ASTR 610
Theory of Galaxy Formation

Lecture 12: Large Scale Structure

Frank van den Bosch
Yale University, Spring 2024

Large Scale Structure

In this lecture we discuss how to characterize the large scale distribution of matter and galaxies using n-point correlation functions both in the continuous and the discrete limit. We also discuss galaxy samples and redshift surveys, and what they teach us about the large scale distribution of galaxies.

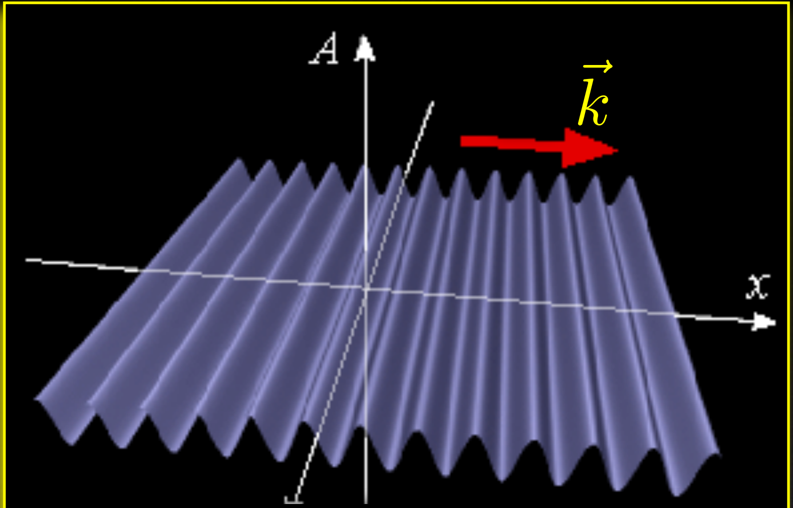
Topics that will be covered include:

- Ergodic Principle
- Gaussian Random Fields
- n-point correlation functions
- Poisson sampling
- Galaxy surveys
- The Limber equations
- Redshift Space Distortions

Notation & Convention: Fourier modes

$$\delta_{\vec{k}} = \frac{1}{V}\int \delta(\vec{x})\, e^{-i\vec{k}\cdot\vec{x}}\, \mathrm{d}^3\vec{x} \qquad [\delta_{\vec{k}}] = \text{unitless}$$

$$\delta(\vec{x}) = \sum \delta_{\vec{k}}\, e^{+i\vec{k}\cdot\vec{x}} \qquad [\delta(\vec{x})] = \text{unitless}$$



2D analog of plane-wave mode

Interpretation:
the cosmological density field is the sum over a discrete number of modes $\delta_{\vec{k}} = A_{\vec{k}} + i\, B_{\vec{k}} = |\delta_{\vec{k}}|\, e^{i\,\phi_{\vec{k}}}$, where $\vec{k} = \frac{2\pi}{L}(i_x, i_y, i_z)$ $\quad i = (1, 2, ...)$

amplitude $\quad$ phase

Note: since $\delta(\vec{x})$ is real, we have that the complex conjugate $\delta^*_{\vec{k}} = \delta_{-\vec{k}}$ and thus $A_{\vec{k}} = A_{-\vec{k}}$ and $B_{\vec{k}} = -B_{-\vec{k}}$. This implies that one only needs Fourier modes in the upper-half space to fully specify $\delta(\vec{x})$

Remember:

Dirac Delta function: $$\delta^{\rm D}(\vec{k} - \vec{k}') = \frac{1}{(2\pi)^3}\int e^{\pm i(\vec{k}-\vec{k}')\cdot\vec{x}}\, \mathrm{d}^3\vec{x}$$

Kronecker Delta function: $$\delta^{\rm D}_{\vec{k}\vec{k}'} = \frac{1}{V}\int e^{\pm i(\vec{k}-\vec{k}')\cdot\vec{x}}\, \mathrm{d}^3\vec{x}$$

The Cosmological Density Field

How can we describe the cosmological (over)density field, $\delta(\vec{x},t)$, without having to specify the actual value of δ at each location in space-time, $(\vec{x},t)$?

Since $\delta(\vec{x})$ is believed to be the outcome of some random process in the early Universe (i.e., quantum fluctuations in inflaton), our goal is to describe the probability distribution

$$\mathcal{P}(\delta_1, \delta_2, ..., \delta_N)\, \mathrm{d}\delta_1\, \mathrm{d}\delta_2 \, ... \, \mathrm{d}\delta_N$$

where $\delta_1 = \delta(\vec{x}_1)$, etc. For now we will focus on the cosmological density field at some particular (random) time. It's time evolution has been addressed in Lectures 4 - 8.

This probability distribution is completely specified by the moments

$$\boxed{\langle \delta_1^{l_1} \delta_2^{l_2} ... \delta_N^{l_N} \rangle = \int \delta_1^{l_1} \delta_2^{l_2} ... \delta_N^{l_N} \mathcal{P}(\delta_1, \delta_2, ..., \delta_N)\, \mathrm{d}\delta_1\, \mathrm{d}\delta_2 \, ... \, \mathrm{d}\delta_N}$$

Cosmological Principle: Universe is homogenous & isotropic.

- ⇒ all positions/directions are equivalent
- ⇒ all moments are invariant under spatial translation & rotation

The Ergodic Hypothesis

NOTE: $\langle ... \rangle$ denotes an ensemble average. For instance, $\langle \delta(\vec{x}) \rangle$ means the average overdensity at $\vec{x}$ for many realizations of the random process...

