ASTR 610
Theory of Galaxy Formation

Lecture 14: Galaxy Interactions

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Yale University, Fall 2022

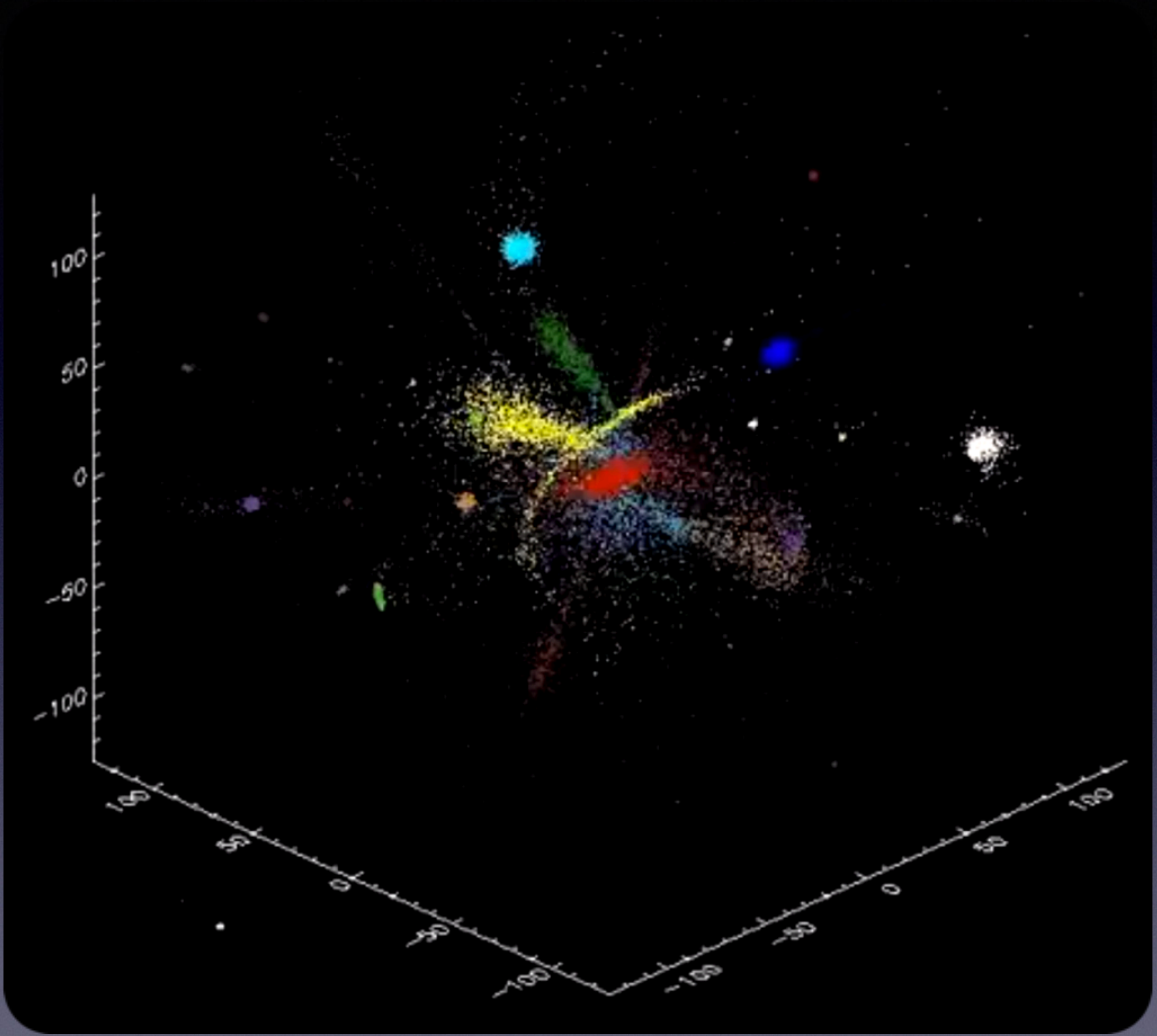
Gravitational Interactions

In this lecture we discuss galaxy interactions and transformations. After a general introduction regarding gravitational interactions, we focus on high-speed encounters, tidal stripping, dynamical friction and mergers. We end with a discussion of various environment-dependent satellite-specific processes such as galaxy harassment, strangulation & ram-pressure stripping.

Topics that will be covered include:

- Impulse Approximation
- Distant Tide Approximation
- Tidal Shocking & Stripping
- Tidal Radius
- Dynamical Friction
- Orbital Decay
- Core Stalling

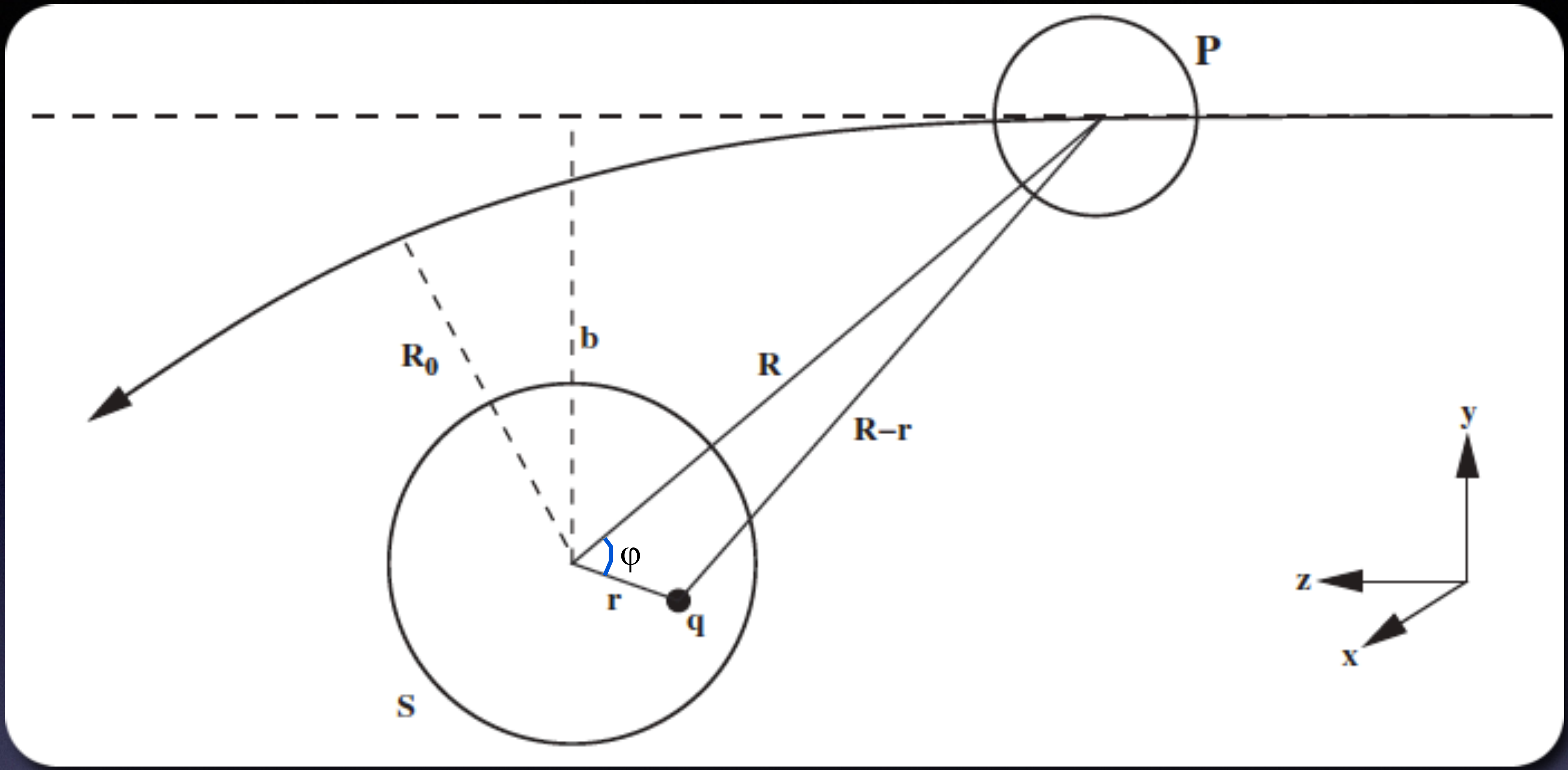
Visual Introduction



This simulation, presented in Bullock & Johnston (2005), nicely depicts the action of tidal (impulsive) heating and stripping. Different colors correspond to different satellite galaxies, orbiting a host halo reminiscent of that of the Milky Way....

