ASTR 610
Theory of Galaxy Formation

Lecture 2: Overview of Cosmology I
(Riemannian Geometry & FRW metric)

Frank van den Bosch
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The Friedmann-Robertson-Walker Metric

In this first part of our brief review of cosmology we focus on geometry. Using Riemannian geometry, and the Cosmological Principle, we show how one arrives at the Friedmann-Robertson-Walker (FRW) metric, which features predominantly in modern cosmology.

Topics that will be covered include:

- Riemannian Geometry
- Concept of metric
- Geometry of Space-Time
- Cosmological Principle
- Fundamental Observers
- Distance Measures
- Thermodynamics of expanding space-time

Cosmology

NOTE: what follows is a very brief review of cosmology.
Students are strongly encouraged to read Chapter 3 of MBW

- Cosmology is the study of the structure & evolution of the Universe as a whole.

- Modern cosmology is founded upon Einstein's GR, according to which the structure of space-time is governed by its matter/energy density.

- Note that this is very different from classical physics, where space and time are eternal and absolute, independent of the existence of matter.

- Since cosmology (without perturbations) is a very simple application of GR, it can be understood without a detailed knowledge of GR.

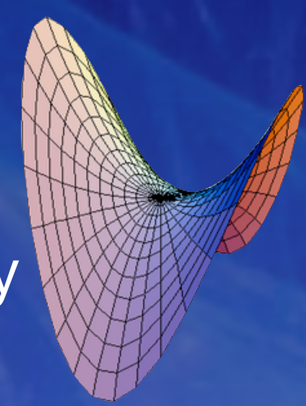
- In this review we focus on geometry (how to describe a curved space-time), which we use to derive the Friedmann-Robertson-Walker (FRW) metric, and on GR, which we use to derive the Einstein equation.
  Substitution of FRW metric in Einstein equation yields the Friedmann equations.

Cosmology in a Nutshell

Lecture 2

Cosmological Principle

Universe is homogeneous & Isotropic

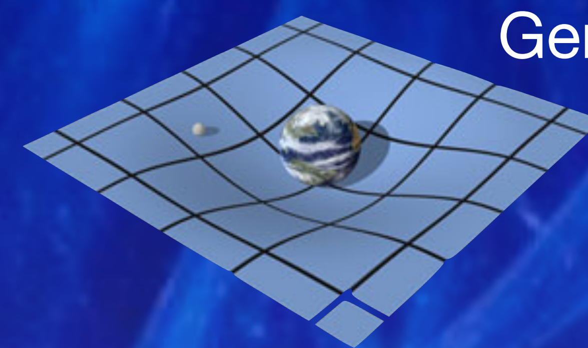


Riemannian Geometry

Friedmann-Robertson-Walker Metric

$$\mathrm{d}s^2 = a^2(\tau)\left[\mathrm{d}\tau^2 - \mathrm{d}\chi^2 - f_K^2(\chi)\,\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right]$$

Lecture 3



General Relativity

Einstein's Field Equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - g_{\mu\nu}\Lambda = \frac{8\pi G}{c^4} T_{\mu\nu}$$

Friedmann Equations

$$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{Kc^2}{a^2} + \frac{\Lambda c^2}{3}$$

Geometry of Space Time

- To specify where and when an event occurs in any space or space-time we typically require a coordinate system x^μ, where μ is an index.

- Note that this coordinate system has no physical meaning: any physical law should be independent of the choice of this coordinate system.

- We can make physical laws manifest invariant (valid for any coordinate system) by writing them in tensor form.

Tensors are geometric objects that can be represented as multi-dimensional arrays of numerical values. The rank (or order) of a tensor is the dimensionality of the array (i.e., the number of indices needed to label a component).

Newtonian potential: $\Phi(x^\mu)$ tensor of rank 0 (= scalar)

Electrical field: $\vec{E}(x^\mu)$ tensor of rank 1 (= vector)

Metric: $g_{\alpha\beta}(x^\nu)$ tensor of rank 2

Geometry of Space Time

The defining properties of tensors are their transformation rules:
i.e., how do their values change under a coordinate transformation $x^i \rightarrow x'^j$

- Scalar: $\Phi'(x'^j) = \Phi(x^i)$

thus, the value of a scalar field at a given point is independent of coordinate system used...

Contra-variant vector: $A'^k = \frac{\partial x'^k}{\partial x^i} A^i$

example of a contra-variant vector is the tangent to a curve...

Covariant vector: $A'_k = \frac{\partial x^i}{\partial x'^k} A_i$

example of a covariant vector is the normal to a surface...

Covariant tensor (rank 2): $T'_{ik} = \frac{\partial x^m}{\partial x'^i} \frac{\partial x^n}{\partial x'^k} T_{mn}$

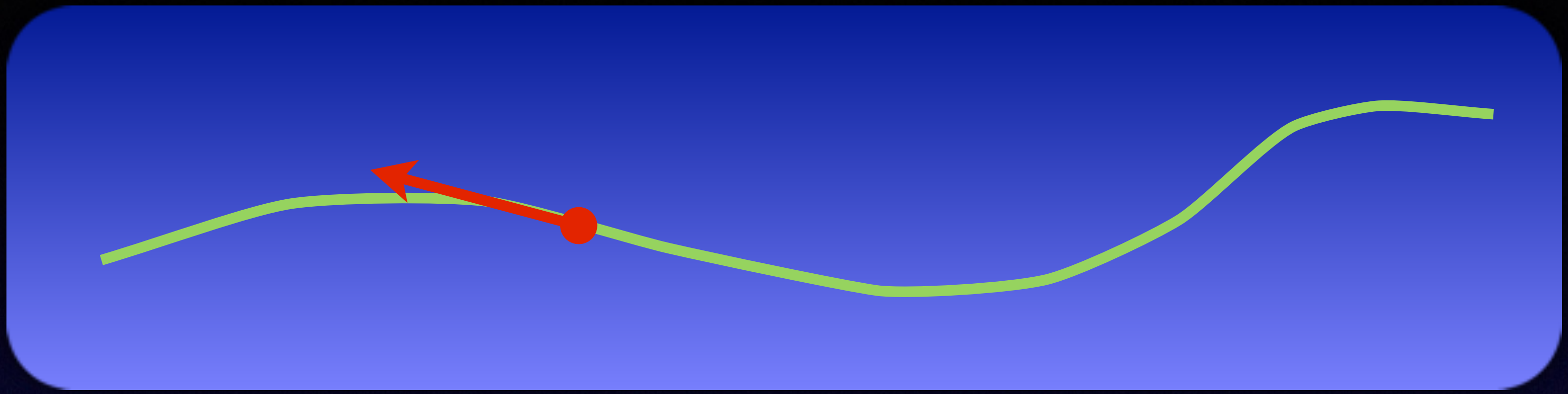
Mixed tensor (rank 2): $T'^i_k = \frac{\partial x'^i}{\partial x^m} \frac{\partial x^n}{\partial x'^k} T^m_n$

etc.

