a non-inertial frame the motion of objects is affected by fictitious forces (e.g, centrifugal & coriolis force).

An invariant is a property or quantity that remains unchanged under some transformation of the frame of reference (i.e., charge of an electron, Planck's constant, any scalar)

Covariance is the invariance of the physical laws or equations under some transformation of the frame of reference.

Cosmology

What have we learned thus far?

Cosmological Principle
Riemannian Geometry } → FRW metric → { $a(t)$ expansion/contraction; K (constant) curvature

Note that thus far we have said NOTHING about gravity!!!

Newtonian Gravity $$\boxed{\vec{F}_{\rm g} = -\frac{G\, m_1\, m_2}{r^2}}$$

$\vec{F}_{\rm g}$ is a central force → can be written as the gradient of a scalar $\vec{g} = -\nabla\Phi$

This scalar is called the (Newtonian) gravitational potential, and it is related to the matter density distribution according to the Poisson equation:

$$\boxed{\nabla^2\Phi = 4\pi G\rho}$$

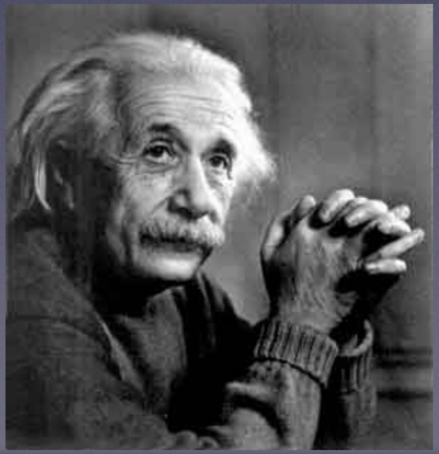
Problems with Gravity around 1905

- Newton's law of gravity appears to give an accurate description of what happens, but gives no explanation of gravity

- Newton's law of gravity only holds in inertial systems, and is covariant under Galilean transformations; however, according to SR inertial systems transform according to Lorentz transformations, which leave Maxwell equations invariant.

- Since there is matter in the Universe, and you can not shield yourself from it (i.e., no equivalent to Faraday cage), inertial systems do not exist...

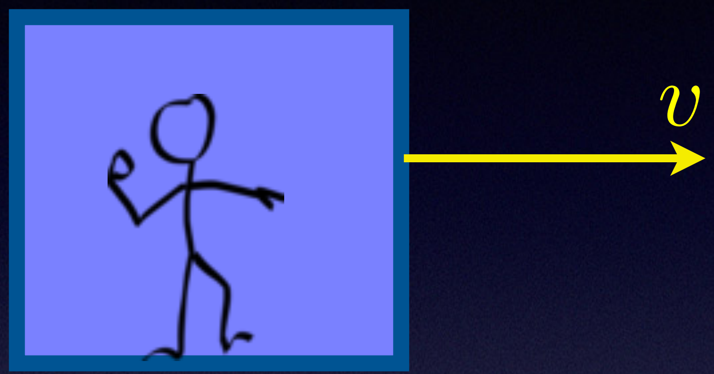
- According to Newton's law of gravity, moving a distant object has an immediate effect all throughout space; violation of Special Relativity.



These issues deeply disturbed Einstein. In 1907, beginning with a simple thought experiment involving an observer in free fall, he embarked on what would be an eight-year search for a manifest covariant, relativistic theory of gravity (GR). This culminated in November 1915 when he presented what are now known as the Einstein Field Equations to the Prussian Academy of Science.

Newtonian & Special Relativity

Galileo, and later Newton, realized that in an inertial frame there is no physical experiment that can reveal the velocity of that inertial frame.



The windowless Lab

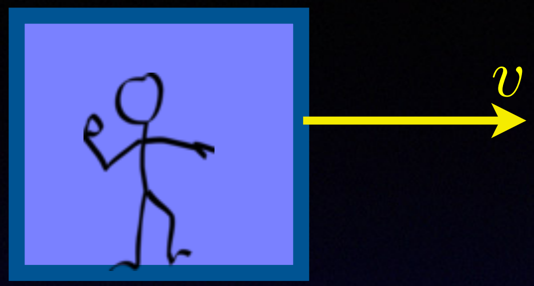
The outcome of every experiment done by stickman is completely independent of the velocity of his inertial frame. When he throws his ball up in the air, it looks exactly the same as if he was at rest wrt distant stars...

This argues against the notion of absolute velocity; only relative motion is measurable in physics. This concept that there is no such thing as absolute velocity is called Newtonian Relativity.

All uniform motion is relative

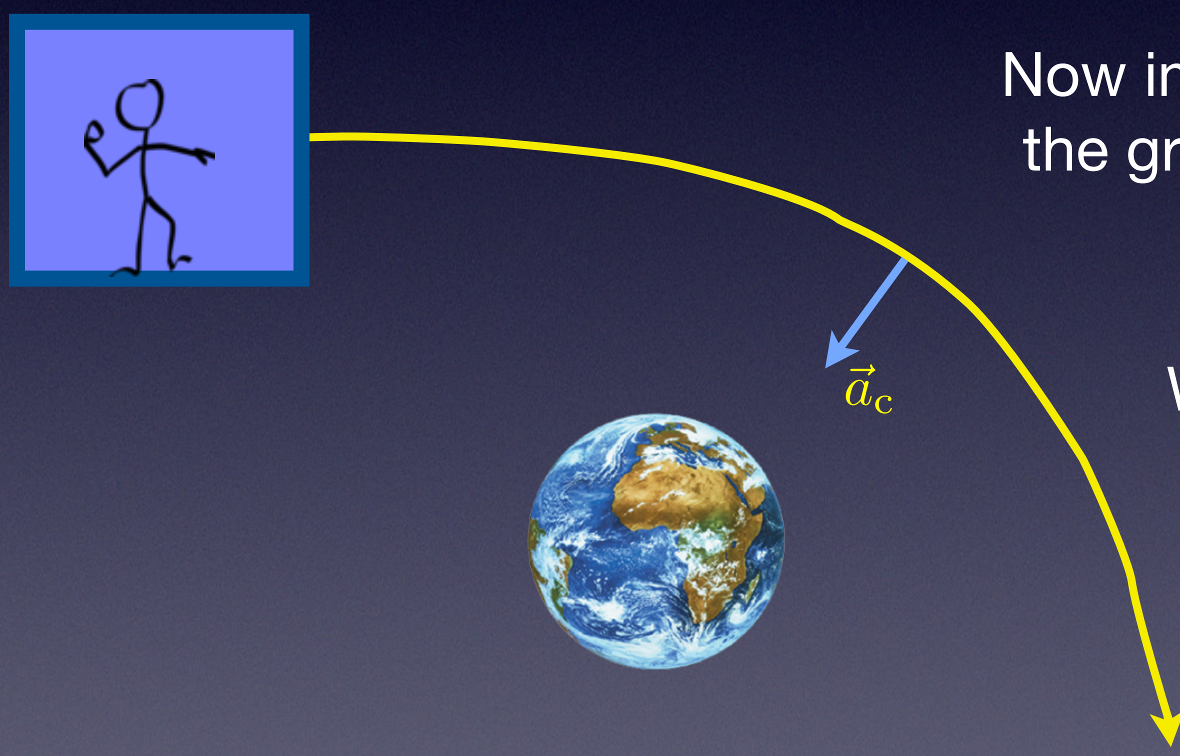
This principle is still valid in Special Relativity. Main change is that inertial frames now transform according to Lorentz transformation (to satisfy constancy of speed of light, required by Michelson-Morley experiment)

Einstein's thought experiments



Consider stick-man in a windowless lab, moving with constant speed (i.e., his lab is an inertial frame)

According to Special Relativity, stick-man can perform no experiment from which he can determine his velocity!