PROBLEM: Theory specifies ensemble average, but observationally we have only access to one realization of the random process....

Ergodic Hypothesis: Ensemble average is equal to spatial average taken over one realization of the random field...

$$\boxed{\langle \delta \rangle = \int \delta \, \mathcal{P}(\delta) \, \mathrm{d}\delta = \frac{1}{V} \int_V \delta(\vec{x}) \, \mathrm{d}^3\vec{x}}$$

Essentially, the ergodic hypothesis requires spatial correlations to decay sufficiently rapidly with increasing separation so that there exists many statistically independent volumes in one realization....

The ergodic hypothesis is proven for Gaussian random fields, which are our main focus in what follows....

Gaussian Random Fields

A random field $\delta(\vec{x})$ is said to be Gaussian if the distribution of the field values at an arbitrary set of N points is an N-variate Gaussian:

$$\mathcal{P}(\delta_1, \delta_2, ..., \delta_N) = \frac{\exp(-Q)}{[(2\pi)^N \det(\mathcal{C})]^{1/2}}$$

$$Q \equiv \frac{1}{2} \sum_{i,j} \delta_i \, (\mathcal{C}^{-1})_{ij} \delta_j$$

$$\mathcal{C}_{ij} = \langle \delta_i \delta_j \rangle \equiv \xi(r_{12})$$

where we have defined the two-point correlation function $\xi(\vec{r}) = \langle \delta(\vec{x}) \, \delta(\vec{x} + \vec{r}) \rangle$

NOTE: because of invariance to spatial translation & rotation, we have that $\langle \delta_i \delta_j \rangle = \xi(r_{12})$, where $r_{12} = |\vec{x}_i - \vec{x}_j|$.

In particular, the one-point distribution function of the field is

$$\mathcal{P}(\delta) \, \mathrm{d}\delta = \frac{1}{\sqrt{2\pi} \, \sigma} \exp\left(-\frac{\delta^2}{2\sigma^2}\right) \mathrm{d}\delta$$

where $\sigma^2 = \langle \delta^2 \rangle = \xi(0)$ is the variance of the density perturbation field.

As you can see, for Gaussian random field the N-point probability function $\mathcal{P}(\delta_1, \delta_2, ..., \delta_N)$ is completely specified by the two-point correlation function.

Gaussian Random Fields

Rather than specifying $\mathcal{P}(\delta_1, \delta_2, ..., \delta_N)\,\mathrm{d}\delta_1\,\mathrm{d}\delta_2 \ldots \mathrm{d}\delta_N$, it is equivalent to specify

$$\boxed{\mathcal{P}(\delta_{\vec{k}_1}, \delta_{\vec{k}_2}, ..., \delta_{\vec{k}_N})\,\mathrm{d}|\delta_{\vec{k}_1}|\,\mathrm{d}|\delta_{\vec{k}_2}| \ldots \mathrm{d}|\delta_{\vec{k}_N}|\,\mathrm{d}\phi_1\,\mathrm{d}\phi_2 \ldots \mathrm{d}\phi_N}$$

which gives the probability that the modes $\delta_{\vec{k}_i}$ have amplitudes in the range $|\delta_{\vec{k}_i}| \pm \mathrm{d}|\delta_{\vec{k}_i}|/2$ and phases in the range $\phi_i \pm \mathrm{d}\phi_i/2$

For a Gaussian random field,

$$\boxed{\mathcal{P}(\delta_{\vec{k}_1}, \delta_{\vec{k}_2}, ..., \delta_{\vec{k}_N})\,\mathrm{d}|\delta_{\vec{k}_1}|\,\mathrm{d}|\delta_{\vec{k}_2}| \ldots \mathrm{d}|\delta_{\vec{k}_N}|\,\mathrm{d}\phi_1\,\mathrm{d}\phi_2 \ldots \mathrm{d}\phi_N = \prod_i \mathcal{P}(\delta_{\vec{k}_i})\,\mathrm{d}|\delta_{\vec{k}_i}|\,\mathrm{d}\phi_i}$$

which makes it explicit that all modes are independent. Furthermore, for each mode $A_{\vec{k}}$ and $B_{\vec{k}}$ are independent, which implies that $\phi_{\vec{k}}$ is distributed uniformly over $[0, 2\pi]$.

Hence

$$\boxed{\mathcal{P}(\delta_{\vec{k}})\,\mathrm{d}|\delta_{\vec{k}}|\,\mathrm{d}\phi_{\vec{k}} = \exp\left[-\frac{|\delta_{\vec{k}}|^2}{2\langle|\delta_{\vec{k}}|^2\rangle}\right] \frac{|\delta_{\vec{k}}|\,\mathrm{d}|\delta_{\vec{k}}|}{\langle|\delta_{\vec{k}}|^2\rangle}\,\frac{\mathrm{d}\phi_{\vec{k}}}{2\pi}}$$

which makes is explicit that the Gaussian random field is completely specified by the power spectrum $P(k) = V\,\langle|\delta_{\vec{k}}|^2\rangle$...

Higher-Order Correlation Functions

The n-point correlation function is defined as $\boxed{\xi^{(n)} \equiv \langle \delta_1 \, \delta_2 \ldots \delta_n \rangle}$

The reduced (or irreducible) n-point correlation function is defined as

$$\boxed{\xi^{(n)}_{\rm red} \equiv \langle \delta_1 \, \delta_2 \ldots \delta_n \rangle_{\rm c}}$$

where $\langle \ldots \rangle_{\rm c}$ is the cumulant or connected moment.