Einstein summation convention

when an index appears twice in a single term, it implies summation of that term over all the values of the index.

Intermezzo: covariant vs. contravariant

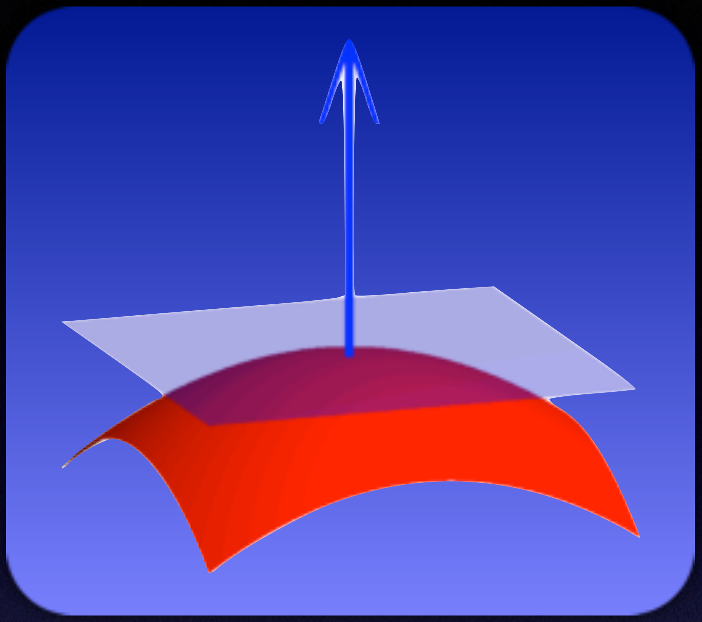


Consider a curve $x^\mu = x^\mu(\lambda)$ where λ is called the affine parameter (which is used to parameterize the curve). The direction of the curve's tangent at any point along the curve is given by the contravariant vector: $A^\nu = \mathrm{d}x^\nu/\mathrm{d}\lambda$

$$A'^\mu = \frac{\mathrm{d}x'^\mu}{\mathrm{d}\lambda} = \frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}\,\frac{\partial x'^\mu}{\partial x^\nu} = A^\nu\,\frac{\partial x'^\mu}{\partial x^\nu}$$

Clearly, it transforms as a tensor, which is consistent with the fact that the concept `tangent to a curve' is physical, and therefore has to be an invariant.

Intermezzo: covariant vs. contravariant



Consider a scalar function $\phi(x^\nu)$. The equation ϕ = constant describes a hyper-surface whose normal is given by the covariant vector: $A_\mu = \partial\phi/\partial x^\mu$

$$A'_\mu = \frac{\partial\phi}{\partial x'^\mu} = \frac{\partial\phi}{\partial x^\nu}\frac{\partial x^\nu}{\partial x'^\mu} = A_\nu \frac{\partial x^\nu}{\partial x'^\mu}$$

Clearly, it transforms as a tensor, which is consistent with the fact that the concept `normal to a surface' is physical, and therefore an invariant.

Geometry of Space Time

Q: How can we describe space-time in a coordinate independent way???

A: We need to focus on the physical invariants, which are the actual \`distances' between events/locations.

space-time: $\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu$

space: $\mathrm{d}l^2 = g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j$

Einstein summation convention

Thus, for a given coordinate system x^α, the geometry of a space or space-time is described by the metric, which in general depends on location: $g_{\mu\nu} = g_{\mu\nu}(x^\alpha)$

NOTE: the numerical values of the metric tensor depend on the choice of the coordinate system!

Example 1: 2D Euclidean Space

Consider the Cartesian coordinate system $x^i = (x, y)$

$$\mathrm{d}l^2 = \mathrm{d}x^2 + \mathrm{d}y^2 \quad \Rightarrow \quad g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

In general: For any Cartesian coordinate system in Euclidean space, E_n, we have that $g_{ij} = \delta_{ij}$, where (i,j) = (1,2,...,n) and δ_{ij} is the Kronecker delta function

Note: Euclidean manifolds are a subset of more general Riemannian manifolds. They are characterized by having zero curvature everywhere.

Example 1: 2D Euclidean Space

Next, consider the curvi-linear, polar coordinate system $x^i = (r, \theta)$

In order to compute $\mathrm{d}l^2$ for this new coordinate system, we use that $\mathrm{d}l^2$ is an invariant. Coordinate transformations wrt Cartesian:

$$\boxed{\begin{aligned} x &= r\,\cos\theta \\ y &= r\,\sin\theta \end{aligned}}$$

$$\left.\begin{aligned} \mathrm{d}x &= \frac{\partial x}{\partial r}\mathrm{d}r + \frac{\partial x}{\partial \theta}\mathrm{d}\theta \\ \mathrm{d}y &= \frac{\partial y}{\partial r}\mathrm{d}r + \frac{\partial y}{\partial \theta}\mathrm{d}\theta \end{aligned}\right\} \quad \mathrm{d}l^2 = \mathrm{d}x^2 + \mathrm{d}y^2 \; = \mathrm{d}r^2 + r^2\mathrm{d}\theta^2$$

$$\Rightarrow \quad g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

Note that $g_{ij} \neq \delta_{ij}$ even though the space *is* Euclidean. This demonstrates that the metric also depends on the coordinate system.
However, it is *always* true that $\mathrm{d}l^2 = g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j$

A space is Euclidean if a coordinate systems exists for which $g_{ij} = \delta_{ij}$ at each location. If so, this is the Cartesian coordinate system.

Geometry of Space Time

Example 2: 4D Minkowski Space

Consider the coordinate system $x^\mu = (ct, x, y, z)$

In Minkowski space, M_4 we have that $\mathrm{d}s^2 = c^2\mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2$

$$\Rightarrow \quad g_{\mu\nu} = \eta_{\mu\nu} \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

NOTE: one is also allowed to define $\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$
The choice of this signature has no impact on the physics as long as one is consistent.