Now imagine stick-man's lab being accelerated due to the gravitational field of the Earth (i.e., stick-man's lab is a non-inertial frame in free-fall)

What experiment(s) can stick-man do from which he can determine his acceleration?

NONE: Stick-man doesn't notice acceleration since gravitational force is exactly balanced by centrifugal force.

Gravity can be transformed away by going to a non-inertial, free-fall frame

Einstein's thought experiments

acceleration

Stick-man's lab is accelerated. He experiences an inertial force, which gives him a non-zero weight.

gravity

Stick-man's lab is inhibited in its free-fall due to the normal force of the Earth. Stick-man experiences the gravitational force, giving him a non-zero weight.

Einstein realized that there is no experiment that Stick-man can do that tells him the difference between gravity and acceleration.

Principle of Relativity is really a principle of impotence: you are unable to tell the difference between being at rest, moving at constant speed or being in free-fall, and you're unable to tell the difference between being in a gravitational field or being accelerated.

Einstein, who had this revelation in 1907, describes it as `the happiest thought of my life'.

Weak Equivalence Principle

Newton's 2nd law of motion: $\vec{F} = m_{\rm i}\,\vec{a}$

Newton's law of gravity: $\vec{F}_{\rm g} = m_{\rm g}\,\vec{g}$

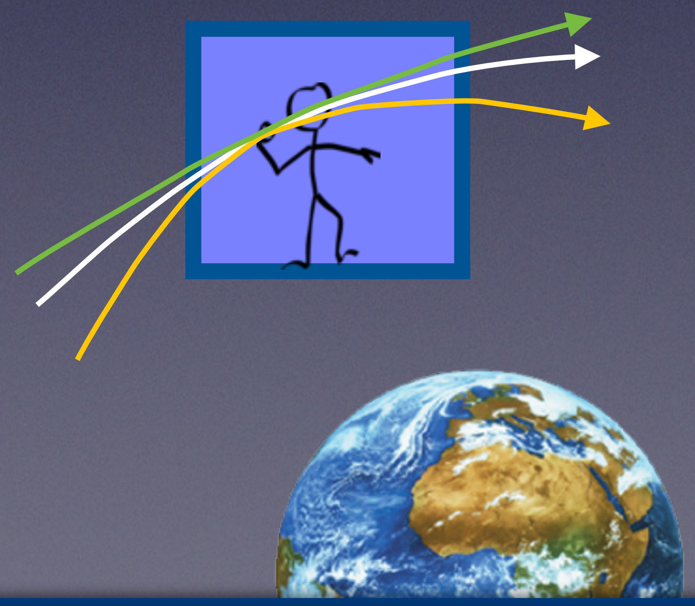
in a gravitational field $\vec{a} = \frac{m_{\rm g}}{m_{\rm i}}\,\vec{g}$

Weak Equivalence Principle:

all material objects in free-fall undergo the same acceleration in a gravitational field, regardless of their mass and composition

inertial mass, $m_{\rm i}$, is equal to gravitational mass, $m_{\rm g}$

If Weak Equivalence Principle were violated, different objects would be on different free-fall trajectories (e.g., objects in Stick-man's lab would fly against walls)



Weak Equivalence Principle

Newton's 2nd law of motion: $\vec{F} = m_{\rm i}\,\vec{a}$

Newton's law of gravity: $\vec{F}_{\rm g} = m_{\rm g}\,\vec{g}$

in a gravitational field

$$\vec{a} = \frac{m_{\rm g}}{m_{\rm i}}\,\vec{g}$$

Weak Equivalence Principle:

all material objects in free-fall undergo the same acceleration in a gravitational field, regardless of their mass and composition

inertial mass, $m_{\rm i}$, is equal to gravitational mass, $m_{\rm g}$

WEP has been confirmed experimentally to high precision

Already in 1889 Eötvös was able to show that there is no difference between inertial and gravitational masses to an accuracy of 1 part in 20 million

Modern versions of the Eötvös torsion balance experiment show that

$$\left|\frac{m_{\rm i}}{m_{\rm g}} - 1\right| < 5 \times 10^{-14}$$

Strong Equivalence Principle

According to WEP, there is no gravity (locally) in free-falling system.

Einstein's SR (Minkowksi space) applies to systems in absence of gravity

space time of freely falling observer is Minkowski space (M_4)

Strong Equivalence Principle: in free-fall in an arbitrary gravitational field, all physical processes (not just the trajectories of material objects) take place in the same way that they would if the gravitational field was absent (i.e., in uniform motion)

Alternative formulations:

For an observer in free-fall in a gravitational field, the results of all (local) experiments are independent of the magnitude of the gravitational field.

All local, freely falling, non-rotating laboratories are fully equivalent for the performance of any physical experiment

- there is a fundamental equivalence between acceleration and gravity
- G is constant (the same at every space-time point)

Strong Equivalence Principle

According to WEP, there is no gravity (locally) in free-falling system.

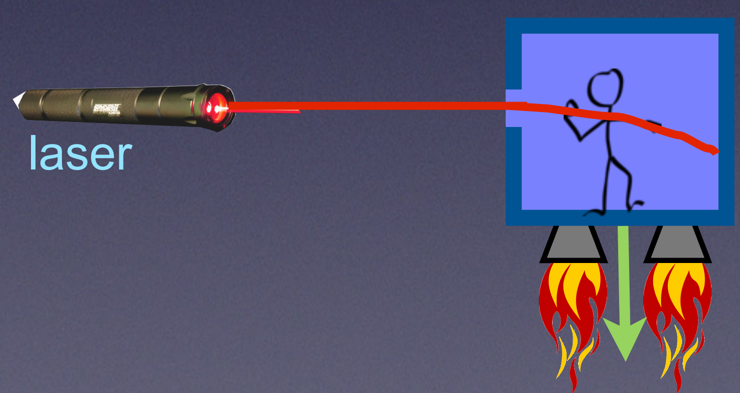
Einstein's SR (Minkowksi space) applies to systems in absence of gravity

space time of freely falling observer is Minkowski space (M_4)