Movie: https://www.youtube.com/watch?v=DhrrcdSjroY

Gravitational Interactions

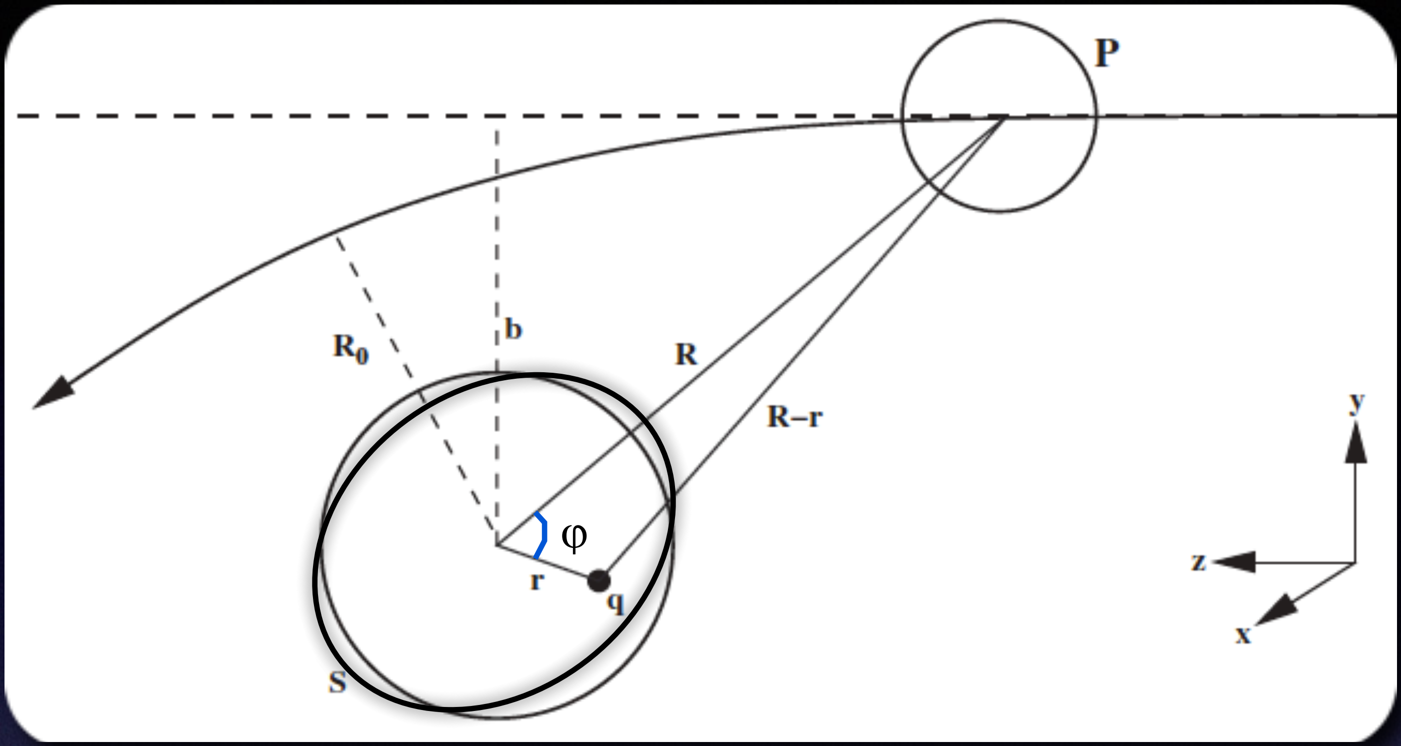


Consider a body S which has an encounter with a perturber P with impact parameter b and initial velocity v_∞

Let q be a particle in S, at a distance r(t) from the center of S, and let R(t) be the position vector of P wrt S.

The gravitational interaction between S and P causes tidal distortions, which in turn causes a back-reaction on their orbit...

Gravitational Interactions



Let $t_{\rm tide} \simeq R_S/\sigma$ be time scale on which tides rise due to a tidal interaction, where R_S and σ are the size and velocity dispersion of the system that experiences the tides.

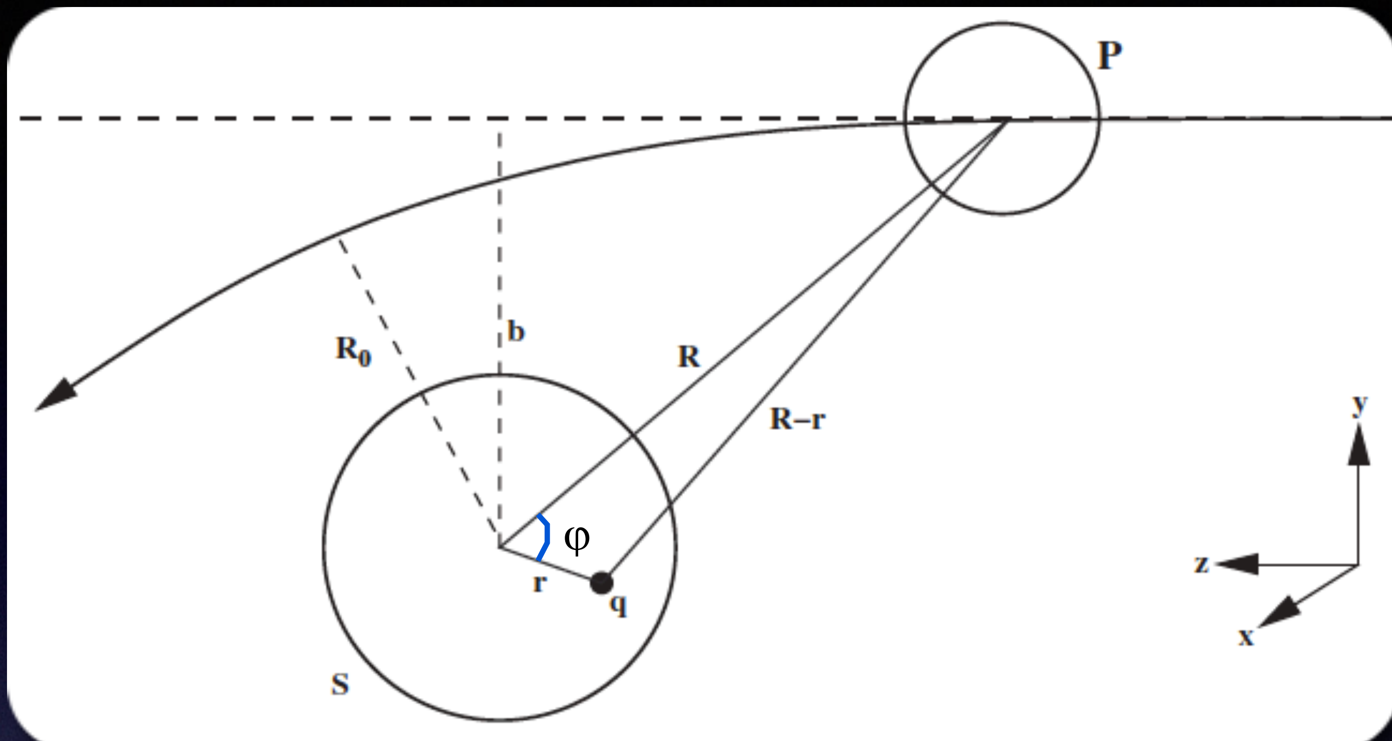
Let $t_{\rm enc} \simeq b/V$ be the characteristic time scale of the encounter

If $t_{\rm enc} \gg t_{\rm tide}$ we are in the adiabatic limit (no net effect)

the system has sufficient time to respond to tidal deformations; deformations during approach and departure cancel each other...

Note. $t_{\rm enc} \gg t_{\rm tide}$ implies that $V \ll (b/R_S)\,\sigma$. Typically, $b > R_S$ and since V can't be much smaller than σ, after all P is accelerated by same gravitational field that is responsible for σ, the situation $t_{\rm enc} \gg t_{\rm tide}$ is extremely rare.

Gravitational Interactions



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Let $t_{\rm enc} \simeq b/V$ be the characteristic time scale of the encounter

If $t_{\rm enc} < t_{\rm tide}$, the response of the system lags behind the instantaneous tidal force, causing a back reaction on the orbit.

transfer of orbital energy to internal energy (of both S and P)

- Under certain conditions, if enough orbital energy is transferred, the two bodies can becomes gravitationally bound to each other, which is called gravitational capture.
- If orbital energy continues to be transferred, capture will ultimately result in merger.
- When internal energy gain is large, particles may become unbound: mass loss

High Speed Encounters

In general, N-body simulations are required to investigate outcome of a gravitational encounter. However, in case of V >> σ (encounter velocity is much large than internal velocity dispersion of perturbed system; e.g., galaxies in clusters) the change in the internal energy can be obtained analytically using impulse approximation