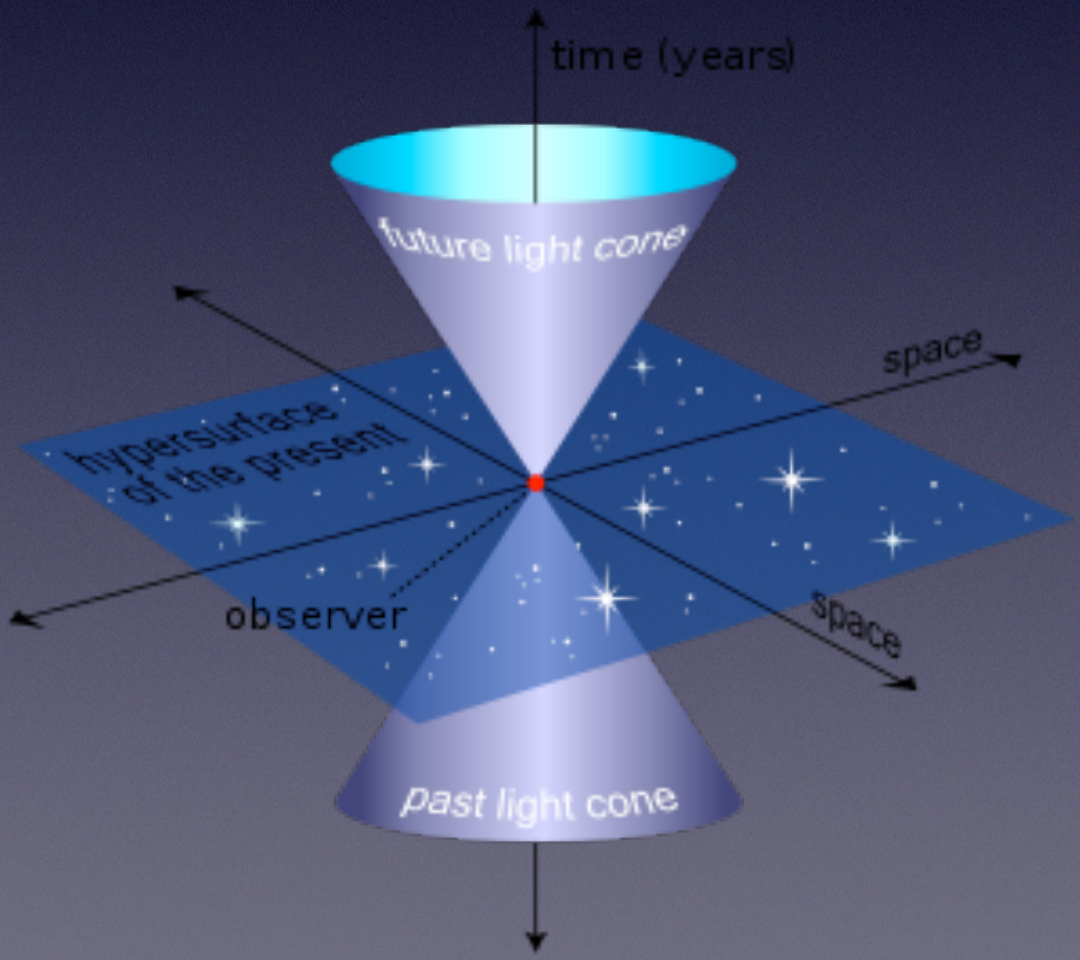
Minkowski space is an example of a pseudo-Euclidean space.

Geometry of Space Time

Example 2: 4D Minkowski Space

The Minkowski metric is used in Special Relativity (SR).

Any two observers will agree on the interval ds between two events A and B, even if they use different coordinate systems. Hence, in SR ds represents the absolute (=invariant) quantity that replaces the Newtonian concepts of absolute space and time. It has the following meaning:



For photons travelling at speed of light, $\mathrm{d}s = 0$

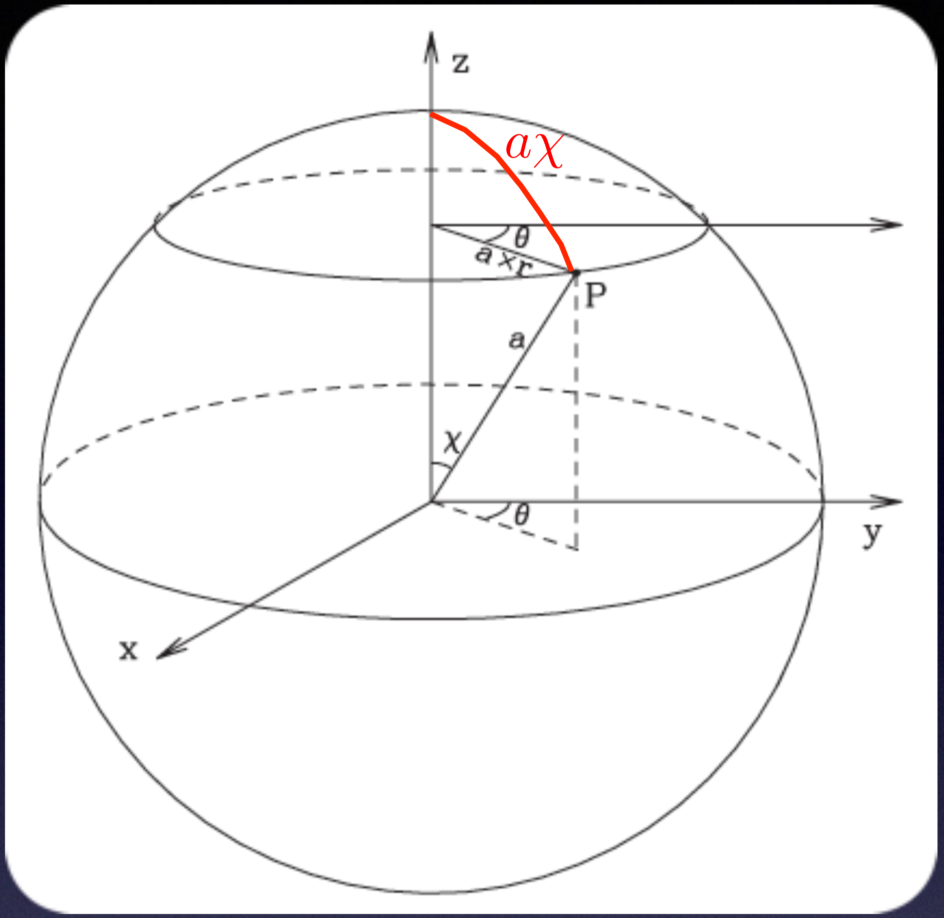
If $\mathrm{d}s^2 > 0$ then ds is the measured time interval between A and B in the restframe of an observer who sees both events occurring at his location.