Strong Equivalence Principle: in free-fall in an <u>arbitrary</u> gravitational field, <u>all</u> physical processes (not just the trajectories of material objects) take place in the same way that they would if the gravitational field was absent (i.e., in uniform motion)

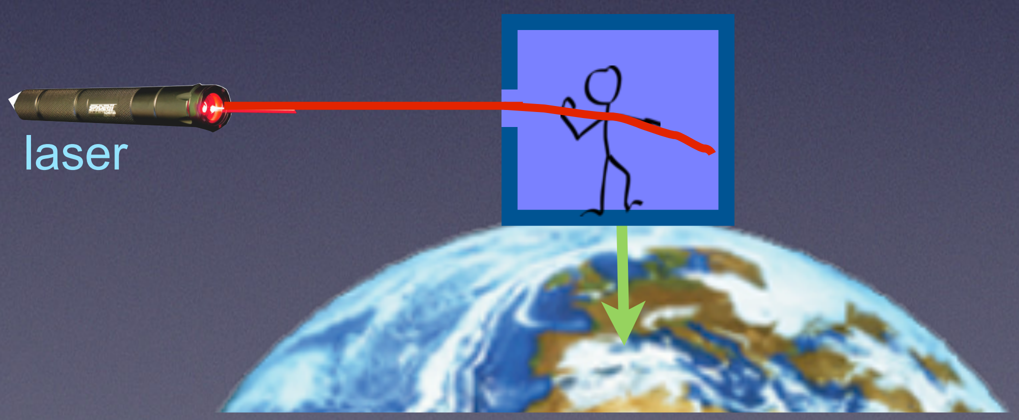
Implication: Gravitational Lensing

<u>acceleration</u>



In stick-man's accelerated lab, the laser-beam appears to follow a curved trajectory, which is simply a reflection of upwards acceleration

<u>gravity</u>



Based on the strong equivalence principle, the laser-beam must follow same trajectory in gravitational field.

Strong Equivalence Principle

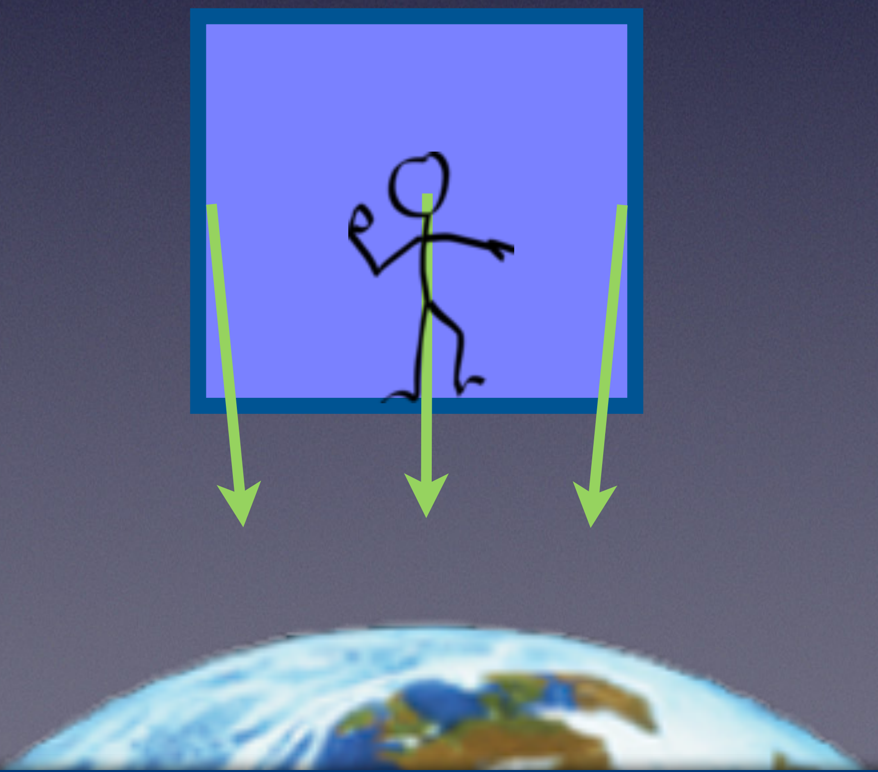
According to WEP, there is no gravity (locally) in free-falling system.

Einstein's SR (Minkowksi space) applies to systems in absence of gravity

⇒ space time of freely falling observer is Minkowski space (M_4)

Strong Equivalence Principle: in free-fall in an arbitrary gravitational field, all physical processes (not just the trajectories of material objects) take place in the same way that they would if the gravitational field was absent (i.e., in uniform motion)

NOTE: gravity can be transformed away by going to a non- inertial, free-falling reference frame.
BUT: this is only true locally, because of the tidal field arising from a non-zero $(\partial^2\Phi/\partial x\,\partial y)$, one can only transform away the effects of gravity on scales that are small compared to variations in the gravitational field.



Towards a Manifest Covariant Theory

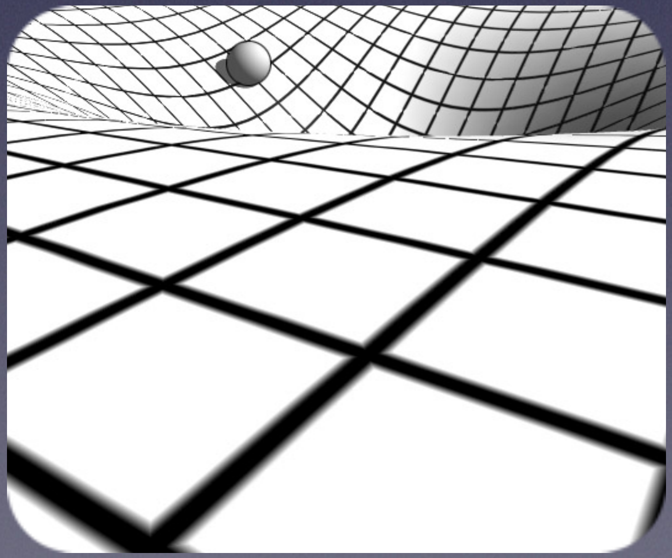
Next Einstein realized that `permanence' of gravity (can only be transformed away locally), implies it must be related to some intrinsic property of space-time itself.

→ gravity manifests itself via $g_{\mu\nu}$

At any point in any Riemannian space-time R_4 one can find a coordinate system for which $g_{\mu\nu} = \eta_{\mu\nu}$ locally. This must then be the coordinate system of a freely falling observer.

Hence, if $g_{\mu\nu} = \eta_{\mu\nu}$ everywhere (i.e., $R_4 = M_4$) then there can be no gravity anywhere. After all, gravity only allows you to have $g_{\mu\nu} = \eta_{\mu\nu}$ locally.

In other words, flat space-time means no gravity...