The cumulants κ_n of a probability distribution are a set of quantities that provide an alternative to the moments μ_n. They are related via the following recursion formula:

$$\kappa_n = \mu_n - \sum_{m=1}^{n-1} \binom{n-1}{m-1} \kappa_m \, \mu_{n-m}$$

In the case of $\mathcal{P}(\delta_1, \delta_2, \ldots, \delta_n)$ the moments are central (i.e., $\mu_1 = \langle \delta \rangle = 0$) so that

$$\langle \delta_1 \rangle = \langle \delta_1 \rangle_{\rm c} = 0$$

$$\langle \delta_1 \delta_2 \rangle = \langle \delta_1 \rangle_{\rm c} \, \langle \delta_2 \rangle_{\rm c} + \langle \delta_1 \delta_2 \rangle_{\rm c} = \langle \delta_1 \delta_2 \rangle_{\rm c}$$

$$\langle \delta_1 \delta_2 \delta_3 \rangle = \langle \delta_1 \rangle_{\rm c} \, \langle \delta_2 \rangle_{\rm c} \, \langle \delta_3 \rangle_{\rm c} + \langle \delta_1 \rangle_{\rm c} \, \langle \delta_2 \delta_3 \rangle_{\rm c} \; (3\,\text{terms}) + \langle \delta_1 \delta_2 \delta_3 \rangle_{\rm c} = \langle \delta_1 \delta_2 \delta_3 \rangle_{\rm c}$$

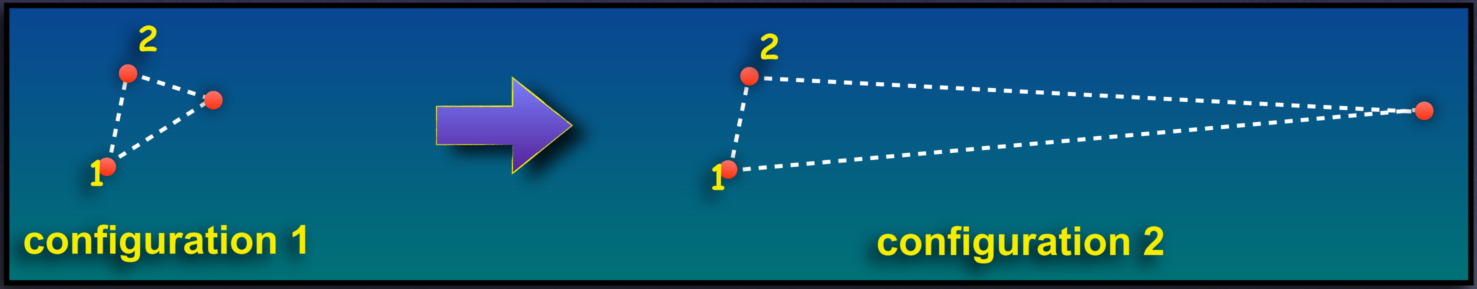
$$\begin{aligned} \langle \delta_1 \delta_2 \delta_3 \delta_4 \rangle = {} & \langle \delta_1 \rangle_{\rm c} \, \langle \delta_2 \rangle_{\rm c} \, \langle \delta_3 \rangle_{\rm c} \, \langle \delta_4 \rangle_{\rm c} + \langle \delta_1 \rangle_{\rm c} \, \langle \delta_2 \delta_3 \delta_4 \rangle_{\rm c} \; (4\,\text{terms}) + \langle \delta_1 \delta_2 \rangle_{\rm c} \, \langle \delta_3 \delta_4 \rangle_{\rm c} \; (3\,\text{terms}) \\ & + \langle \delta_1 \delta_2 \rangle_{\rm c} \, \langle \delta_3 \rangle_{\rm c} \, \langle \delta_4 \rangle_{\rm c} \; (6\,\text{terms}) + \langle \delta_1 \delta_2 \delta_3 \delta_4 \rangle_{\rm c} = \langle \delta_1 \delta_2 \rangle \, \langle \delta_3 \delta_4 \rangle \; (3\,\text{terms}) + \langle \delta_2 \delta_2 \delta_3 \delta_4 \rangle_{\rm c} \end{aligned}$$

Higher-Order Correlation Functions

Hence, we have that

$$\begin{aligned}
\langle \delta_1 \rangle_c &= \langle \delta_1 \rangle = 0 \\
\langle \delta_1 \delta_2 \rangle_c &= \langle \delta_1 \delta_2 \rangle = \xi(r_{12}) \\
\langle \delta_1 \delta_2 \delta_3 \rangle_c &= \langle \delta_1 \delta_2 \delta_3 \rangle = \xi^{(3)}_{123} \\
\langle \delta_1 \delta_2 \delta_3 \delta_4 \rangle_c &= \langle \delta_1 \delta_2 \delta_3 \delta_4 \rangle - \langle \delta_1 \delta_2 \rangle_c \, \langle \delta_3 \delta_4 \rangle_c \; (3\,\mathrm{terms}) \\
&= \xi^{(4)}_{1234} - \xi^{(2)}_{12}\,\xi^{(2)}_{34} - \xi^{(2)}_{13}\,\xi^{(2)}_{24} - \xi^{(2)}_{14}\,\xi^{(2)}_{23}
\end{aligned}$$