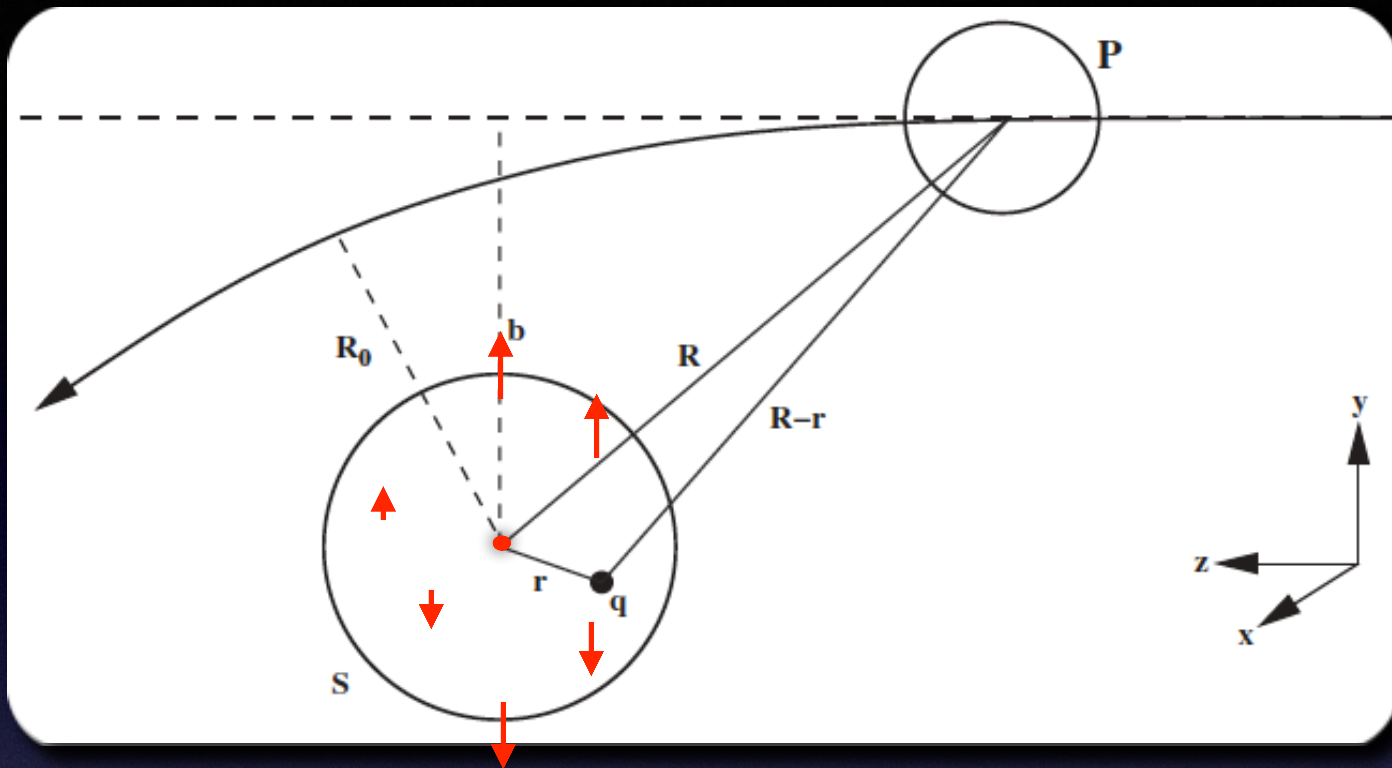
Consider the encounter between S and P. In impulse approximation we may consider a particle q in S to be stationary (wrt center of S) during the encounter; q only experiences a velocity change Δv, but its potential energy remains unchanged,

$$\Delta E_q = \frac{1}{2}(\vec{v} + \Delta\vec{v})^2 - \frac{1}{2}\vec{v}^2 = \vec{v}\cdot\Delta\vec{v} + \frac{1}{2}|\Delta\vec{v}|^2$$

We are interested in computing ΔE_S, which is obtained by integrating ΔE_q over the entire system S. Because of symmetry, the $\vec{v}\cdot\Delta\vec{v}$-term will equal zero

$$\Delta E_S = \frac{1}{2}\int |\Delta\vec{v}(\vec{r})|^2 \, \rho(r) \, \mathrm{d}^3\vec{r}$$

High Speed Encounters



In the large v_∞ limit, $R_0 \rightarrow b$ and we have that $v_{\rm P}(t) \simeq v_\infty\,\hat{e}_z \equiv v_{\rm P}\,\hat{e}_z$

$$\Rightarrow \boxed{R(t) = (0, b, v_{\rm P} t)}$$

In the distant encounter approximation, $b >> \mathrm{MAX}[R_S, R_P]$ and the perturber P may be considered a point mass.

The potential due to P at r is $\boxed{\Phi_{\rm P} = -\frac{G\,M_{\rm P}}{|\vec{r}-\vec{R}|}}$ where $|\vec{r}-\vec{R}| = \sqrt{R^2 - 2rR\cos\phi + r^2}$

with ϕ the angle between $\vec{r}$ and $\vec{R}$.

Using that $(1+x)^{-1/2} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{15}{48}x^3 + \ldots$ we can write this as

$$\Rightarrow \boxed{\Phi_{\rm P} = -\frac{G\,M_{\rm P}}{R} - \frac{G\,M_{\rm P}\,r}{R^2}\cos\phi - \frac{G\,M_{\rm P}\,r^2}{R^3}\left(\frac{3}{2}\cos\phi - \frac{1}{2}\right) + \mathcal{O}(r^3/R^3)}$$

constant term; does not yield any force

describes how center of mass of S changes: not of interest to us

describes tidal force per unit mass. This is term that we want...

dropping higher order terms is called the tidal approximation.

Impulse Approximation

NOTE: high-speed encounters for which both the impulsive and the tidal approximations are valid are called tidal shocks.

Taking the gradient of the potential, and dropping the second (constant acceleration) term, yields the tidal force per unit mass: $\vec{F}_{\rm tid}(\vec{r}) = \nabla\Phi_{\rm P}$

Integrating $\vec{F}_{\rm tid}(\vec{r}) = {\rm d}\vec{v}/{\rm d}t$ over time then yields the cumulative change in velocity wrt the center of S. After some algebra (see MBW §12.1) one finds that

$$\boxed{\Delta\vec{v} = \frac{2\,G\,M_{\rm P}}{v_{\rm P}\,b^2}\,(-x, y, 0)}$$

Substituting in the expression for the total change in energy of S yields

$$\boxed{\Delta E_{\rm S} = \frac{1}{2}\int |\Delta\vec{v}|^2\,\rho(r)\,{\rm d}^3\vec{r} = \frac{2\,G^2\,M_{\rm P}^2}{v_{\rm P}^2\,b^4}\int \rho(r)\,(x^2+y^2)\,{\rm d}^3\vec{r} = \frac{2\,G^2\,M_{\rm P}^2}{v_{\rm P}^2\,b^4}\,M_{\rm S}\,\langle x^2+y^2\rangle}$$

Assuming spherical symmetry for S, so that $\langle x^2+y^2\rangle = \frac{2}{3}\langle x^2+y^2+z^2\rangle = \frac{2}{3}\langle r^2\rangle$

Impulse Approximation

$$\boxed{\Delta E_{\rm S} = \frac{4}{3}\,G^2\,M_{\rm S}\left(\frac{M_{\rm P}}{v_{\rm P}}\right)^2\frac{\langle r^2\rangle}{b^4}}$$

Impulse Approximation

Impulse Approximation

$$\Delta E_{\rm S} = \frac{4}{3} G^2 M_{\rm S} \left(\frac{M_{\rm P}}{v_{\rm P}}\right)^2 \frac{\langle r^2 \rangle}{b^4}$$

This approximation is surprisingly accurate, even for relatively slow encounters with $v_P \sim \sigma_S$, as long as the impact parameter $b \gtrsim 5\,\mathrm{MAX}[R_P, R_S]$. For smaller impact parameters one needs to account for the detailed mass distribution of P (see MBW §12.1 for details).

Note that $\Delta E_{\rm S} = b^{-4}$ → closer encounters have a much greater impact, but recall that results are only valid for a distant tide…

For a non-perturbative treatment also valid for small b, see Banik & vdBosch 2019

If $\Delta E_{\rm S}$ is sufficiently large, the system may be tidally disrupted.

Tidal disruption at work: comet Schoemaker-Levy

Impulsive Heating

In the impulse approximation, the encounter only changes the kinetic energy of S, but leaves its potential energy intact.

after the encounter, S is no longer in virial equilibrium

Consequently, after encounter S undergoes relaxation to re-establish virial equilibrium

Let $K_{\rm S}$ be the original (pre-encounter) kinetic energy of S:

Virial Equilibrium: $E_{\rm S} = -K_{\rm S}$

After encounter: $E_{\rm S} \rightarrow E_{\rm S} + \Delta E_{\rm S}$

Since all new energy is kinetic: $K_{\rm S} \rightarrow K_{\rm S} + \Delta E_{\rm S}$

After relaxation: $K_{\rm S} = -(E_{\rm S} + \Delta E_{\rm S}) = -E_{\rm S} - \Delta E_{\rm S}$