→ gravity originates from curvature in space-time

Towards a Manifest Covariant Theory

Once Einstein realized that gravity is a manifestation of curved space-time, he was ready for the next step: to find a manifest covariant version of Poisson equation.

$$\boxed{\nabla^2\Phi = 4\pi G\rho}$$

Step 1: if $\Phi \leftrightarrow g_{\mu\nu}$ then it makes sense that $\nabla^2\Phi \leftrightarrow B_{\mu\nu}$. Here $B_{\mu\nu}$ is any tensor that is made out of second derivatives of the metric $g_{\mu\nu}$

Step 2: replace density with the energy momentum tensor of a fluid

$$T^{\mu\nu} = (\rho + P/c^2)\, U^\mu\, U^\nu - P g^{\mu\nu}$$

Here we have simply replaced $\eta^{\mu\nu}$ (SR) with the general metric $g^{\mu\nu}$ (GR)

Hence, our manifest covariant version of the Poisson equation is going to looking something like

$$\boxed{B^{\mu\nu} = kT^{\mu\nu}}$$

where k is some constant.

Riemannian Geometry

To make progress (i.e., determine $B_{\mu\nu}$) let us look at Riemannian geometry. The geometry of any manifold is fully described by the metric tensor $g_{\mu\nu}(\mathbf{x})$ where once again we make it explicit that in general $g_{\mu\nu}$ will be a function of location.

The metric tensor is also used to raise or lower indices

$$A^\mu = g^{\mu\nu} A_\nu \qquad\qquad g^{\mu\alpha}\, g_{\nu\alpha} = \delta^\mu_\nu$$
$$A_\mu = g_{\mu\nu} A^\nu$$

From the metric tensor, one can also construct a number of other quantities that are useful to describe geometry, such as the Christoffel Symbols

$$\Gamma_{\alpha\beta\gamma} \equiv \frac{1}{2}\left(g_{\alpha\beta,\gamma} + g_{\alpha\gamma,\beta} - g_{\beta\gamma,\alpha}\right)$$

Christoffel Symbol of 1st kind

$$\Gamma^\alpha_{\beta\gamma} \equiv \frac{1}{2} g^{\alpha\mu} \left(g_{\mu\beta,\gamma} + g_{\mu\gamma,\beta} - g_{\beta\gamma,\mu}\right)$$

Christoffel Symbol of 2nd kind
= affine connection

where we have used the notation $(\ldots)_{,\mu} \equiv \partial_\mu(\ldots) \equiv \partial(\ldots)/\partial x^\mu$

Covariant Derivatives

Important property of affine connection is in defining covariant derivatives:

On the previous page we defined $A_{\mu,\nu} = \partial A_\mu / \partial x^\nu$

Now consider a new coordinate system $\bar{x}^\alpha = \bar{x}^\alpha(x)$

We have that

$$\bar{A}_{\mu,\nu} = \frac{\partial \bar{A}_\mu}{\partial \bar{x}^\nu} = \frac{\partial}{\partial \bar{x}^\nu}\left[\frac{\partial x^\alpha}{\partial \bar{x}^\mu} A_\alpha\right]$$

$$= \frac{\partial x^\alpha}{\partial \bar{x}^\mu}\frac{\partial A_\alpha}{\partial \bar{x}^\nu} + \frac{\partial^2 x^\alpha}{\partial \bar{x}^\nu \partial \bar{x}^\mu} A_\alpha$$

$$= \frac{\partial x^\alpha}{\partial \bar{x}^\mu}\frac{\partial x_\beta}{\partial \bar{x}^\nu}\frac{\partial A_\alpha}{\partial x^\beta} + \boxed{\frac{\partial^2 x^\alpha}{\partial \bar{x}^\nu \partial \bar{x}^\mu} A_\alpha}$$

Because of this term, $\bar{A}_{\mu,\nu}$ is not a tensor

➡ The operator ∂_μ can not be used in physical laws. Rather, we need to find a covariant derivative, which properly transforms as a tensor, so that our equations can be made manifest covariant.

Covariant Derivatives

As it turns out (without proof), the covariant derivatives are:

$$A_{\mu;\nu} \equiv D_\nu A_\mu \equiv \frac{D\,A_\mu}{D\,x^\nu} = A_{\mu,\nu} - \Gamma^{\alpha}_{\mu\nu}\,A_\alpha$$

$$A^{\mu}{}_{;\nu} \equiv D_\nu A^\mu \equiv \frac{D\,A^\mu}{D\,x^\nu} = A^{\mu}{}_{,\nu} - \Gamma^{\mu}_{\alpha\nu}\,A^\alpha$$

which contain the affine connection.

NOTE: For a scalar field, one has that $\phi_{;\alpha} = \phi_{,\alpha}$

NOTE: The metric tensor obeys $g_{\mu\nu;\alpha} = g^{\mu\nu}{}_{;\alpha} = 0$

NOTE: in a Cartesian coordinate system in (pseudo)-Euclidean space, and thus also in M_4 , one has that $\Gamma^{\alpha}_{\mu\nu} = 0$

NOTE: Christoffel symbols are NOT tensors (they don't transform as such)

Riemannian Geometry

From the affine connection, one can construct the Riemann tensor:

$$\boxed{R^{\alpha}{}_{\beta\gamma\delta} = -\Gamma^{\alpha}_{\beta\gamma,\delta} + \Gamma^{\alpha}_{\beta\delta,\gamma} + \Gamma^{\sigma}_{\beta\delta}\Gamma^{\alpha}_{\sigma\gamma} - \Gamma^{\sigma}_{\beta\gamma}\Gamma^{\alpha}_{\sigma\delta}}$$

Note that this Riemann tensor contains derivatives of the affine connection, and therefore is related to the second derivative of the metric tensor.

Using contraction, once can construct the Ricci tensor from the Riemann tensor

$$\boxed{R_{\beta\delta} \equiv R^{\gamma}_{\beta\gamma\delta} = g^{\alpha\gamma} R_{\alpha\beta\gamma\delta}}$$

This is the only tensor of rank 2 that can be constructed from the Riemann tensor by contraction!

Finally, by contracting the Ricci tensor one obtains the curvature scalar

$$\boxed{R \equiv g^{\beta\delta} R_{\beta\delta}}$$

The Einstein Tensor

Now let's head back to our suggestion for the manifest covariant Poisson equation:

$$\boxed{B^{\mu\nu} = kT^{\mu\nu}}$$

Conservation of energy & momentum in SR implies that $T^{\mu\nu}{}_{;\nu} = 0$

This implies that we seek a tensor $B^{\mu\nu}$ that obeys $B^{\mu\nu}{}_{;\nu} = 0$

Einstein used this knowledge to seek a tensor that is covariantly conserved. Using the Ricci tensor, the metric tensor, and the curvature scalar, Einstein constructed what is nowadays known as the Einstein tensor:

$$\boxed{G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu}}$$

which is a tensor constructed from second-order derivatives of the metric tensor, that obeys

$$\boxed{G^{\mu\nu}{}_{;\nu} = 0}$$

The Einstein Field Equation

Based on the above considerations, Einstein proposed the following tensor equation (= manifest covariant), as the GR replacement of the Poisson equation

Einstein Field Equation

$$\boxed{G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}}$$

where the constant comes from constraint that it reduces to Poisson equation in Newtonian limit: gravitational field is static and weak ($\Phi/c^2 \ll 1$) and test-particles move at low speed ($v/c \ll 1$).