If $\mathrm{d}s^2 < 0$ then no observer can experience both events, and |ds| is the space-interval between A and B in a frame in which the events are simultaneous.

Geometry of Space Time

Example 3: 2D surface of a sphere

It is useful (though not necessary) to imagine this 2D sphere embedded in 3D Euclidean space



That means we can use (x,y,z) to specify a point P on this surface, where we have the constraint that $x^2 + y^2 + z^2 = a^2$. Alternatively, we can define the coordinate system $x^i = (\chi, \theta)$

Note that unlike (x,y,z), the coordinates (χ, θ) are intrinsic to the surface.
The parameter a is just a scale-factor, which appears in the transformation relations, and whose relevance becomes clear later.

$$\boxed{\begin{aligned} x &= a \sin\chi \sin\theta \\ y &= a \sin\chi \cos\theta \\ z &= a \cos\chi \end{aligned}}$$

$$\mathrm{d}l^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 = a^2(\mathrm{d}\chi^2 + \sin^2\chi \, \mathrm{d}\theta^2)$$

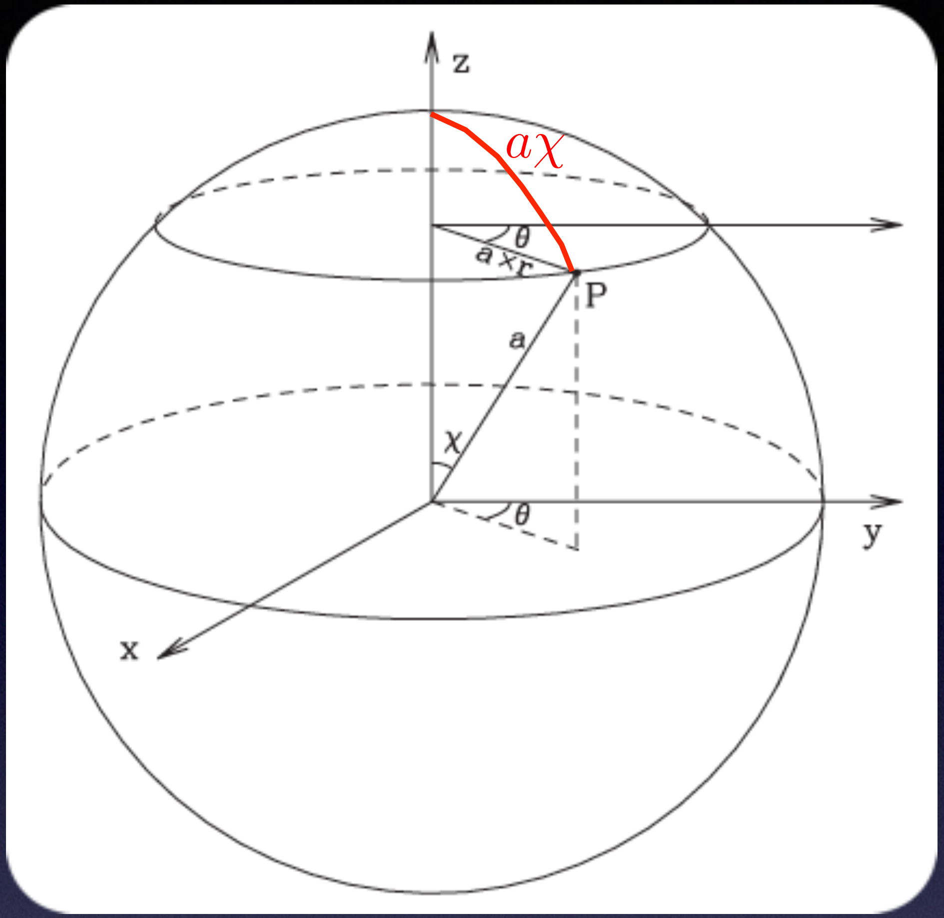
$$\Rightarrow \quad g_{ij} = \begin{pmatrix} a^2 & 0 \\ 0 & a^2 \sin^2\chi \end{pmatrix}$$

Geometry of Space Time

Example 3: 2D surface of a sphere

Alternatively, one can define the unitless parameter $r \equiv \sin\chi$ (which obeys $r \in [0,1]$). In terms of this parameter, we have that

$$\boxed{\begin{aligned} x &= a\,r\sin\theta \\ y &= a\,r\cos\theta \\ z &= a\,\sqrt{1-r^2} \end{aligned}}$$



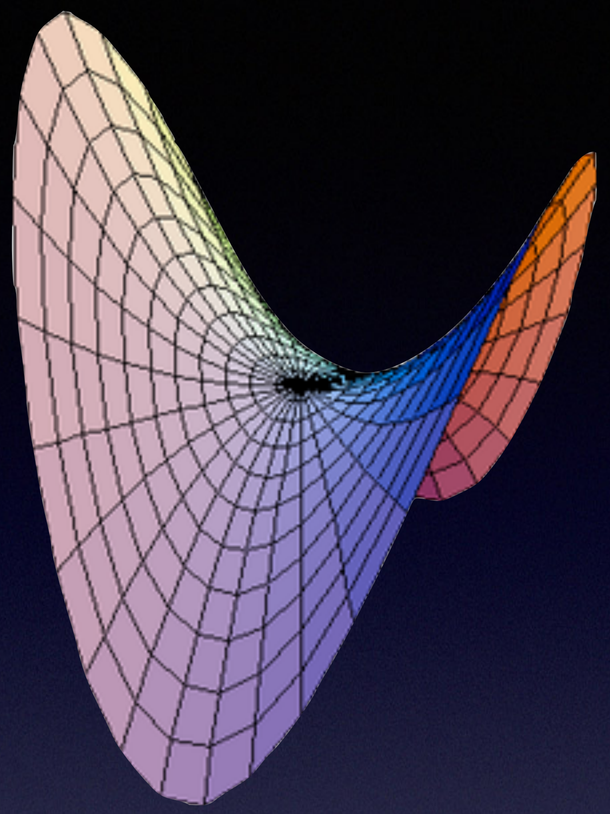
which implies that:

$$\mathrm{d}l^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 = a^2\left[\frac{\mathrm{d}r^2}{1-r^2} + r^2\mathrm{d}\theta^2\right] \Rightarrow g_{ij} = \begin{pmatrix} \frac{a^2}{1-r^2} & 0 \\ 0 & a^2r^2 \end{pmatrix}$$

The 2D sphere is a Riemann space with a constant, positive curvature.