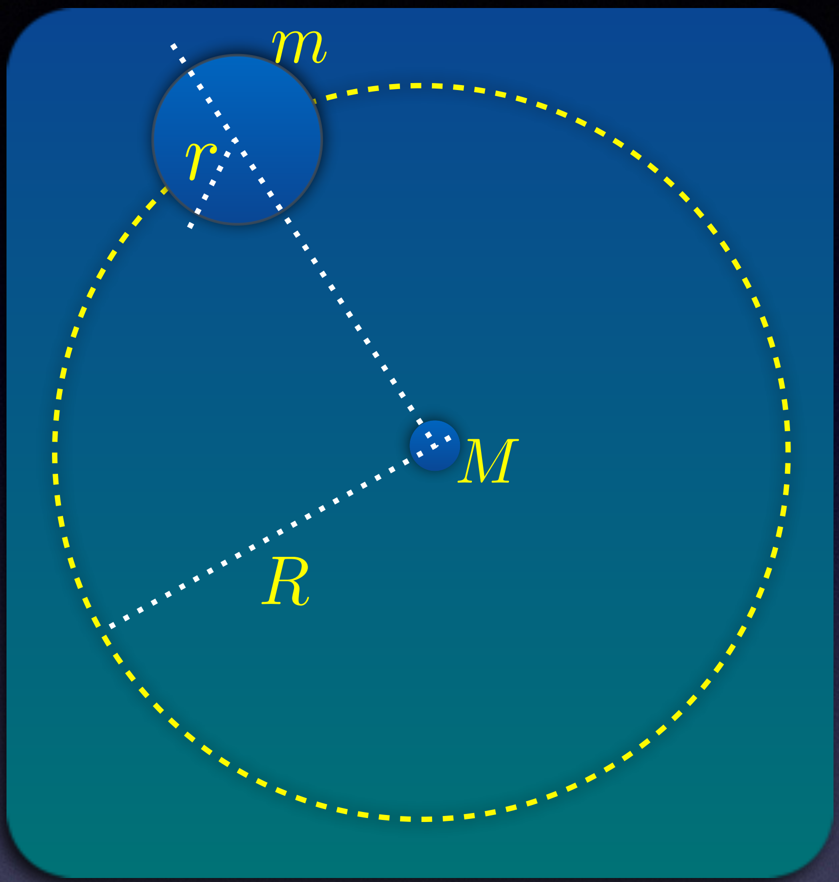
Relaxation decreases the kinetic energy by $2\Delta E_{\rm S}$

Manifestation of negative heat capacity: add heat to system, and it gets colder

This energy is transferred to potential energy, which becomes less negative. Hence, tidal shocks ultimately cause the system to expand (make it less bound).

Tidal Stripping

Even in the non-impulsive case, tidal forces can strip matter (=tidal stripping).



Consider a mass m, with radius r, orbiting a point mass M on a circular orbit of radius R.

The mass m experiences a gravitation acceleration due to M equal to $\vec{g} = \frac{G\,M}{R^2}$

The gravitational acceleration at the point in m closest to M is equal to $\vec{g} = \frac{G\,M}{(R-r)^2}$

Hence, the tidal acceleration at the edge of m is:

$$\vec{g}_{\rm tid}(r) = \frac{GM}{R^2} - \frac{GM}{(R-r)^2} \simeq \frac{2GMr}{R^3} \qquad (r \ll R)$$

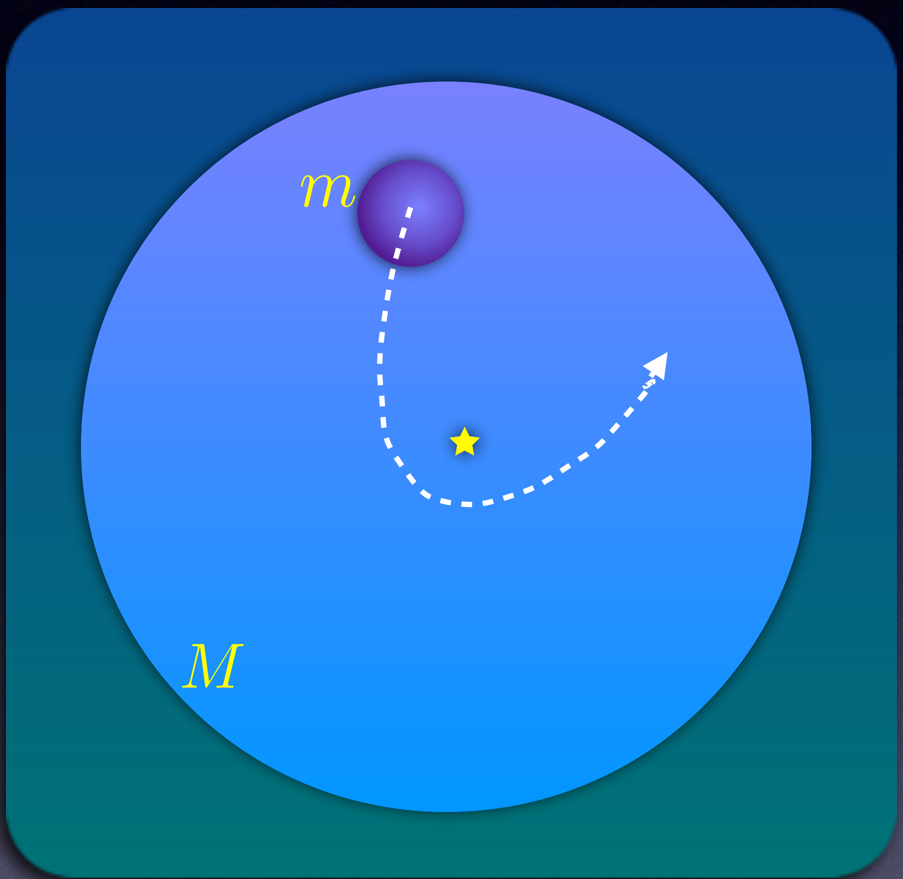
If this tidal acceleration exceeds the binding force per unit mass, $\frac{Gm}{r^2}$, the material at distance r from the center of m will be stripped. This defines the

tidal radius $$r_{\rm t} = \left(\frac{m}{2M}\right)^{1/3} R$$

Tidal Stripping

Taking account of centrifugal force associated with the circular motion results in a somewhat modified (more accurate) tidal radius:

$$r_{\rm t} = \left(\frac{m/M}{3 + m/M}\right)^{1/3} R$$



More realistic case:
object m is on eccentric orbit within an extended mass M (i.e., a satellite galaxy orbiting inside a massive host halo).

The tidal radius in this case is conveniently defined as distance from center of m at which a point on line connecting centers of m and M experiences zero acceleration when m is located at the pericentric distance R_0. This yields

$$r_{\rm t} \simeq \left[\frac{m(r_{\rm t})/M(R_0)}{2 + \frac{\Omega^2 R_0^3}{G\,M(R_0)} - \left.\frac{{\rm d}\ln M}{{\rm d}\ln R}\right|_{R_0}}\right]^{1/3} R_0$$

Here Ω is the circular speed at $R=R_0$

CAUTION: the concept tidal radius is poorly defined in this case, and the expression to the right therefore has to be taken with a grain of salt. At pericentric passage, it may be more appropriate to resort to impulse approximation...

Dynamical Friction

When an object of mass M_S ("subject mass") moves through a large collisionless system whose contituent particles ("field particles") have mass $m << M_S$, it experiences a drag force, called dynamical friction.

Dynamical friction transfers the orbital energy of satellite galaxy (and dark matter subhaloes) to the dark matter particles that make up the host halo, causing the satellite (subhalo) to "sink" to the center of the potential well, where it can ultimately merge with the central galaxy (cannibalism)

Intuitive Picture 1: Equipartition

two-body encounters move systems towards equipartition

$$\boxed{m_1 \langle v_1^2 \rangle = m_2 \langle v_2^2 \rangle = m_3 \langle v_3^2 \rangle = \text{etc.}}$$

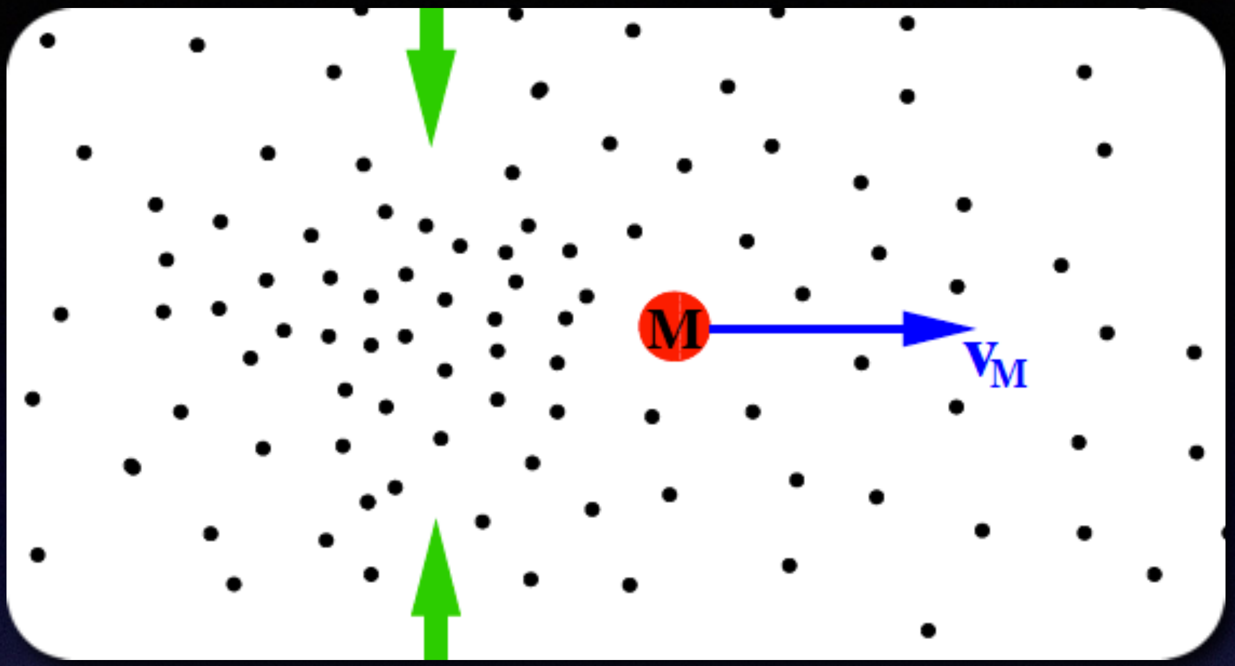
since initially $v_S \sim \langle v_{\rm field}^2 \rangle^{1/2}$ and $M_S \gg m$ the subject mass will (on average) loose energy to the field particles. Hence, the subject mass will slow down...