Since ${g^{\mu\nu}}_{;\nu} = 0$, it is clear that rather than using $G^{\mu\nu}$ one could also use any combination $G^{\mu\nu} + k\, g^{\mu\nu}$, and still obey the constraint ${B^{\mu\nu}}_{;\nu} = 0$.

Hence, there are plausible alternatives to the above Field Equation. Einstein himself used that freedom at a later stage: in order to obtain a static Universe he decided to change his Field Equation to

$$\boxed{G_{\mu\nu} - \Lambda\, g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}}$$

where Λ is the cosmological constant.

The Friedmann Equation

Cosmological Principle $\Rightarrow$ $$ds^2 = c^2 dt^2 - a^2(t)\left[\frac{dr^2}{1-Kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right]$$

General Relativity $\Rightarrow$ $$G_{\mu\nu} - \Lambda\, g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

For homogeneous & isotropic Universe: $T^{\mu\nu} = \mathrm{diag}(\rho c^2, -P, -P, -P)$

Substitute FRW metric in $g_{\mu\nu} \longrightarrow \Gamma^{\alpha}_{\beta\gamma} \longrightarrow R^{\alpha}_{\beta\gamma\delta} \longrightarrow \underbrace{R_{\mu\nu} \longrightarrow R}_{G_{\mu\nu}}$

What you get out is:

00- or time-time component: $$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right) + \frac{\Lambda c^2}{3}$$

ii- or space-space components: $$\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + 2\frac{Kc^2}{a^2} = 4\pi G\left(\rho - \frac{P}{c^2}\right) + \Lambda c^2$$

Substituting the 00-component in the ii-component yields

The Friedmann Equation $$\boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{Kc^2}{a^2} + \frac{\Lambda c^2}{3}}$$

Students: do this at home; great exercise

Friedmann Equation

The Friedmann equation relates $\dot{a}$, and hence $a(t)$, to energy density & curvature.

We can rewrite the Friedmann equation as

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{Kc^2}{a^2}}$$

where we have absorbed the term with the cosmological constant in ρ, i.e., we have "interpreted" Λ as an energy component with $\rho_\Lambda = \Lambda c^2/8\pi G$

As we have seen in Lecture 2:

$$\boxed{\begin{aligned} \rho_{\rm m} &= \rho_{\rm m,0}(a/a_0)^{-3} \\ \rho_{\rm r} &= \rho_{\rm r,0}\,(a/a_0)^{-4} \\ \rho_\Lambda &= \rho_{\Lambda,0} \end{aligned}}$$

Which allows us to write the Friedmann equation in the following form:

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 = H^2(t) = \frac{8\pi G}{3}\left[\rho_{\rm m,0}\left(\frac{a}{a_0}\right)^{-3} + \rho_{\rm r,0}\left(\frac{a}{a_0}\right)^{-4} + \rho_{\Lambda,0}\right] - \frac{Kc^2}{a^2}}$$

Friedmann Equation

Interestingly, the Friedmann eq. can also be derived (almost) from Newtonian physics....

Consider a small spherical region of mass M and radius R. Since the Universe is homogeneous and isotropic, Newton's first theorem implies that

$$\ddot{R} = -\frac{GM}{R^2}$$

Newton's first theorem: a body that is inside a spherical shell of matter experiences no net gravitational force from that shell.

Integrating this equation once gives $\frac{1}{2}\dot{R}^2 - \frac{GM}{R} = C$, where C is the integration constant.

Note that C is the sum of kinetic and potential energy per unit mass. It is the specific energy of the spherical shell of radius R. Let's write $R = a(t)R_0$, and use that $M = 4\pi R^3 \bar{\rho}/3$
Then:

$$\frac{1}{2}\dot{a}^2 R_0^2 = \frac{4\pi G}{3}\bar{\rho}_{\rm m} a^2 R_0^2 + E$$

$$\Leftrightarrow \quad \dot{a}^2 = \frac{8\pi G}{3}\bar{\rho}_{\rm m} a^2 + \frac{2E}{R_0^2} \quad \Rightarrow \quad \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\bar{\rho}_{\rm m} - \frac{Kc^2}{a^2}$$

where $K = -\frac{2E}{c^2 R_0^2}$

The only difference wrt the relativistic version is that the matter density, $\bar{\rho}_{\rm m}$, is now replaced by the energy density/c^2, where the energy density includes radiation as well as the contribution from the cosmological constant, $\rho_\Lambda = \Lambda c^2/8\pi G$

The Critical Density

Using the Friedmann equation $$H^2(t) = \frac{8\pi G}{3}\rho(t) - \frac{Kc^2}{a^2(t)}$$

we see that a flat Universe (i.e. $K = 0$) implies that $$\rho(t) = \frac{3H^2(t)}{8\pi G} \equiv \rho_{\rm crit}(t)$$

This is called the **critical density**.

At z=0 we have that $\rho_{\rm crit} = 2.78 \times 10^{11} h^{-1} M_\odot/(h^{-1}{\rm Mpc})^3$

In cosmology, it is customary to write the various density components in unitless form:

$$\Omega_{\rm x}(t) \equiv \frac{\rho_{\rm x}(t)}{\rho_{\rm crit}(t)}$$

where x can be matter (DM, baryons, or both), radiation, neutrinos, cosmological constant, etc. For the total density, we use

$$\Omega(t) = \sum_{\rm x} \Omega_{\rm x}(t)$$

Friedmann Equation

Now let us rewrite the Friedmann equation in terms of the Ω's

$$H^2(t) = \frac{8\pi G}{3}\left[\rho_{\mathrm{m},0}\left(\frac{a}{a_0}\right)^{-3} + \rho_{\mathrm{r},0}\left(\frac{a}{a_0}\right)^{-4} + \rho_{\Lambda,0}\right] - \frac{Kc^2}{a^2}$$
$$= H_0^2\left[\Omega_{\mathrm{m},0}(1+z)^3 + \Omega_{\mathrm{r},0}(1+z)^4 + \Omega_{\Lambda,0}\right] - \frac{Kc^2}{a^2}$$