These reduced (or irreducible) correlation functions express the part of the n-point correlation functions that cannot be obtained from lower-order reduced correlation functions:



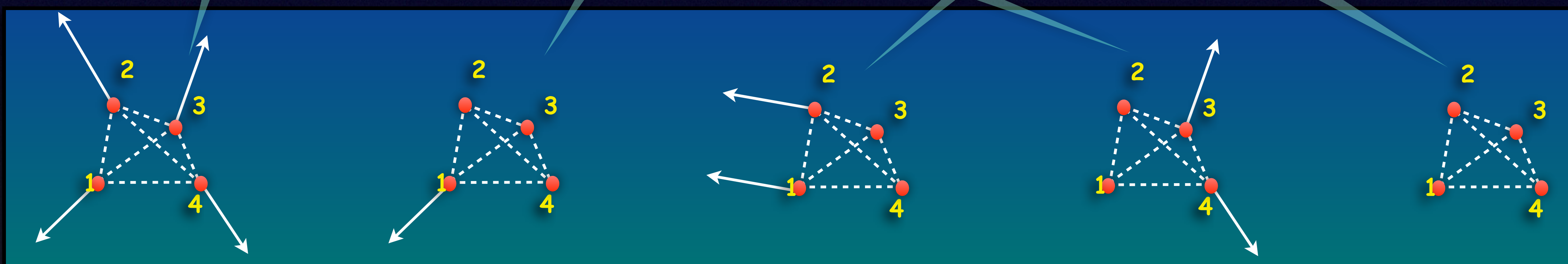
In the limit where r_{13} goes to infinity, the correlation between the three points in configuration 2 is entirely due to that between points 1 and 2. The reduced correlation function subtracts the correlations due to these configurations from the total correlation function. ...

Higher-Order Correlation Functions

Consider once more the four point correlation function:

$$\langle\delta_1\delta_2\delta_3\delta_4\rangle = \langle\delta_1\rangle_c\,\langle\delta_2\rangle_c\,\langle\delta_3\rangle_c\,\langle\delta_4\rangle_c + \langle\delta_1\rangle_c\,\langle\delta_2\delta_3\delta_4\rangle_c\;(4\,\text{terms}) + \langle\delta_1\delta_2\rangle_c\,\langle\delta_3\delta_4\rangle_c\;(3\,\text{terms})$$
$$+\langle\delta_1\delta_2\rangle_c\,\langle\delta_3\rangle_c\,\langle\delta_4\rangle_c\;(6\,\text{terms}) + \langle\delta_1\delta_2\delta_3\delta_4\rangle_c$$

Using similar diagrams we can understand the origin of each of these terms



Here •⟶ means: "this point moving to infinity"

Notation:

$$\xi(\vec{x}_1,\vec{x}_2) = \langle\delta_1\delta_2\rangle_c \qquad \zeta(\vec{x}_1,\vec{x}_2,\vec{x}_3) = \langle\delta_1\delta_2\delta_3\rangle_c \qquad \eta(\vec{x}_1,\vec{x}_2,\vec{x}_3,\vec{x}_4) = \langle\delta_1\delta_2\delta_3\delta_4\rangle_c$$

For a Gaussian random field, all connected moments (=reduced correlation functions) of $n > 2$ are equal to zero (i.e., $\zeta = \eta = 0$).

One can use ζ and η to test whether the density field is Gaussian or not...

The Wiener-Khinchin Theorem

The Wiener-Khinchin Theorem states that the power spectrum is the Fourier transform of the two-point auto-correlation function. In what follows we provide the proof:

$$\delta_{\vec{k}} = \frac{1}{V}\int \delta(\vec{x}) e^{-i\vec{k}\cdot\vec{x}}\, \mathrm{d}^3\vec{x}$$

$$\delta_{-\vec{k}} = \frac{1}{V}\int \delta(\vec{x}) e^{+i\vec{k}\cdot\vec{x}}\, \mathrm{d}^3\vec{x}$$