dynamical friction is a manifestation of mass segregation

Dynamical Friction

Intuitive Picture 2: Gravitational Wake

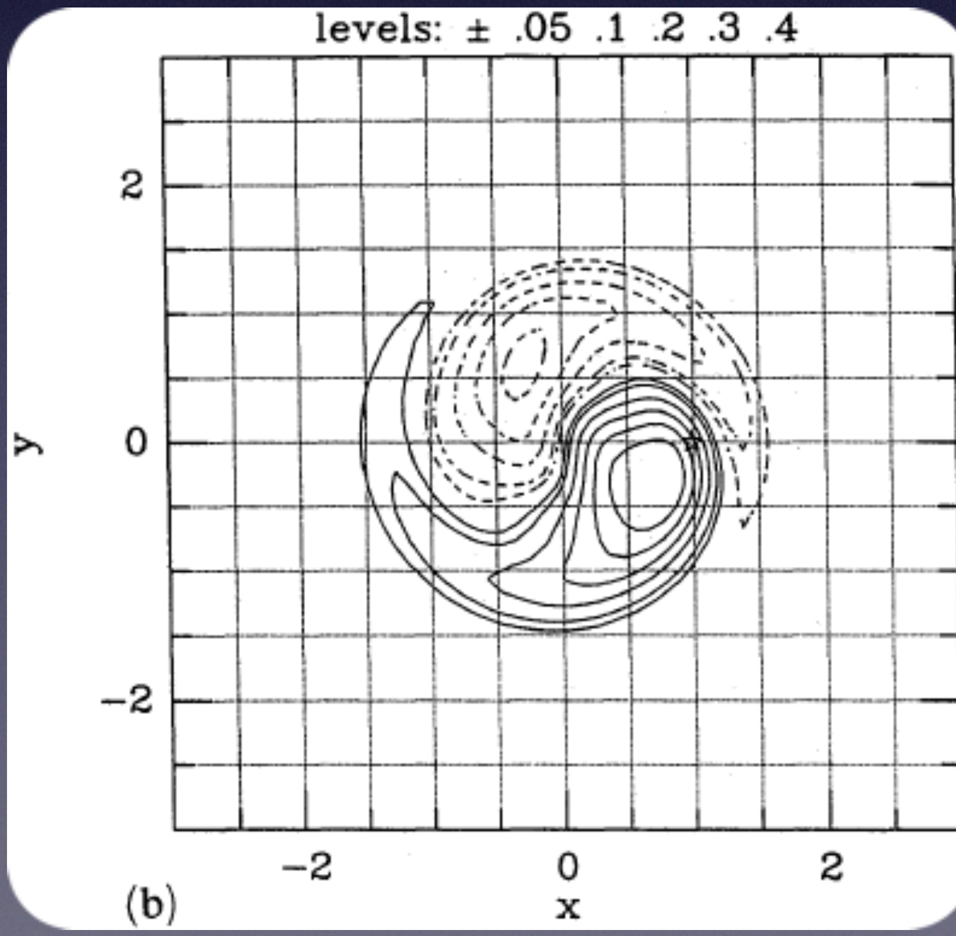
The moving subject mass perturbs distribution of field particles creating a trailing density enhancement ("wake") . The gravitational force of this wake on M_S slows it down.



Although a very "popular" view of dynamical friction, the assumption that dynamical friction is due to the back reaction arising from a local wake is wrong...So be careful!!

Intuitive Picture 3: Linear Response Theory

The moving subject mass perturbs the gravitational potential; this introduces a response density (perturbation), whose back reaction on the subject mass causes it to slow down.



Source: Weinberg, 1989, MNRAS, 239, 549

Although similar to the wake-picture above, the important different is that the response density is a global, rather than a local, distortion, Also, in linear response theory the self-gravity of the field particles is properly accounted for.

Dynamical Friction

Chandrasekhar (1943) derived an analytical expression for the dynamical friction force:

$$\vec{F}_{\rm df} = M_{\rm S} \frac{{\rm d}\vec{v}_{\rm S}}{{\rm d}t} = -4\pi \left(\frac{G M_{\rm S}}{v_{\rm S}}\right)^2 \ln\Lambda \, \rho(< v_{\rm S}) \, \frac{\vec{v}_{\rm S}}{v_{\rm S}}$$

[see MBW §12.3 for derivation]

Here $\rho(< v_{\rm S})$ is the (local) density of field particles with speeds less than $v_{\rm S}$, and $\ln\Lambda$ is called the Coulomb logarithm, which can be approximated as

$$\ln\Lambda \approx \ln\left(\frac{b_{\rm max}}{b_{90}}\right)$$

$b_{\rm max} \sim R$ maximum impact parameter ~ size of system in which subject mass is orbiting

$b_{90} \sim \frac{G M_{\rm S}}{\langle v_{\rm m}^2 \rangle^{1/2}}$ impact parameter for which field particle is deflected by 90 degrees...

$\vec{F}_{\rm df} \propto M_{\rm S}^2$ and is independent of the mass of the field particles.

$\vec{F}_{\rm df}$ points in direction opposite to motion (similar to the case of frictional drag in fluid mechanics). However, whereas hydrodynamical friction always increases with $v_{\rm S}$, this is NOT the case with dynamical friction for which

$$F_{\rm df} \propto v_{\rm S} \qquad (v_{\rm S} \text{ small})$$
$$F_{\rm df} \propto v_{\rm S}^{-2} \qquad (v_{\rm S} \text{ large})$$

Dynamical Friction

Chandrasekhar's expression for the dynamical friction force is based on the following three assumptions:

- subject mass and field particles are point masses
- self-gravity of field particles can be ignored
- distribution of field particles is infinite, homogeneous & isotropic

The latter of these is the reason why the Coulomb logarithm has to be introduced; the maximum impact parameter is needed to prevent divergence....

Chandrasekhar dynamical friction is considered as the sum of uncorrelated two-body interactions between a field particle and the subject mass. However, this ignores collective effects due to self-gravity of the field particles.

Chandrasekhar dynamical friction is considered a purely local effect, which is evident from the fact that $\vec{F}_{\rm df} \propto \rho(< v_{\rm S})$. However, dynamical friction is a global phenomenon, which is evident from fact that subject mass experiences dynamical friction even if it orbits beyond the outer edge of a finite host system. Hence, a proper treatment of dynamical friction requires linear response theory...

Orbital Decay

Consider the subject mass on a circular orbit in spherical, singular isothermal host halo with density distribution

$$\rho(r) = \frac{V_{\rm c}^2}{4\pi G r^2}$$

Note: the circular velocity $V_{\rm c}$ is independent of radius...