At $t = t_0$ (present), this becomes $H_0^2 = H_0^2\,\Omega_0 - \frac{Kc^2}{a_0^2}$

which allows us to write $-\frac{Kc^2}{a^2} = -\frac{Kc^2}{H_0^2\,a_0^2}\left(\frac{a}{a_0}\right)^{-2} H_0^2 = (1-\Omega_0)\,H_0^2\,(1+z)^2$

$$H^2(z) = H_0^2\,E^2(z)$$

where

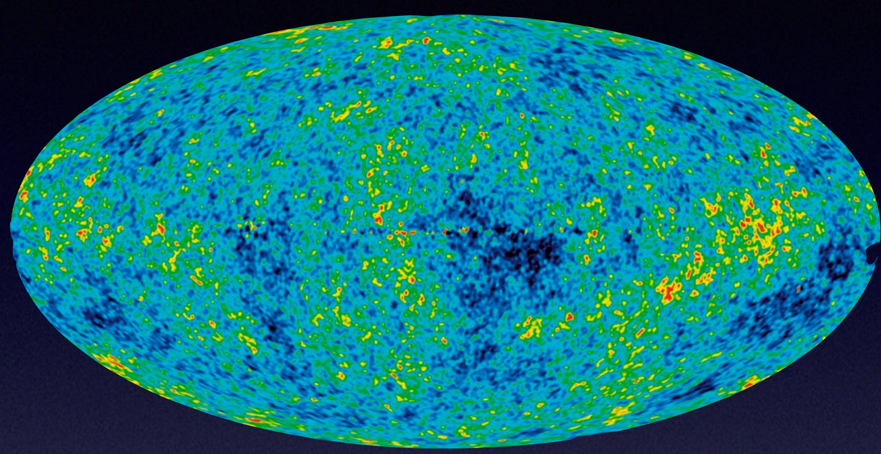
$$E(z) = \left[\Omega_{\Lambda,0} + (1-\Omega_0)\,(1+z)^2 + \Omega_{\mathrm{m},0}(1+z)^3 + \Omega_{\mathrm{r},0}(1+z)^4\right]^{1/2}$$

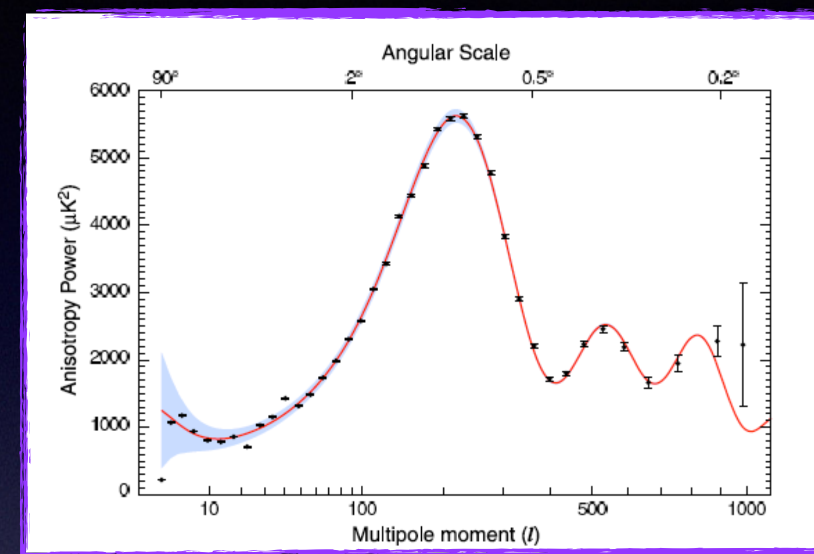
This is the commonly used, compact form of the Friedmann equation.

The Critical Density

See MBW §3.2.2 for details

Some of the best constraints on the energy densities in the present-day Universe come from the temperature fluctuations in the cosmic microwave background (CMB)





Location of first accoustic peak → $\Omega_0 \simeq 1.0$ (i.e., Universe is flat: $K \simeq 0$)

Ratio of 1st to 2nd peak heights → $\Omega_{\rm b,0} \sim 0.044$

Ratio of 2nd to 3rd peak heights → $\Omega_{\rm m,0} \sim 0.26$

$\left.\begin{matrix}\Omega_{\rm b,0} \sim 0.044 \\ \Omega_{\rm m,0} \sim 0.26\end{matrix}\right\} \rightarrow f_{\rm bar} \sim 0.17$

Supernovae Ia → $\Omega_{\Lambda,0} \sim 0.74$

Cepheid stars → $H_0 \sim 72\,{\rm km/s/Mpc}$

CMB temperature → $\Omega_{\gamma,0} \sim 5 \times 10^{-5}$ $\quad (\rho_\gamma \propto T^4)$

\+ neutrinos → $\Omega_{\rm r,0} \sim 8 \times 10^{-5}$ $\quad (T_\nu = (4/11)^{1/3} T_\gamma)$

The Critical Density

See MBW §3.2.2 for details

Some of the best constraints on the energy densities in the present-day Universe come from the temperature fluctuations in the cosmic microwave background (CMB)

Location of first accoustic peak → $\Omega_0 \simeq 1.0$ (i.e., Universe is flat: $K \simeq 0$)

Ratio of 1st to 2nd peak heights → $\Omega_{\rm b,0} \sim 0.044$

Ratio of 2nd to 3rd peak heights → $\Omega_{\rm m,0} \sim 0.26$

$\left.\begin{matrix}\Omega_{\rm b,0} \sim 0.044 \\ \Omega_{\rm m,0} \sim 0.26\end{matrix}\right\} \rightarrow f_{\rm bar} \sim 0.17$

Supernovae Ia → $\Omega_{\Lambda,0} \sim 0.74$

Cepheid stars → $H_0 \sim 72\,{\rm km/s/Mpc}$

CMB temperature → $\Omega_{\gamma,0} \sim 5 \times 10^{-5}$ $(\rho_\gamma \propto T^4)$

\+ neutrinos → $\Omega_{\rm r,0} \sim 8 \times 10^{-5}$ $(T_\nu = (4/11)^{1/3} T_\gamma)$

This implies that the redshift of equality, at which the energy density of radiation is equal to that of non-relativistic matter, is given by (see Problem Set 1)

$$1 + z_{\rm eq} \simeq 2.4 \times 10^4\, \Omega_{\rm m,0} h^2$$

Evolution of the Density Parameters

We can write $\Omega_{\rm x}(t) = \frac{\rho_{\rm x}(t)}{\rho_{\rm crit}(t)} = \frac{\rho_{\rm x}(t)}{\rho_{\rm x,0}} \frac{\rho_{\rm x,0}}{\rho_{\rm crit,0}} \frac{\rho_{\rm crit,0}}{\rho_{\rm crit}(t)} = \left(\frac{a}{a_0}\right)^{-\gamma_{\rm x}} \Omega_{\rm x,0} \frac{H_0^2}{H^2(t)}$

$$\boxed{\Omega_{\rm x}(z) = \frac{\Omega_{\rm x,0}(1+z)^{\gamma_{\rm x}}}{E^2(z)} \quad \text{where} \quad \begin{array}{ll} \gamma_{\rm m} = 3 & \gamma_\Lambda = 0 \\ \gamma_{\rm r} = 4 & \gamma_{\rm K} = 2 \end{array}}$$