$$\vec{x}_1 \equiv \vec{x}_2 + \vec{r}$$

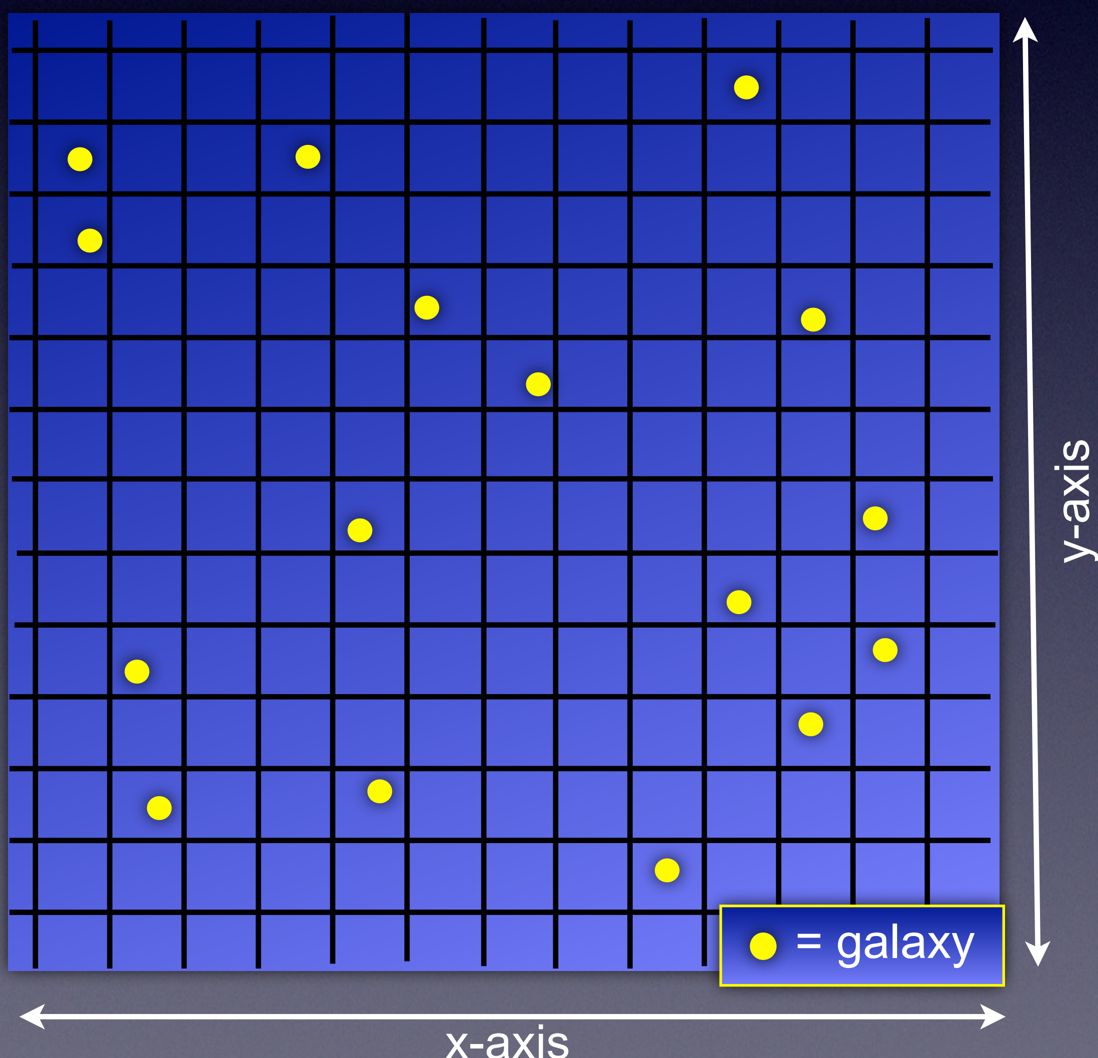
$$\begin{aligned}
P(k) &= V\,\langle|\delta_{\vec{k}}|^2\rangle = V\langle\delta_{\vec{k}}\delta^*_{\vec{k}}\rangle = V\langle\delta_{\vec{k}}\delta_{-\vec{k}}\rangle \\
&= V\left\langle \frac{1}{V^2}\int \mathrm{d}^3\vec{x}_1 e^{-i\vec{k}\cdot\vec{x}_1}\delta(\vec{x}_1)\int \mathrm{d}^3\vec{x}_2 e^{+i\vec{k}\cdot\vec{x}_2}\delta(\vec{x}_2)\right\rangle \\
&= \frac{1}{V}\int \mathrm{d}^3\vec{x}_1 e^{-i\vec{k}\cdot\vec{x}_1}\int \mathrm{d}^3\vec{x}_2 e^{+i\vec{k}\cdot\vec{x}_2}\langle\delta(\vec{x}_1)\delta(\vec{x}_2)\rangle \\
&= \frac{1}{V}\int \mathrm{d}^3\vec{r}\,\xi(r)\, e^{-i\vec{k}\cdot\vec{r}}\int \mathrm{d}^3\vec{x}_2\, e^{-i\vec{k}\cdot\vec{x}_2}\, e^{+i\vec{k}\cdot\vec{x}_2} \\
&= \int \mathrm{d}^3\vec{r}\,\xi(r)\, e^{-i\vec{k}\cdot\vec{r}}
\end{aligned}$$

QED

Discrete N-point statistics

Thus far we focussed on the continuous overdensity field, $\delta(\vec{x})$. We have seen that $\delta(\vec{x})$ can be described by the n-point correlation function, or, equivalently by the mass moments κ_n

We now consider a discrete distribution of points (i.e., galaxies) and use similar statistics to describe their distribution in space.



Imagine space divided into many small volumes, δV_i, which are so small that none of them contain more than one galaxy...

Let $\mathcal{N}_i$ be the occupation number of galaxies in cell i

Then we have that $\mathcal{N}_i = 0, 1$ and therefore $\mathcal{N}_i = \mathcal{N}_i^2 = \mathcal{N}_i^3 =$ etc..

Discrete N-point statistics

We now `replace' $\mathcal{P}(\delta_1, \delta_2, ..., \delta_N)\, \mathrm{d}\delta_1\, \mathrm{d}\delta_2 \ldots \mathrm{d}\delta_N$ with the probability $\mathcal{P}(\mathcal{N}_1, \mathcal{N}_2, ..., \mathcal{N}_N)$ that we have the realization $\{\mathcal{N}_1, \mathcal{N}_2, ..., \mathcal{N}_N\}$.