Geometry of Space Time

Example 4: 2D `saddle' surface (pringle chips)



This geometry differs from the 2D sphere in that it has a negative curvature. Unlike the 2D sphere,
it cannot be embedded in 3D Euclidean space (which is why it is difficult to draw/imagine). Note, though, that it *can* be embedded in a 3D pseudo-Euclidean space, i.e., it obeys $x^2 + y^2 - z^2 = -a^2$, and has that

$$\boxed{\mathrm{d}l^2 = \mathrm{d}x^2 + \mathrm{d}y^2 - \mathrm{d}z^2}$$

Pick coordinate system $x^i = (\chi, \theta)$ that is intrinsic to the surface, and related to (x,y,z) via

$$\boxed{\begin{aligned} x &= a\,\sinh\chi\,\sin\theta \\ y &= a\,\sinh\chi\,\cos\theta \\ z &= a\,\cosh\chi \end{aligned}}$$

If we define the unitless parameter $r = \sinh\chi$ we obtain that

$$\mathrm{d}l^2 = \mathrm{d}x^2 + \mathrm{d}y^2 - \mathrm{d}z^2 = a^2\left[\frac{\mathrm{d}r^2}{1+r^2} + r^2\mathrm{d}\theta^2\right] \Rightarrow g_{ij} = \begin{pmatrix} \frac{a^2}{1+r^2} & 0 \\ 0 & a^2 r^2 \end{pmatrix}$$

Geometry of Space Time

Summary: 2D surfaces

2D Euclidean	$\mathrm{d}l^2 = a^2\left[\mathrm{d}r^2 + r^2\mathrm{d}\theta^2\right]$
2D Sphere	$\mathrm{d}l^2 = a^2\left[\frac{\mathrm{d}r^2}{1-r^2} + r^2\mathrm{d}\theta^2\right]$
2D Saddle	$\mathrm{d}l^2 = a^2\left[\frac{\mathrm{d}r^2}{1+r^2} + r^2\mathrm{d}\theta^2\right]$

Note that for all these surfaces each point on the surface is equivalent, which means that the metric is independent of the location on the surface....

By introducing the curvature parameter K = (+1,0,-1), we obtain the general metric:

$$g_{ij} = \begin{pmatrix} \frac{a^2}{1-Kr^2} & 0 \\ 0 & a^2r^2 \end{pmatrix}$$

Geometry of Space Time

Now that we have seen how to derive the metric of a general Riemann space, let's focus on the metric of space-time.

Cosmological Principle:

The Universe is homogeneous & isotropic on large scales.

- Copernican principle: our location in the Universe is in no way special
- In that case, why would ANY location be special (Occam's razor)
- Consistent with observations

Isotropy ☛ the only motion possible is global expansion or contraction; $a = a(t)$

Homogeneity ☛ the metric is independent of location; $g_{\mu\nu}(x^\alpha) = g_{\mu\nu}$

Thus, we seek a general 3D Riemann space, to be embedded in a (3+1)D space-time. The metric of a homogeneous and isotropic 3D Riemann space is

$$\boxed{\mathrm{d}l^2 = a^2(t)\left[\frac{\mathrm{d}r^2}{1-Kr^2} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\right]}$$

Note: (r, K, θ, ϕ) are all unitless. Only the scale-factor $a(t)$ has the dimensions of length!

The Friedmann-Robertson-Walker Metric

Upon embedding this 3D Riemann sphere in a (3+1)D space-time, we obtain the so-called Friedmann-Robertson-Walker (FRW) metric:

$$\boxed{\mathrm{d}s^2 = c^2\mathrm{d}t^2 - a^2(t)\left[\frac{\mathrm{d}r^2}{1-Kr^2} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\right]}$$

Fundamental Observers: A fundamental observer is an observer who, in a (unperturbed) FRW metric, observes the universe to be isotropic. The set of all fundamental observers defines a cosmological `rest-frame' at each location in space.

- The coordinates (r, θ, ϕ) label fundamental observers and are called comoving coordinates (i.e., they don't change under expansion/contraction).
- The parameter t in the FRW metric is called the proper time. It is the time on a standard clock of a fundamental observer.
- The parameter a in the FRW metric is called the scale factor. It relates the coordinates to true physical distances [recall; $r \in [0,1]$ is dimensionless].
- The parameter K in the FRW metric is called the curvature parameter. It indicates the global curvature of space-time, and can take on the values +1, 0, -1.

The Friedmann-Robertson-Walker Metric

- The proper distance is defined as the distance between two fundamental observers at some proper time t

$$l = \int \mathrm{d}l = a(t) \int_0^r \frac{\mathrm{d}r'}{\sqrt{1 - K r'^2}} = a(t)\,\chi(r)$$ *

Here $\chi(r)$ is the comoving distance between the fundamental observers

$$\chi(r) = \begin{cases} \sin^{-1} r & \text{if } K = +1 \\ r & \text{if } K = 0 \\ \sinh^{-1} r & \text{if } K = -1 \end{cases}$$

NOTE: proper distance = scale factor x comoving distance

* We have assumed here that one of the fundamental observers is located at $(r, \theta, \phi) = (0, 0, 0)$, while the other is at $(r, \theta, \phi) = (r, 0, 0)$: I can always pick my coordinate system such that this is the case....