Under the assumption that the velocity distribution of field particles is a Maxwell-Boltzmann distribution with velocity dispersion $\sigma = V_{\rm c}/\sqrt{2}$, the DF force reduces to:

$$\boxed{F_{\rm df} = -0.428 \, \frac{G M_{\rm S}^2}{r^2} \, \ln\Lambda \, \frac{\vec{v}_{\rm S}}{v_{\rm S}}}$$

being on a circular orbit, the rate at which the subject mass loses its orbital angular momentum $L_{\rm S} = r\, v_{\rm S}$ is

$$\boxed{\frac{{\rm d}L_{\rm S}}{{\rm d}t} = r\,\frac{{\rm d}v_{\rm S}}{{\rm d}t} = r\,\frac{F_{\rm df}}{M_{\rm S}} = -0.428 \, \frac{G M_{\rm S}}{r} \, \ln\Lambda}$$

Orbital Decay

Since the circular speed is independent of radius, the subject mass continues to orbit with a speed $v_{\rm S}$ as it spirals inwards, so that the orbit radius changes as

$$v_{\rm S}\frac{{\rm d}r}{{\rm d}t} = -0.428\frac{GM_{\rm S}}{r}\ln\Lambda \quad \xrightarrow{v_{\rm S}=V_{\rm c}} \quad r\frac{{\rm d}r}{{\rm d}t} = -0.428\frac{GM_{\rm S}}{V_{\rm c}}\ln\Lambda$$

This allows us to compute how long it takes for the orbit to decay from some initial radius $r_{\rm i}$ to $r=0$. This time is called the dynamical friction time

$$t_{\rm df} = \frac{1.17}{\ln\Lambda}\frac{r_{\rm i}^2\,V_{\rm c}}{G\,M_{\rm S}} \qquad V_{\rm c} = \sqrt{\frac{G\,M_{\rm h}}{r_{\rm h}}} \quad \Longrightarrow \quad t_{\rm df} = \frac{1.17}{\ln\Lambda}\left(\frac{r_{\rm i}}{r_{\rm h}}\right)^2\left(\frac{M_{\rm h}}{M_{\rm S}}\right)\frac{r_{\rm h}}{V_{\rm c}}$$

Finally, using that $r_{\rm h}/V_{\rm c} \sim 1/[10H(z)] = 0.1\,t_{\rm H}$ and that $\ln\Lambda \sim \ln(M_{\rm h}/M_{\rm S})$, yields

$$t_{\rm df} \simeq 0.117\frac{(M_{\rm h}/M_{\rm S})}{\ln(M_{\rm h}/M_{\rm S})}\,t_{\rm H}$$

Only systems, with $M_{\rm S}/M_{\rm h} > 0.03$ experience significant mass segregation

Orbital Decay

$$t_{\rm df} \simeq 0.117 \frac{(M_{\rm h}/M_{\rm S})}{\ln(M_{\rm h}/M_{\rm S})} t_{\rm H}$$

CAUTION: this estimate is based on a number of questionable assumptions...In general

- haloes are not singular, isothermal spheres
- orbits are not circular
- tidal stripping implies mass loss; $M_{\rm S} = M_{\rm S}(t)$

When orbits are eccentric, dynamical friction may cause the obit's eccentricity to evolve as function of time. In fact, as shown by van den Bosch et al. (1999)

$$\frac{{\rm d}e}{{\rm d}t} = \frac{\eta}{v} \frac{{\rm d}e}{{\rm d}\eta} \left[1 - \left(\frac{v}{V_{\rm c}}\right)^2\right] \frac{{\rm d}v}{{\rm d}t}$$

eccentricity $e = \frac{r_+ - r_-}{r_+ + r_-}$

r_+ = apocenter

r_- = pericenter

circularity $\eta = L/L_{\rm c}(E)$

L = orbital angular momentum

$L_C(E)$ = orbital angular momentum of circular orbit with same energy

circular orbit: e=0 and η=1

radial orbit: e=1 and η=0

Orbital Decay

eccentricity $e = \frac{r_+ - r_-}{r_+ + r_-}$

r_+ = apocenter

r_- = pericenter

$$\frac{\mathrm{d}e}{\mathrm{d}t} = \frac{\eta}{v}\frac{\mathrm{d}e}{\mathrm{d}\eta}\left[1 - \left(\frac{v}{V_\mathrm{c}}\right)^2\right]\frac{\mathrm{d}v}{\mathrm{d}t}$$

circularity $\eta = L/L_\mathrm{c}(E)$

L = orbital angular momentum

$L_C(E)$ = orbital angular momentum of circular orbit with same energy

circular orbit: $e=0$ and $\eta=1$

radial orbit: $e=1$ and $\eta=0$

Since $de/d\eta < 0$, and since dynamical friction causes $dv/dt < 0$, we have that

$de/dt < 0$ for $v > V_C$ → orbit circularizes near pericenter

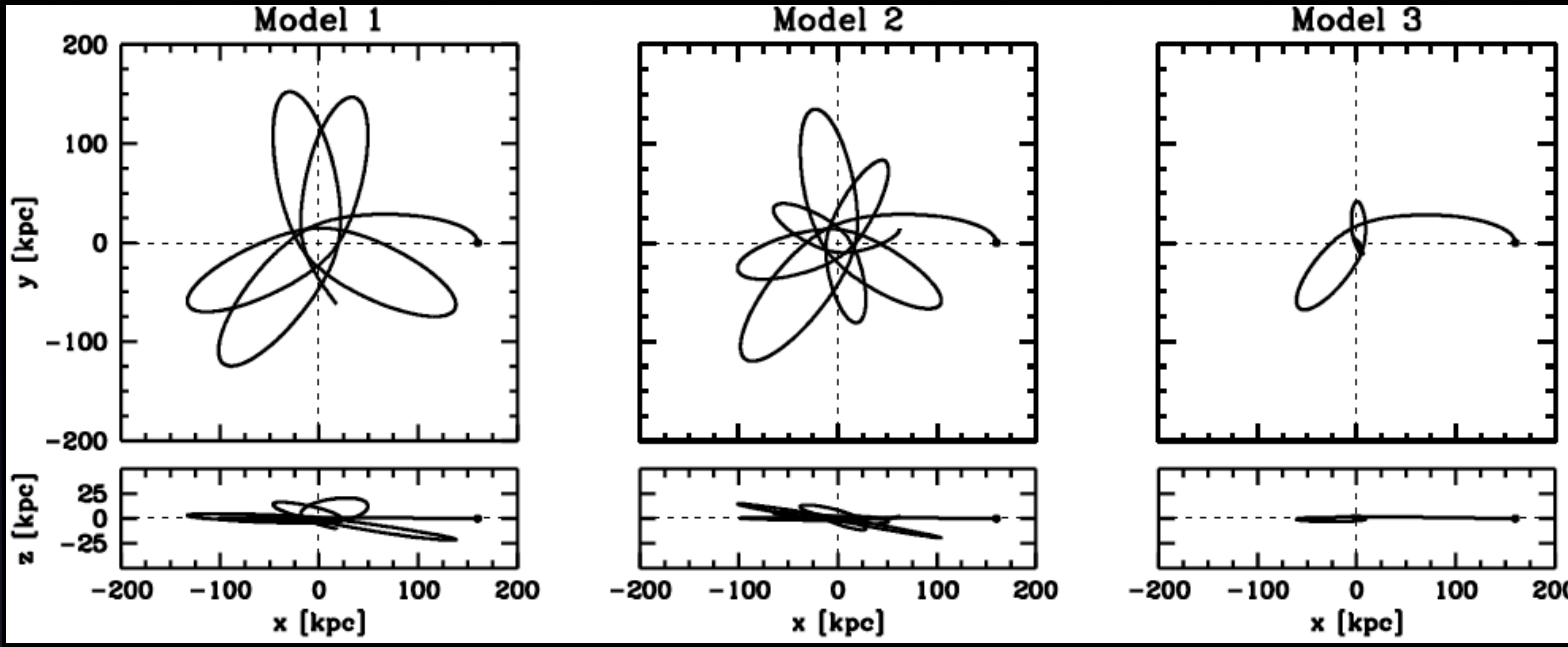
$de/dt > 0$ for $v < V_C$ → orbit gains eccentricity near apocenter

Numerical simulations of DF in realistic potentials (but ignoring mass loss) find that the effects largely cancel so that $de/dt \sim 0$ when integrated over an entire orbit. Contrary to urban myth, dynamical friction does not lead to orbit circularization.

The same simulations also show that $t_\mathrm{df} \propto \eta^{0.53}$.

→ **more eccentric orbits decay faster** !