As before, we will characterize $\mathcal{P}(\mathcal{N}_1, \mathcal{N}_2, ..., \mathcal{N}_N)$ by its moments $\langle \mathcal{N}_1^{l_1}\, \mathcal{N}_2^{l_2} \ldots \mathcal{N}_N^{l_N} \rangle$

Using that $\mathcal{N}_i = \mathcal{N}_i^2 = \ldots = \mathcal{N}_i^n$ we have that

$$\boxed{\langle \mathcal{N}_1^{l_1}\, \mathcal{N}_2^{l_2} \ldots \mathcal{N}_N^{l_N} \rangle = \langle \mathcal{N}_1\, \mathcal{N}_2 \ldots \mathcal{N}_N \rangle = \delta\mathcal{P}_{12\ldots N}}$$

where we have defined the probability $\delta\mathcal{P}_{12\ldots N}$ that there is a galaxy in δV_1, <u>and</u> there is a galaxy in δV_2, . . . , <u>and</u> there is a galaxy in δV_N

Let $\bar{n}$ be the average number density of galaxies, then $\delta\mathcal{P}_1 = \langle \mathcal{N}_1 \rangle = \bar{n}\, \delta V_1$

If the `point process' (i.e., the random process that puts down the points) is a random Poisson process, then $\delta\mathcal{P}_{12} = \langle \mathcal{N}_1\, \mathcal{N}_2 \rangle \equiv \bar{n}^2\, \delta V_1\, \delta V_2$, i.e., the probability to have a galaxy at δV_1 is independent of probability to have one at δV_2...

In the more general case where the point process is not Poisson we define

$$\boxed{\delta\mathcal{P}_{12} = \langle \mathcal{N}_1\, \mathcal{N}_2 \rangle \equiv \bar{n}^2\, \delta V_1\, \delta V_2\, [1 + \xi_{12}]}$$

Discrete N-point statistics

$$\delta\mathcal{P}_{12} = \langle \mathcal{N}_1\, \mathcal{N}_2 \rangle \equiv \bar{n}^2\, \delta V_1\, \delta V_2\, [1 + \xi_{12}]$$

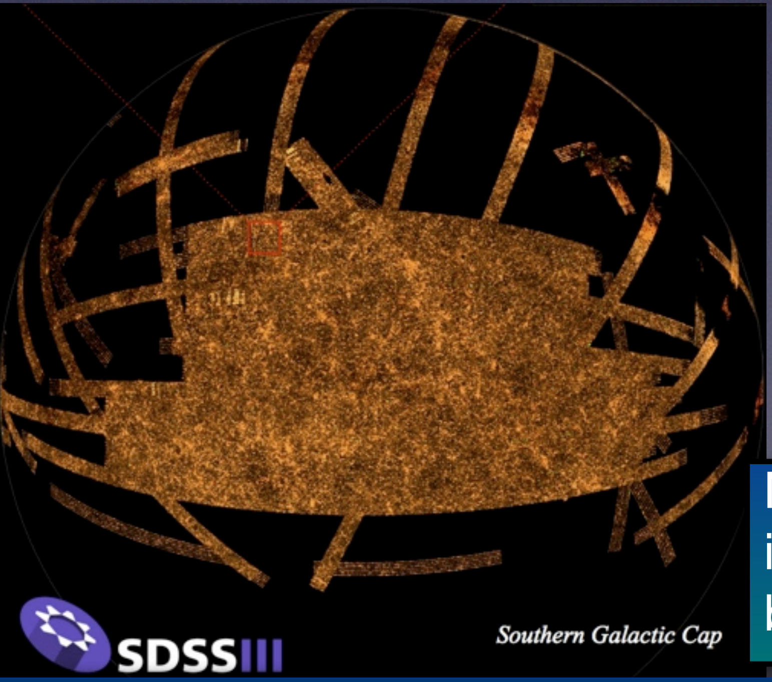
The above relation <u>defines</u> the two-point correlation function $\xi_{12} = \xi(r_{12})$

As is immediate evident from its definition, ξ_{12} is the excess probability, relative to Poisson, that two galaxies (points) are separated by a distance r_{12}

The two-point correlation function of galaxies is typically measured using

$$\xi(r) = \frac{DD(r)\,\Delta r}{RR(r)\,\Delta r} - 1$$

Here $DD(r)\,\Delta r$ is the number of pairs with separations $r \pm \Delta r/2$ in the data, and $RR(r)\,\Delta r$ is the corresponding number of pairs if point process is random.



Other \`estimators' for $\xi(r)$ are also available in the literature, but as long as the data sample is sufficiently large, the above is more than adequate...

NOTE: when constructing the random sample, it is important that one carefully models the survey boundary (\`footprint') of the data sample...

Discrete N-point statistics

Clearly, this notation looks very similar to what we used in the continuous limit, suggesting a close link. To see this connection, we write $\delta\mathcal{P}_{12} = \langle \mathcal{N}_1\,\mathcal{N}_2 \rangle \equiv \langle n_1\, n_2 \rangle\, \delta V_1\, \delta V_2$ where we have introduced the continuous number density field $n(\vec{x})$ for which $\langle n(\vec{x}) \rangle = \bar{n}$

Writing $n(\vec{x}) = \bar{n}\,[1 + \delta_{\rm g}(\vec{x})]$, where we introduced the galaxy overdensity field $\delta_{\rm g}(\vec{x})$ we can write that

$$\boxed{\delta\mathcal{P}_{12} = \bar{n}^2 \delta V_1 \delta V_2 \langle (1+\delta_1)\,(1+\delta_2) \rangle = \bar{n}^2 \delta V_1 \delta V_2 [1 + \langle \delta_1 \delta_2 \rangle]}$$

where from now on we consider it understood that $\delta = \delta_{\rm g}$ (for the sake of brevity).