The Friedmann-Robertson-Walker Metric

- In addition to the proper time, t , one can also define the conformal time:

$$\tau(t) \equiv \int_0^t \frac{c\,\mathrm{d}t'}{a(t')}$$

$[\tau]$ = unitless

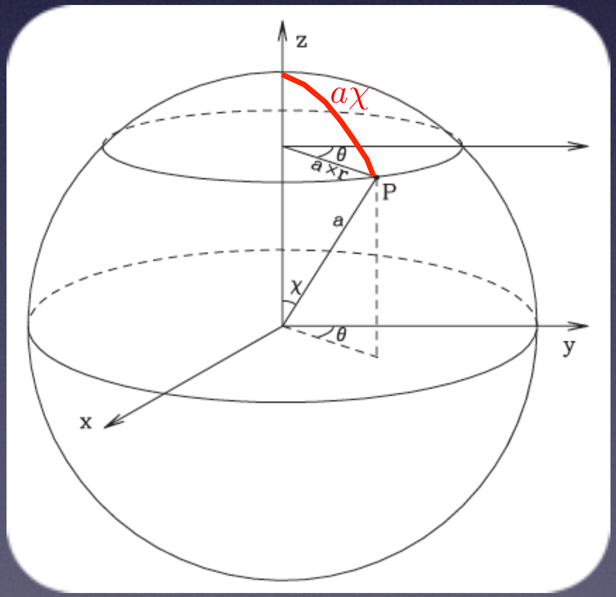
The conformal time is the total comoving distance χ light could have travelled.
The proper time a photon travels in a proper time interval Δt is simply $\Delta l = c\Delta t$.
In a conformal time interval $\Delta\tau$ the photon has travelled a comoving distance $\Delta\chi = \Delta\tau$

- In terms of χ and τ , the FRW metric can be rewritten as:

$$\mathrm{d}s^2 = a^2(\tau)\left[\mathrm{d}\tau^2 - \mathrm{d}\chi^2 - f_K^2(\chi)\,\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right]$$

where

$$f_K(\chi) = r = \begin{cases} \sin\chi & \text{if } K = +1 \\ \chi & \text{if } K = 0 \\ \sinh\chi & \text{if } K = -1 \end{cases}$$



The advantage of this form of the FRW metric is that χ is an actual distance measure. This is NOT the case for r, which is `just' a coordinate.

The Friedmann-Robertson-Walker Metric

A word of caution about units:

In what we discussed above, the coordinates (r, χ, θ, ϕ) are all unitless.

Proper time has the units of time, but conformal time is also unitless.

Only the scale factor $a(t)$ has the units of length, thus making both $\mathrm{d}l$ and $\mathrm{d}s$ have the units of length as well.

However,

It is common practice (and we will adopt this as well), to define the dimensionless scale factor $a(t) = a(t)/a_0$, where a_0 is the present-day value of the scale factor. Hence, this dimensionless scale factor is normalized to have $a(t) = 1$ at the present.

With this new convention, $a(t)$ is unitless, while (r, χ, τ) all carry the units of length! Also, the curvature scalar now has $[K] = \mathrm{length}^{-2}$

The Hubble Parameter

The rate at which the proper distance between two fundamental observers changes as a function of time can be written as

$$\boxed{\frac{\mathrm{d}l}{\mathrm{d}t} \equiv H(t)\, l}$$

which defines the Hubble parameter, $H(t)$. Using that $l = a(t)\chi$, with χ the comoving distance between the fundamental observers, we have that:

$$\boxed{H(t) = \frac{\dot{a}}{a}}$$

where $\dot{a} = \mathrm{d}a/\mathrm{d}t$, and we have used that $\frac{\mathrm{d}l}{\mathrm{d}t} = \dot{a}\chi = \frac{\dot{a}}{a} a\chi = \frac{\dot{a}}{a} l$.

The value of the Hubble parameter at the present is called the Hubble constant $H_0 \simeq (70 \pm 2)\,\mathrm{km\,s^{-1}Mpc^{-1}}$.
It describes the present-day expansion rate of the Universe.

Redshift

See MBW §3.1.3 for details

Photons move along geodesics, $\mathrm{d}s = 0$. Substitution in FRW metric implies that $\mathrm{d}\tau = \mathrm{d}\chi$, where without losing generality we assumed $\mathrm{d}\theta = \mathrm{d}\phi = 0$

$$\boxed{\mathrm{d}\tau = \frac{c\,\mathrm{d}t}{a(t)} = \mathrm{d}\chi}$$

Hence, if the comoving distance between emitter and observer remains fixed (i.e., in the absence of peculiar velocities), proper time intervals scale with the scale factor of the Universe:

$$\boxed{\frac{\delta t_{\rm obs}}{a(t_{\rm obs})} = \frac{\delta t_{\rm em}}{a(t_{\rm em})}}$$

Since the wavelength of a photon is proportional to the period between the arrival of two wavecrests, we have that

$$\boxed{z \equiv \frac{\lambda_{\rm obs} - \lambda_{\rm em}}{\lambda_{\rm em}} = \frac{a(t_{\rm obs})}{a(t_{\rm em})} - 1}$$

NOTE: this has NOTHING to do with Doppler effect. After all, nobody is moving...

Thus, for $a(t_{\rm obs}) = a_0 = 1$ we have that

$$a = \frac{1}{1+z}$$

i.e., a photon from z=1 was emitted when the Universe was half its present size.

Peculiar Velocities

See MBW §3.1.4 for details

The proper velocity of a particle wrt a fundamental observer at the origin is defined as $v = \mathrm{d}l/\mathrm{d}t$, with $l(t)$ the proper distance between particle & observer.

Using that $l(t) = a(t)\chi(t)$ we obtain that $\boxed{v = \dot{a}\chi + a\dot{\chi} \equiv v_{\rm exp} + v_{\rm pec}}$

$v_{\rm exp} = \frac{\dot{a}}{a} a\chi = H\,l$ reflects the "velocity" due to the Hubble expansion

$v_{\rm pec}$ is the "peculiar velocity" wrt a co-spatial fundamental observer

The observed redshift from an object is $\boxed{1 + z_{\rm obs} = (1 + z_{\rm cos})\,(1 + z_{\rm pec})}$

$1 + z_{\rm cos} = 1/a(t_{\rm em})$ is cosmological redshift due to expansion of space-time

$1 + z_{\rm pec} = \sqrt{\frac{1 + v_{\rm pec}/c}{1 - v_{\rm pec}/c}}$ is Doppler redshift due to peculiar velocity (along los)

In non-relativistic limit ($v_{\rm pec} \ll c$), this reduces to: $\boxed{z_{\rm obs} = z_{\rm cos} + \frac{v_{\rm pec}}{c}(1 + z_{\rm cos})}$

Due to the expansion, the peculiar velocities of particles that do not experience an external force decay with time as $v_{\rm pec} \propto a^{-1}$