Orbital Decay



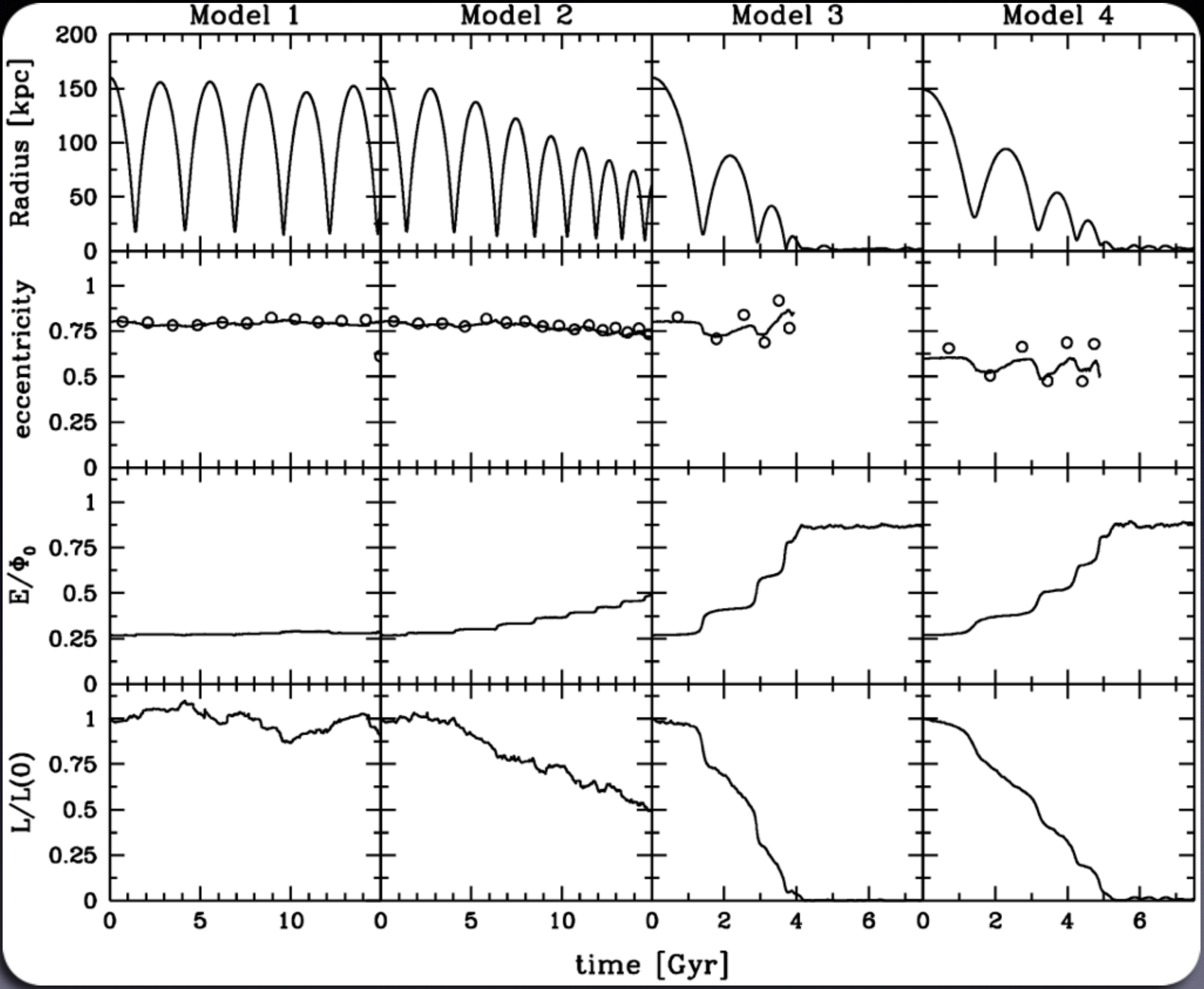
Source: van den Bosch et al., 1999, ApJ, 515, 50

Orbits of solid bodies experiencing dynamical friction in high resolution N body simulations.

Model 1: $M_S/M_h = 2 \times 10^{-4}$
Model 2: $M_S/M_h = 2 \times 10^{-3}$
Model 3: $M_S/M_h = 2 \times 10^{-2}$

- All three orbits have initial eccentricity e=0.8
- Orbits of more massive subjects decay faster
- No (obvious) orbit circularization

Orbital Decay



Source: van den Bosch et al., 1999, ApJ, 515, 50

Dynamical Friction: impact of mass loss

When subject masses are not solid bodies, but N-body systems themselves, they can experience mass loss due to tidal stripping and tidal heating/shocking.

Rough, analytical estimate indicates that mass loss causes average dynamical friction time to increase by factor ~2.8 wrt estimate that does not account for mass loss.

[see MBW §12.3.1]

This is in good agreement with results from numerical simulations...

[e.g., Colpi et al. 1999; Boylan-Kolchin et al. 2008; Jiang et al. 2008]

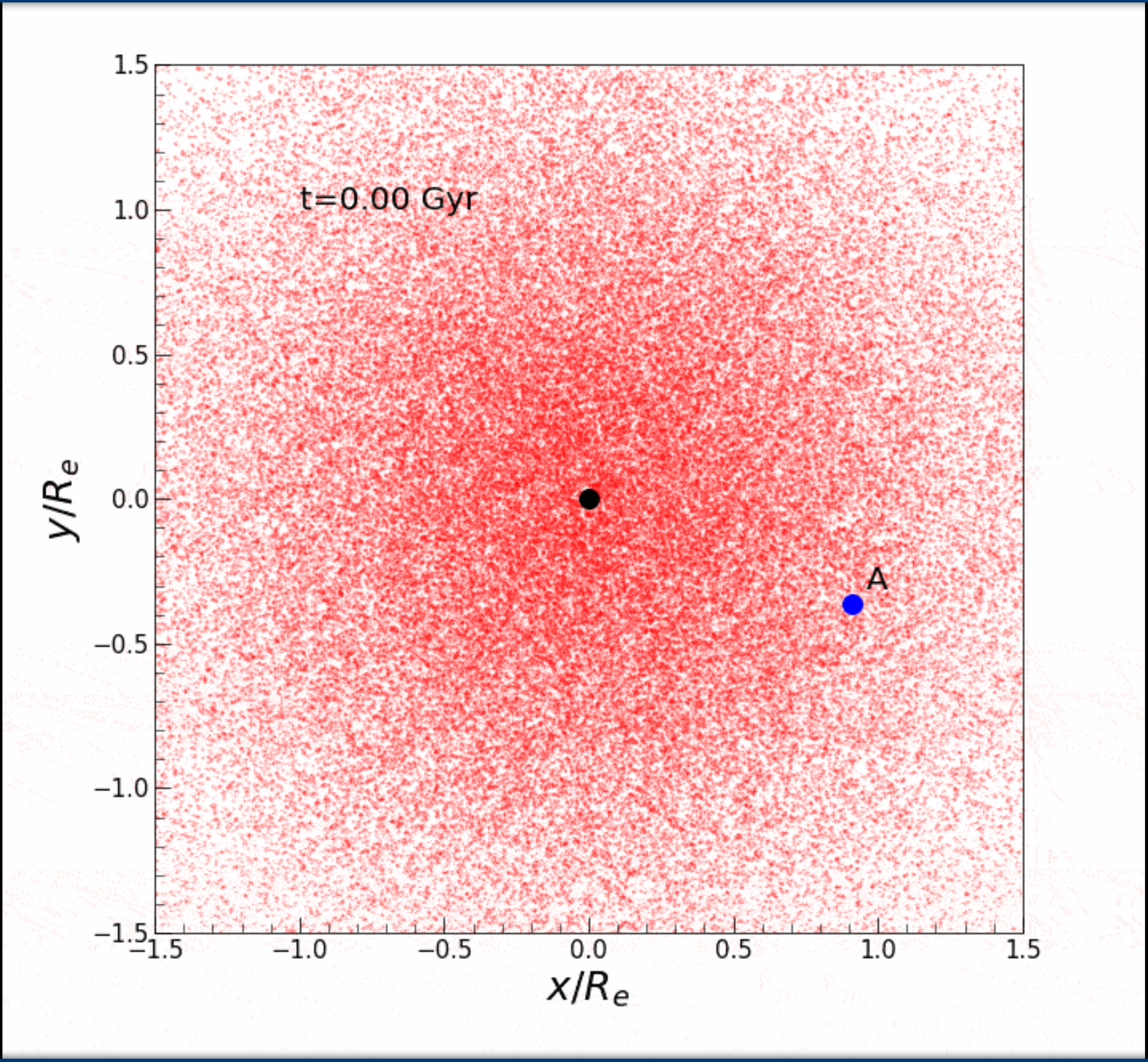
These simulations show that in the presence of mass loss the dependence of the dynamical friction time on orbital circularity becomes