Comparing the above to how we defined the (discrete) two-point correlation function:

$$\boxed{\delta\mathcal{P}_{12} = \langle \mathcal{N}_1\,\mathcal{N}_2 \rangle \equiv \bar{n}^2\, \delta V_1\, \delta V_2\, [1 + \xi_{12}]}$$

$$\boxed{\xi_{12} = \langle \delta_1 \delta_2 \rangle}$$

Thus we see that the two-point correlation function that we defined in the discrete case is the same as that defined in the continuous case.

Discrete N-point statistics

Finally, we derive the power spectrum for our discrete distribution of points:

$$n(\vec{x}) = \bar{n}\,[1+\delta_{\rm g}(\vec{x})] = \sum_i \delta^{\rm D}(\vec{x}-\vec{x}_i) \quad \Rightarrow \quad \delta_{\rm g}(\vec{x}) = \frac{1}{\bar{n}}\sum_i \delta^D(\vec{x}-\vec{x}_i) - 1$$

Hence, we have that

$$\begin{aligned}
\delta_{\vec{k}} &= \frac{1}{V}\int \delta_{\rm g}(\vec{x})\, e^{-i\vec{k}\cdot\vec{x}}\,{\rm d}^3\vec{x} \\
&= \frac{1}{\bar{n}V}\sum_i e^{-i\vec{k}\cdot\vec{x}_i} - \frac{1}{V}\int e^{-i\vec{k}\cdot\vec{x}}\,{\rm d}^3\vec{x} \\
&= \frac{1}{\bar{n}V}\sum_j \mathcal{N}_j\, e^{-i\vec{k}\cdot\vec{x}_j} - \delta^{\rm D}_{\vec{k}}
\end{aligned}$$

Substitution of the above in the expression for the power spectrum yields

$$\boxed{P(k) \equiv V\,\langle|\delta_{\vec{k}}|^2\rangle = P_{\rm gg}(k) + \frac{1}{\bar{n}}}$$

Students: try to derive this at home

where $P_{\rm gg}(k) \equiv \int \xi_{\rm gg}(r)\, e^{-i\vec{k}\cdot\vec{r}}\,{\rm d}^3\vec{r}$. The extra $1/\bar{n}$ -term is due to shot-noise. If the galaxy distribution is Poisson, then $P_{\rm gg}(k)=0$. However, the Fourier modes have non-zero variance $\langle|\delta_{\vec{k}}|^2\rangle = (\bar{n}V)^{-1}$ due to discreteness. Note that this shot noise manifests itself as a white-noise ($P(k) \propto k^0$) contribution.

Poisson Sampling & Galaxy Bias

Suppose galaxy formation is very simple, such that the probability that a cell δV_i contains a galaxy follows a Poisson distribution with a mean proportional to the mean density $\rho_i = \frac{1}{\delta V_i} \int_{\delta V_i} \rho(\vec{x}) \, \mathrm{d}^3\vec{x}$

We say that the galaxies sample the density field $\rho(\vec{x}) = \bar{\rho}[1 + \delta(\vec{x})]$ via a Poisson process

The probability $p^{(1)}(\vec{x})$ that a cell at $\vec{x}$ contains one galaxy is

$$\boxed{p^{(1)}(\vec{x}) = [1 + \delta(\vec{x})] \, \bar{n}\delta V}$$

We can also write that $p^{(1)}(\vec{x}) = \langle \mathcal{N}(\vec{x}) \rangle_{\mathrm{P}}$, where $\langle ... \rangle_{\mathrm{P}}$ indicates an average over the Poisson probability distribution...

The galaxy distribution, in this case, is the outcome of a double stochastic process, with one level of randomness coming from the random density field and the second from the Poisson sampling...

Poisson Sampling & Galaxy Bias

We can now immediately write down the n-point statistics for the galaxy distribution.

Consider the two-point statistic (what follows holds for all n-point statistics though...)

$$\langle \mathcal{N}_1 \mathcal{N}_2 \rangle = \langle p^{(1)}(\vec{x}_1) p^{(1)}(\vec{x}_2) \rangle = (\bar{n}\delta V)^2 \langle (1+\delta(\vec{x}_1))(1+\delta(\vec{x}_2)) \rangle = (\bar{n}\delta V)^2 [1+\xi(r_{12})]$$

- The first step simply expresses that the Poisson samplings at different locations are independent of each other...
- The two-point correlation function $\xi(r_{12})$ is that of the continuous matter field.

We also had that for a point (galaxy) distribution $\langle \mathcal{N}_1 \mathcal{N}_2 \rangle \equiv \delta \mathcal{P}_{12} = (\bar{n}\delta V)^2 [1+\xi_{12}]$

$$\Rightarrow \quad \underbrace{\xi_{12}}_{\text{2-point correlation function of galaxies}} = \underbrace{\xi(r_{12})}_{\text{2-point correlation function of matter}}$$

If galaxy formation is a Poisson sampling of the density field, then all n-point correlation functions of the galaxy distribution are identical to those of the matter distribution

Poisson Sampling & Galaxy Bias

How realistic is it that galaxies are a Poisson sampling with $p^{(1)}(\vec{x}) \propto \rho(\vec{x})$??

- Galaxies are believed to form and reside in dark matter haloes.
- As we have seen before, dark matter haloes are biased tracers of matter distribution. Hence, it seems only logical that the galaxy distribution is also biased.