Here we have defined the density parameter "associated with the curvature" as

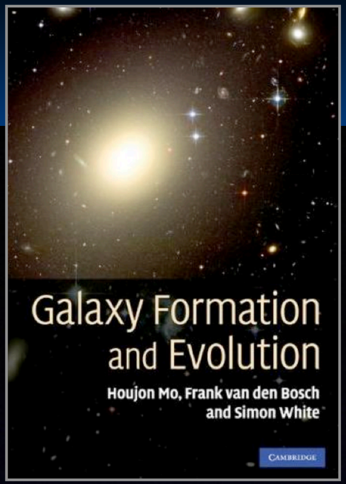
$$\Omega_{\rm K}(z) \equiv 1 - \Omega(z)$$

from which we infer that

$$\boxed{\Omega(z) - 1 = (\Omega_0 - 1)\,\frac{(1+z)^2}{E^2(z)}}$$

All cosmologies with $\rho_{\rm m,0} \neq 0$ or $\rho_{\rm r,0} \neq 0$ have that $\lim_{z\to\infty} \Omega(z) = 1$ and therefore behave similar as an Einstein-de Sitter (EdS) Universe, which is a Universe with $(\Omega_{\rm m,0}, \Omega_{\Lambda,0}) = (1,0)$

For Further Study...

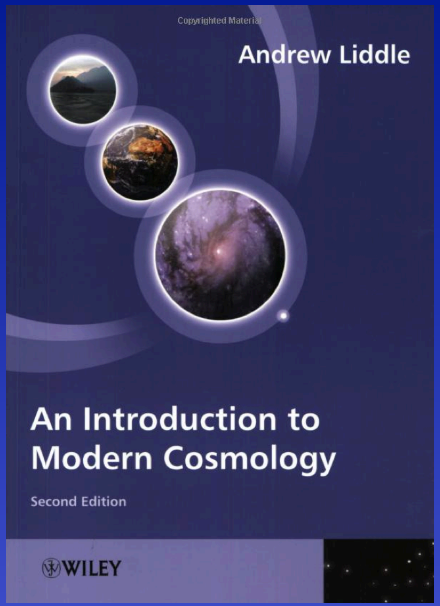


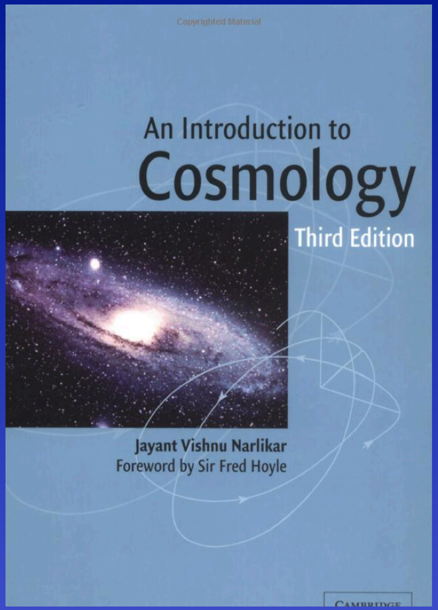
For specific solutions to the Friedmann equation; see MBW §3.2.3

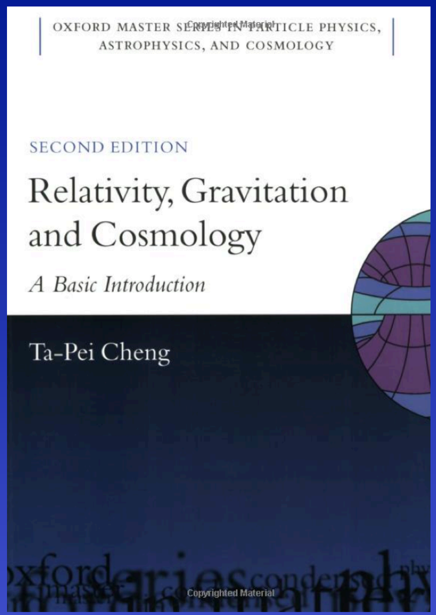
For the relation between time and redshift; see MBW §3.2.5

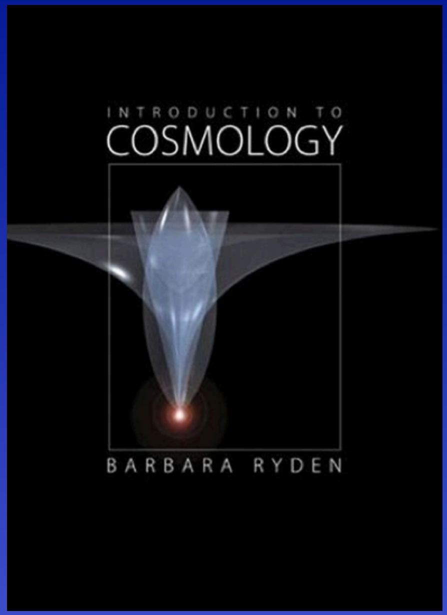
Other recommended textbooks on cosmology include:

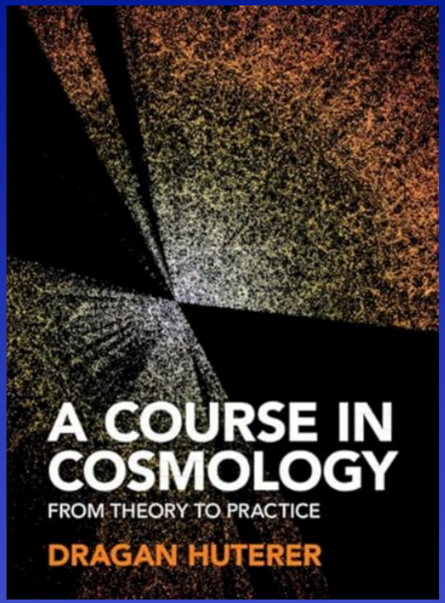
introductory level



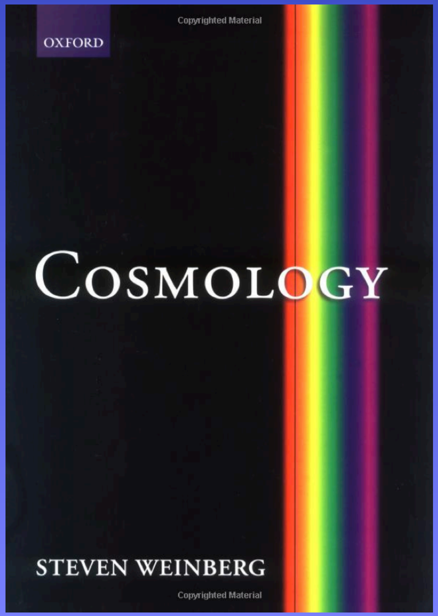


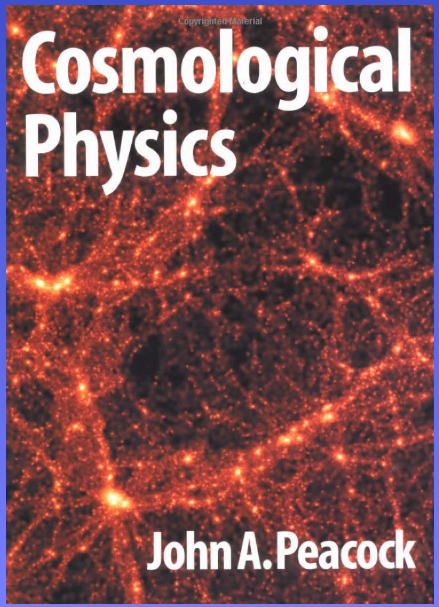


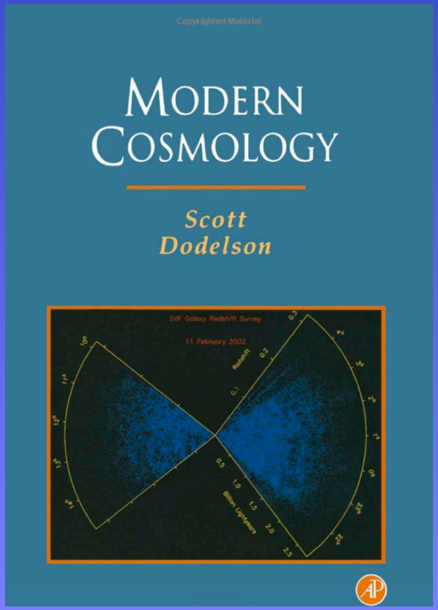


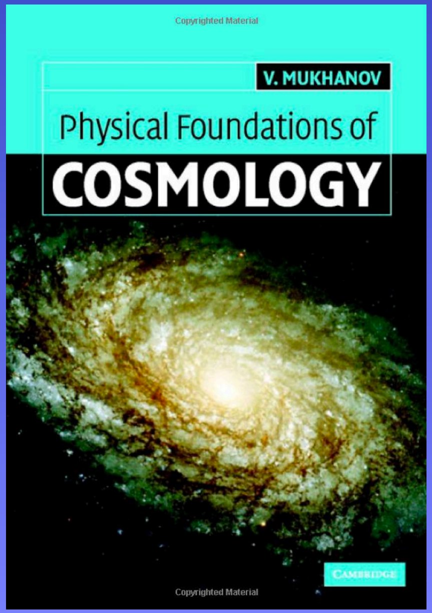


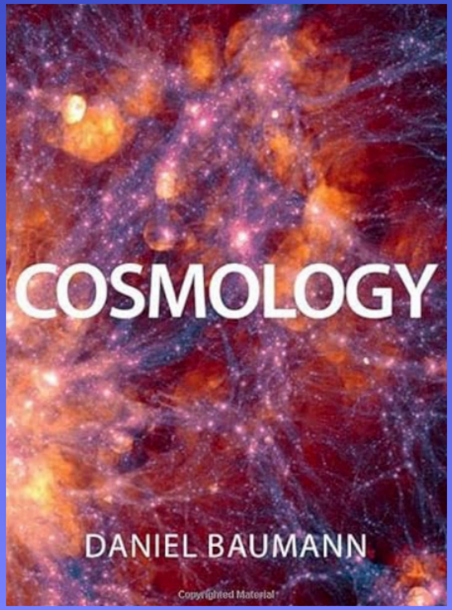
more advanced level











Lecture 3

SUMMARY

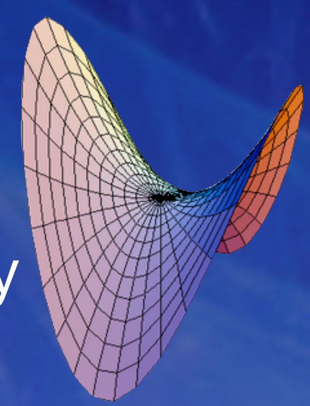
Cosmology in a Nutshell

Lecture 2

Cosmological Principle

Universe is homogeneous & Isotropic

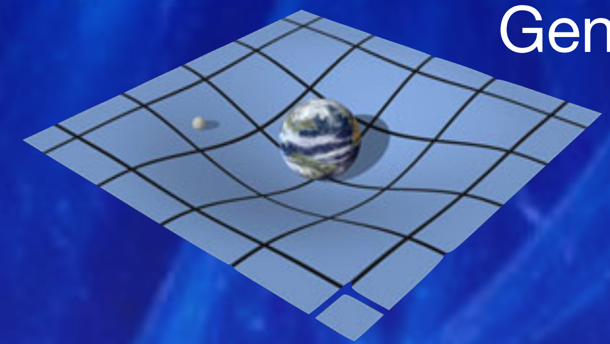
Riemannian Geometry



Friedmann-Robertson-Walker Metric

$$\mathrm{d}s^2 = a^2(\tau)\left[\mathrm{d}\tau^2 - \mathrm{d}\chi^2 - f_K^2(\chi)\,\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right]$$

Lecture 3



General Relativity

Einstein's Field Equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - g_{\mu\nu}\Lambda = \frac{8\pi G}{c^4} T_{\mu\nu}$$

Friedmann Equations

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{Kc^2}{a^2} + \frac{\Lambda c^2}{3}$$

Summary: key words & important facts

Key words

Equivalence Principle	Riemann tensor
Christoffel symbols	Ricci tensor
covariant derivative	Einstein tensor

Why we need GR

- Newtonian gravity only holds in inertial systems, is covariant under Galilean transformations, and moving mass has immediate effect all throughout space.

but

- inertial systems do not exist (you can't shield yourself from gravity)
- SR: inertial systems transform according to Lorentz transformations
- SR: universal speed limit; no information can propagate instantaneously

---

The Key to GR

- Since gravity is `permanent' (can only be transformed away locally), it must be related to an intrinsic property of space-time itself.
- Space-time of freely falling observer (no gravity) is flat Minkowski space; hence, gravity originates from curvature in space-time (Riemann space)
- Einstein Field equation is the manifest covariant version of Poisson equation

Summary: key equations & expressions

Poisson Equation

$$\boxed{\nabla^2 \Phi = 4\pi G \rho}$$

GR ↓

Einstein Field Equation

$$\boxed{R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} - \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}}$$

The Friedmann Equation

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{K c^2}{a^2} + \frac{\Lambda c^2}{3}}$$

$H^2(z) = H_0^2\, E^2(z)$ where

$$E(z) = \left[\Omega_{\Lambda,0} + (1 - \Omega_0)\,(1+z)^2 + \Omega_{\mathrm{m},0}(1+z)^3 + \Omega_{\mathrm{r},0}(1+z)^4\right]^{1/2}$$

Density Parameter

$$\boxed{\Omega(z) - 1 = (\Omega_0 - 1)\,\frac{(1+z)^2}{E^2(z)}}$$