$$\boxed{t_{\rm df} \propto \eta^{0.3-0.4}}$$

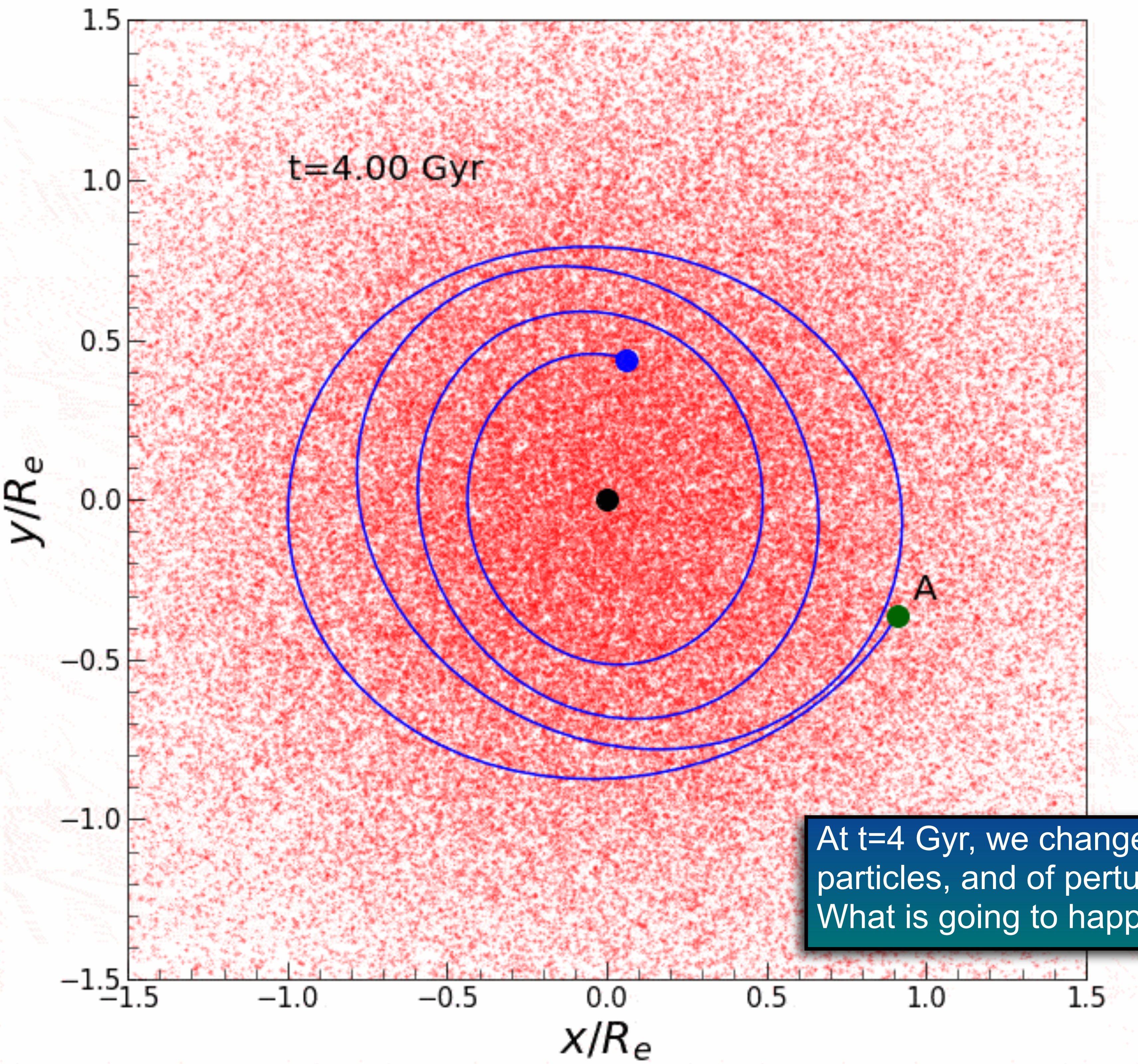
slightly weaker than in the absence of mass loss...

Accurate modeling of tidal stripping, tidal heating, and dynamical friction is important for predicting disruption & merger rates of satellite galaxies.

Fun with Dynamical Friction



Fun with Dynamical Friction



At t=4 Gyr, we change velocity vectors of all particles, and of perturber, by 180 degrees. What is going to happen next???

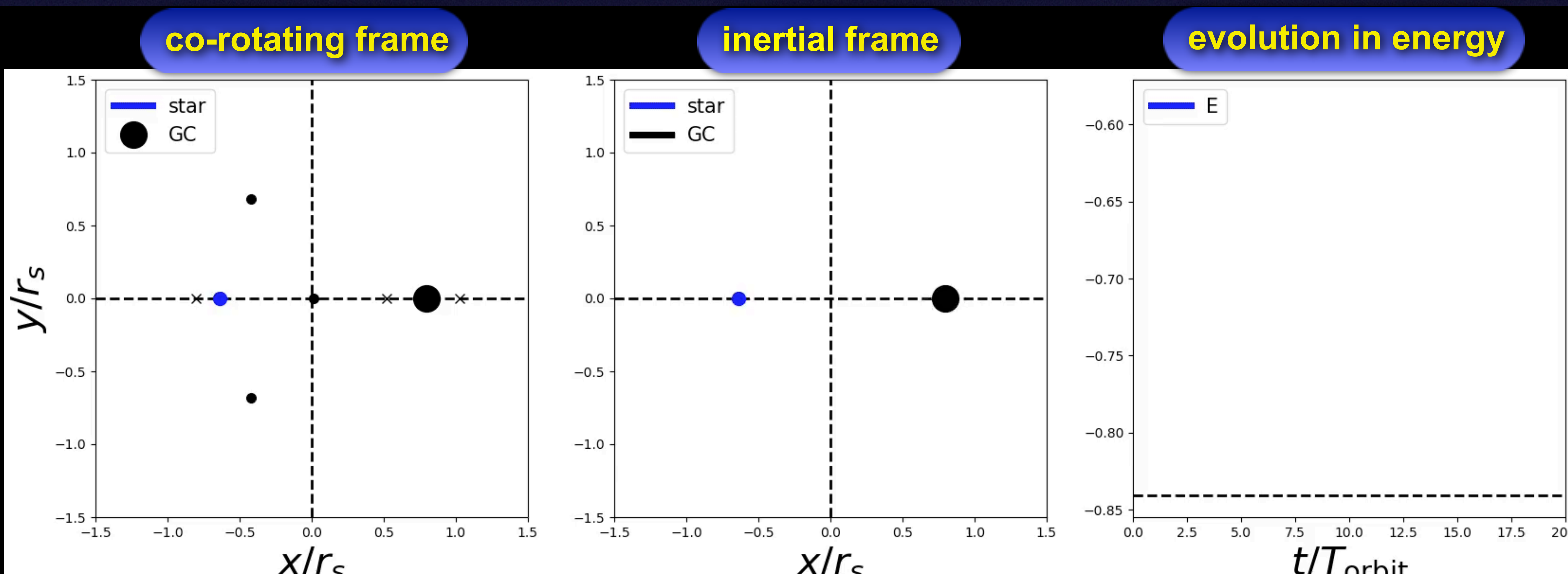
The Inner Workings of Dynamical Friction

In reality, DF is far more complicated than envisioned by Chandrasekhar's sum of uncorrelated two-body interactions in a homogeneous, isotropic background.

Movie shows how one star exchanges energy with a perturber on a circular orbit in a spherically symmetric (inhomogeneous) galaxy….

Note how star first gains energy (=perturber loses energy=dynamical friction), but then loses it again (=perturber gains energy=dynamical heating)…net result is zero

Question: so how, then, does dynamical friction arise???



Source: Banik & van den Bosch, 2020

Lecture 14

SUMMARY

Summary: key words & important facts

Key words

Impulse & tidal approximations	dynamical friction
distant encounter approximation	gravitational capture
tidal shock heating	orbital decay
tidal mass stripping	negative heat capacity

- Gravitational encounter results in tidal distortion. If tidal distortion lags perturber, the resulting torque causes a transfer of orbital energy into internal energy of the objects involved.
- An impulsive encounter that results in an (internal) energy increase ΔE that is larger than the system's binding energy does not necessarily result in the system's disruption
- During re-virialization, following an impulsive encounter, the subject converts $2\mathrm{x}\Delta E$ from kinetic into potential energy, resulting in the system `puffing' up.
- Dynamical friction is a global, rather than a local effect. Unlike hydrodynamical friction, the deceleration decreases with increasing velocity, at least at the high-velocity end.
- Dynamical friction is only important for subjects with a mass larger than a few percent of the host halo mass. For less massive subjects, $t_{df} > t_H$
- Dynamical friction does not generally result in orbital circularization. More eccentric orbits decay faster.

Summary: key equations & expressions

Impulse Approximation

$$\Delta E_{\rm S} = \frac{1}{2}\int |\Delta\vec{v}(\vec{r})|^2\,\rho(r)\,{\rm d}^3\vec{r} = \frac{4}{3}G^2\,M_{\rm S}\left(\frac{M_{\rm P}}{v_{\rm P}}\right)^2\frac{\langle r^2\rangle}{b^4}$$

Tidal Radius

Point masses

$$r_{\rm t} = \left(\frac{m}{2M}\right)^{1/3} R$$

\+ centrifugal force

$$r_{\rm t} = \left(\frac{m/M}{3+m/M}\right)^{1/3} R$$

\+ extended mass distributions

$$r_{\rm t} \simeq \left[\frac{m(r_{\rm t})/M(R_0)}{2+\frac{\Omega^2 R_0^3}{G\,M(R_0)} - \frac{{\rm d}\ln M}{{\rm d}\ln R}\big|_{R_0}}\right]^{1/3} R_0$$

Impulse Approximation

Chandrasekhar dynamical friction force

$$\vec{F}_{\rm df} = M_{\rm S}\frac{{\rm d}\vec{v}_{\rm S}}{{\rm d}t} = -4\pi\left(\frac{GM_{\rm S}}{v_{\rm S}}\right)^2 \ln\Lambda\,\rho(<v_{\rm S})\,\frac{\vec{v}_{\rm S}}{v_{\rm S}}$$

Coulomb logarithm

$$\ln\Lambda = \ln\left(\frac{b_{\rm max}}{b_{90}}\right) \approx \ln\left(\frac{M_{\rm h}}{M_{\rm s}}\right)$$

dynamical friction time scale (isothermal sphere)

$$t_{\rm df} = \frac{1.17}{\ln\Lambda}\left(\frac{r_{\rm i}}{r_{\rm h}}\right)^2\left(\frac{M_{\rm h}}{M_{\rm S}}\right)\frac{r_{\rm h}}{V_{\rm c}}$$

evolution of orbital eccentricity

$$\frac{{\rm d}e}{{\rm d}t} = \frac{\eta}{v}\frac{{\rm d}e}{{\rm d}\eta}\left[1-\left(\frac{v}{V_{\rm c}}\right)^2\right]\frac{{\rm d}v}{{\rm d}t}$$