Distances

See MBW §3.1.6 for details

So far we have encountered two different distances. The comoving distance χ and the proper distance l , which are related according to $l = a\chi$

In a static, Euclidean space the angular extent Θ and flux f of an object are related to its physical size D and luminosity L according to

$$\Theta = \frac{D}{d_{\rm A}} \qquad\qquad f = \frac{L}{4\pi d_{\rm L}^2}$$

and the angular diameter distance $d_{\rm A}$ is equal to the luminosity distance $d_{\rm L}$

However, in an expanding space time, this is no longer the case. If we use the above equations to define $d_{\rm A}$ and $d_{\rm L}$, then it can be shown that:

$$d_{\rm A}(z) = \frac{a_0\, r}{1+z} \qquad\qquad d_{\rm L}(z) = a_0\, r\,(1+z)$$

where $r = r(z) = f_{\rm K}(\chi)$ is the coordinate in the FRW metric. Hence, for an object at redshift z one distinguishes three distances: χ , $d_{\rm A}$, and $d_{\rm L}$.
To compute the z-dependence of these distances requires Friedmann eq.

Energy Density of the Universe

Our cosmological fluid consists of multiple components:

$$\boxed{\rho c^2 = \rho_{\rm m} c^2 + \rho_{\rm m}\varepsilon + \rho_{\rm r} c^2 + \rho_\Lambda c^2}$$

$\rho_{\rm m} c^2$ = contribution due to rest-mass of matter

$\rho_{\rm m}\varepsilon$ = contribution due to internal energy of matter

$\rho_{\rm r} c^2$ = contribution due to energy of radiation

$\rho_\Lambda c^2$ = contribution due to energy of vacuum

$$\rho_{\rm r} c^2 = \frac{4\sigma_{\rm SB}}{c} T^4$$

$\sigma_{\rm SB}$ = Stefan Boltzmann constant

ε = internal energy per unit mass

- As we will see, one can use simple Newtonian thermodynamics to infer how these different energy components evolve in an expanding space-time.

- Since energy densities of baryons & dark matter evolve in the same way, it is sufficient to describe the (non-relativistic) matter as one component .

- The energy density of radiation and any other relativistic component (e.g., neutrinos) only depends on temperature.

Thermodynamics in Expanding Space-Time

Consider a comoving volume $V \propto a^3(t)$ in a homogeneous & isotropic Universe.
We can consider V arbitrarily small --> no need for GR.

1st law of thermodynamics $\mathrm{d}U = \mathrm{d}Q + \mathrm{d}W$

2nd law of thermodynamics $\mathrm{d}S = \mathrm{d}Q/T$

$\mathrm{d}U$ = increase in internal energy
$\mathrm{d}Q$ = heat transfer into system
$\mathrm{d}W$ = work done on system
$\mathrm{d}S$ = increase in entropy

For an isolated, adiabatically expanding volume $\mathrm{d}Q = 0$ $\Rightarrow$ $\mathrm{d}U = -P\mathrm{d}V$, $\mathrm{d}S = 0$

Let $\rho\, c^2$ be the energy density. Then $U = \rho\, c^2\, V$

$$\begin{aligned} \mathrm{d}U &= \frac{\partial U}{\partial \rho}\mathrm{d}\rho + \frac{\partial U}{\partial V}\mathrm{d}V \\ &= c^2\, V\, \mathrm{d}\rho + \rho\, c^2\, \mathrm{d}V \end{aligned}$$

$\Rightarrow$

$$\begin{aligned} \mathrm{d}U + P\mathrm{d}V &= 0 \\ c^2 V \mathrm{d}\rho + \rho c^2 \mathrm{d}V + P\mathrm{d}V &= 0 \\ V\mathrm{d}\rho + \left(\rho + \frac{P}{c^2}\right)\mathrm{d}V &= 0 \end{aligned}$$

Using that $V \propto a^3$ and differentiating with respect to the scale factor yields

$$\boxed{\frac{\mathrm{d}\rho}{\mathrm{d}a} + 3\left(\frac{\rho + P/c^2}{a}\right) = 0}$$

Equation of State

The equation of state (EoS) is a thermodynamic equation describing the interconnection between various macroscopic properties of a system.

For fluids, one often considers EoS of the form $P = P(\rho, T)$

In cosmology, it is convenient, and common, to write $P = w\rho c^2$.
Here $w = w(T)$ is the EoS parameter describing our cosmological `fluid'

Substitution of this general EoS into our first law of thermodynamics yields

$$\boxed{\frac{\mathrm{d}\rho}{\mathrm{d}a} + 3(1+w)\frac{\rho}{a} = 0 \quad \Rightarrow \quad \rho \propto a^{-3(1+w)}}$$

Hence, the EoS parameter of a particular component of the cosmological fluid, determines how its energy density evolves with the scale parameter.

To learn more about EoS; see App I of my **ASTR 501** (*Dynamics of Astrophysical Many-Body Systems*) lecture notes,

Non-Relativistic Matter

Non-relativistic matter can be describes reasonably well as an ideal gas.

An ideal gas is a hypothetical gas that consists of identical particles of zero volume that undergo perfectly elastic collisions and for which intermolecular forces can be neglected.

Ideal Gas Law: $PV = N k_{\rm B} T$
$\rho_{\rm m} = N \mu m_{\rm p} / V$ $\Rightarrow$ $P = \frac{k_{\rm B} T}{\mu m_{\rm p}} \rho_{\rm m}$ (EoS)

See App A of MBW

The EoS parameter for an ideal gas is (see problem set 1)

$$w = w(T) = \frac{k_{\rm B} T}{\mu m_{\rm p} c^2} \left(1 + \frac{1}{\gamma - 1} \frac{k_{\rm B} T}{\mu m_{\rm p} c^2}\right)^{-1}$$

Since for a non-relativistic fluid $k_{\rm B} T \ll \mu m_{\rm p} c^2$ we have that $w \simeq 0$

Hence, non-relativistic fluid can be approximated as zero-pressure fluid (“dust‘’)

$w = 0$ $\Rightarrow$	$\rho \propto a^{-3}$	(conservation of particles)
$k_{\rm B} T \propto m\, v^2 \quad v \propto a^{-1}$ $\Rightarrow$	$T \propto a^{-2}$	
$P = (k_{\rm B} T / \mu m_{\rm p}) \rho$ $\Rightarrow$	$P \propto a^{-5}$	(pressure rapidly drops)

μ = mean molecular weight in units of proton mass $m_{\rm p}$ $k_{\rm B}$ = Boltzmann constant γ = adiabatic index

Radiation & Relativistic Matter

For relativistic matter (mainly photons), we have that $P = \frac{1}{3}\rho c^2$ (see MWB §3.3.2)

Hence we have that the EoS parameter $w = 1/3$, and thus

$$w = 1/3 \Rightarrow \rho \propto a^{-4}$$
$$\rho \propto T^4 \Rightarrow T \propto a^{-1}$$
$$P = \rho c^2/3 \Rightarrow P \propto a^{-4}$$

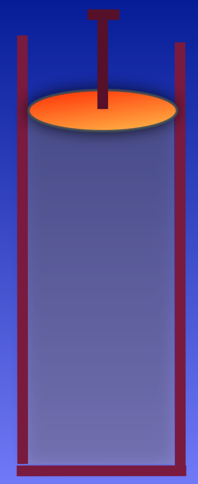
The fact that the energy density of radiation scales with a^{-4} can be understood as the number density of photons scaling as $n_\gamma \propto a^{-3}$ while the energy per photon $E = h\nu \propto a^{-1}$

Also, the fact that the energy density of radiation decreases faster than that of matter implies that radiation dominated at early times.

Vacuum Energy

According to quantum-physics, the vacuum can also have a non-zero energy density. We associate that with the cosmological constant, though we caution that `dark energy' is not understood at all!!!

The pressure associated with the cosmological constant behaves as $P = -\rho_\Lambda c^2$ (see inset), which implies an EoS parameter $w = -1$



Consider a piston filled with vacuum. Increasing its volume dV increases the total energy by $\mathrm{d}U = \rho_\Lambda c^2 \mathrm{d}V$. According to 1st law of thermodynamics $\mathrm{d}U + P\mathrm{d}V = 0$. Hence, we have that $P = -\rho_\Lambda c^2$

$$w = -1 \;\Rightarrow\; \rho \propto a^0$$

no temperature

$$P = -\rho_\Lambda c^2 \;\Rightarrow\; P \propto a^0$$

Since the properties of the vacuum are fixed, the energy density of the vacuum, and its associated pressure are also constant. Note that one cannot speak of the temperature of a vacuum.

Lecture 2

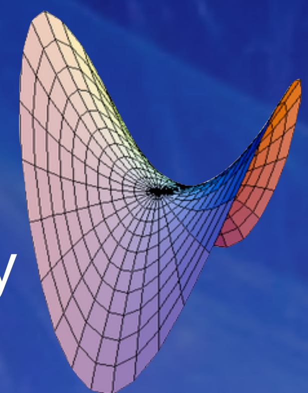
SUMMARY

Cosmology in a Nutshell

Lecture 2

Cosmological Principle

Universe is homogeneous & Isotropic

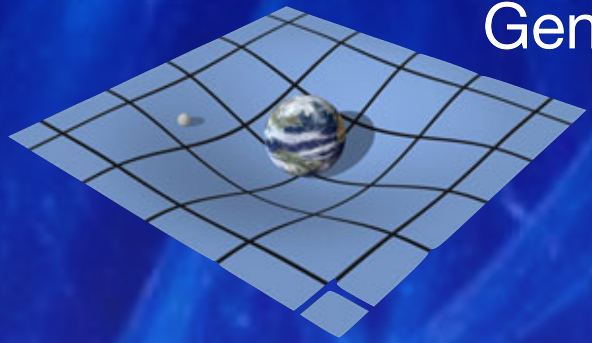


Riemannian Geometry

Friedmann-Robertson-Walker Metric

$$ds^2 = a^2(\tau)\left[d\tau^2 - d\chi^2 - f_K^2(\chi)\,\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right]$$

Lecture 3



General Relativity

Einstein's Field Equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - g_{\mu\nu}\Lambda = \frac{8\pi G}{c^4} T_{\mu\nu}$$

Friedmann Equations

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{Kc^2}{a^2} + \frac{\Lambda c^2}{3}$$

Summary: key words & important facts

Key words	
Fundamental observer	Cosmological Principle
Proper time *vs*. conformal time	FRW metric
Comoving *vs.* proper distance	Hubble parameter
Angular diameter distance	redshift
Luminosity distance	peculiar velocity

- Physical laws can be made manifest invariant by writing them in tensor form.
- The geometry of space-time is described by the metric $g_{\mu\nu} = g_{\mu\nu}(x^\alpha)$
- The FRW-metric is the most general metric consistent with the cosmological principle, that the Universe is homogeneous and isotropic (on large scales).
- Due to the expansion, the peculiar velocities of particles that do not experience an external force decay with time as $v_{\rm pec} \propto a^{-1}$
- Since energy densities of baryons & dark matter evolve in the same way, it is sufficient to describe the (non-relativistic) matter as one component .
- Since energy densities of radiation & relativistic matter (i.e., neutrinos) evolve in the same way, it is sufficient to describe them as one component .

Summary: key equations & expressions

Two ways of writing the FRW-metric

$$ds^2 = c^2 dt^2 - a^2(t)\left[\frac{dr^2}{1-Kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right]$$

$$ds^2 = a^2(\tau)\left[d\tau^2 - d\chi^2 - f_K^2(\chi)\,\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right]$$

Thermodynamics $\rightarrow$ $\frac{d\rho}{da} + 3(1+w)\frac{\rho}{a} = 0$ $\rightarrow$ $\rho \propto a^{-3(1+w)}$

non-relativistic matter (baryons & dark matter) w = 0
relativistic matter (radiation) w = 1/3
cosmological constant (dark energy) w = -1

Redshift, wavelength, scale-factor & peculiar velocity

$$z \equiv \frac{\lambda_{\rm obs} - \lambda_{\rm em}}{\lambda_{\rm em}} = \frac{a(t_{\rm obs})}{a(t_{\rm em})} - 1$$

$v = \dot{a}\chi + a\dot{\chi} \equiv v_{\rm exp} + v_{\rm pec}$ $\rightarrow$ $1 + z_{\rm obs} = (1 + z_{\rm cos})\,(1 + z_{\rm pec})$

angular diameter distance
luminosity distance

$$d_A(z) = \frac{a_0\, r}{1+z} \qquad d_L(z) = a_0\, r\,(1+z)$$