To get some insight into the implications of galaxies being biased tracers of the mass distribution, assume that the sampling is still a Poisson process but with

$$\boxed{p^{(1)}(\vec{x}) = [1 + b\,\delta(\vec{x})]\,\bar{n}\delta V}$$

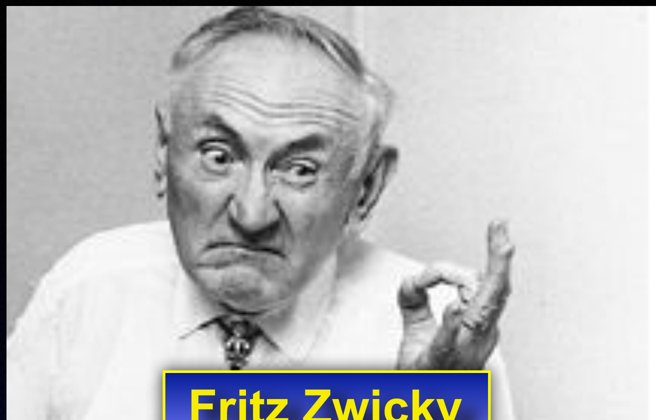
where b is some constant `bias' parameter. We then have that

$$\begin{aligned}\langle \mathcal{N}_1 \mathcal{N}_2 \rangle = \langle p^{(1)}(\vec{x}_1) p^{(1)}(\vec{x}_2) \rangle &= (\bar{n}\delta V)^2 \langle (1 + b\delta(\vec{x}_1))(1 + b\delta(\vec{x}_2)) \rangle \\ &= (\bar{n}\delta V)^2 [1 + b^2 \xi(r_{12})]\end{aligned}$$

$$\Rightarrow \quad \boxed{\xi_{12} = b^2 \xi(r_{12})}$$

- As we will see later, galaxy bias is much more complicated than what is assumed here.
- In general, one cannot infer the matter distribution from the galaxy distribution without detailed knowledge of its bias.

Mapping the Galaxy Distribution



In the early days, mapping the large scale structure was done by counting galaxies on photographic plates, by 

Constructing galaxy catalogues was pioneered by Shapley and Zwicky in 1930s. A milestone was the Lick catalogue ofShane & Wirtanen (1967), which contained over 1 million galaxies identified by eye on the Lick plates, down to a limiting photographic magn. of ~18.3

Zwicky et al., galaxy overdensities map

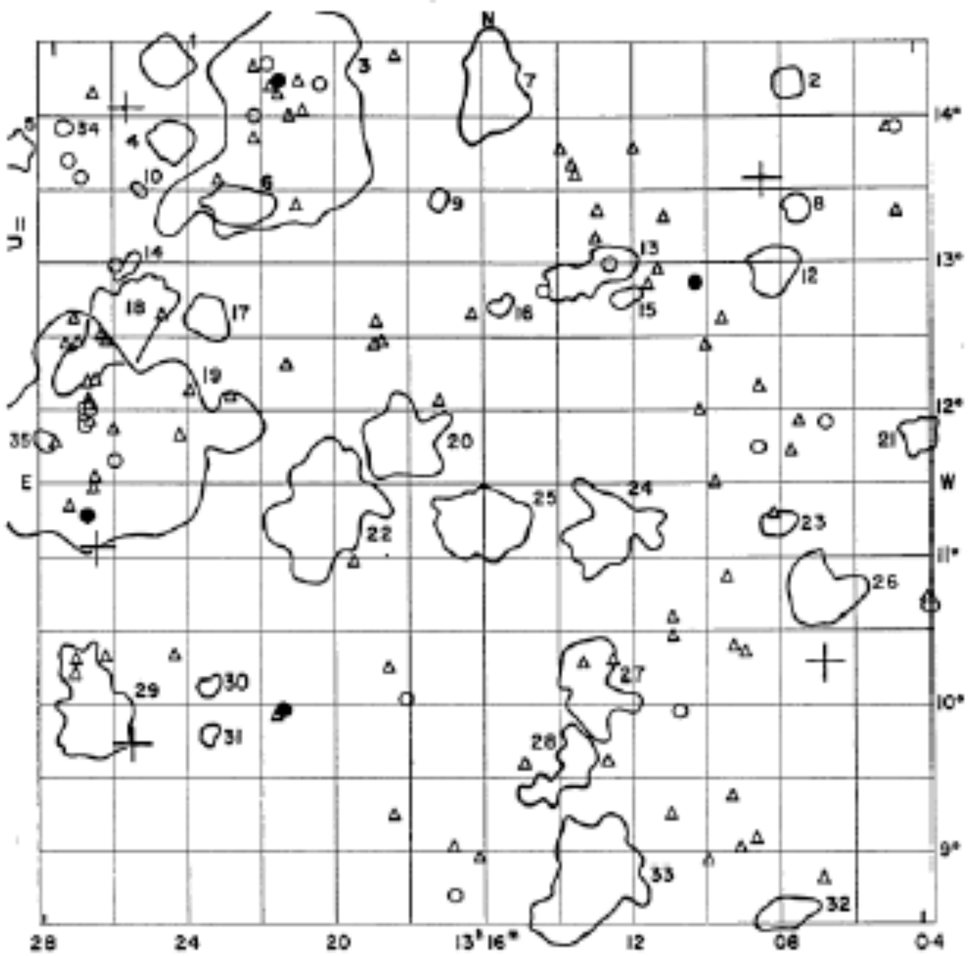
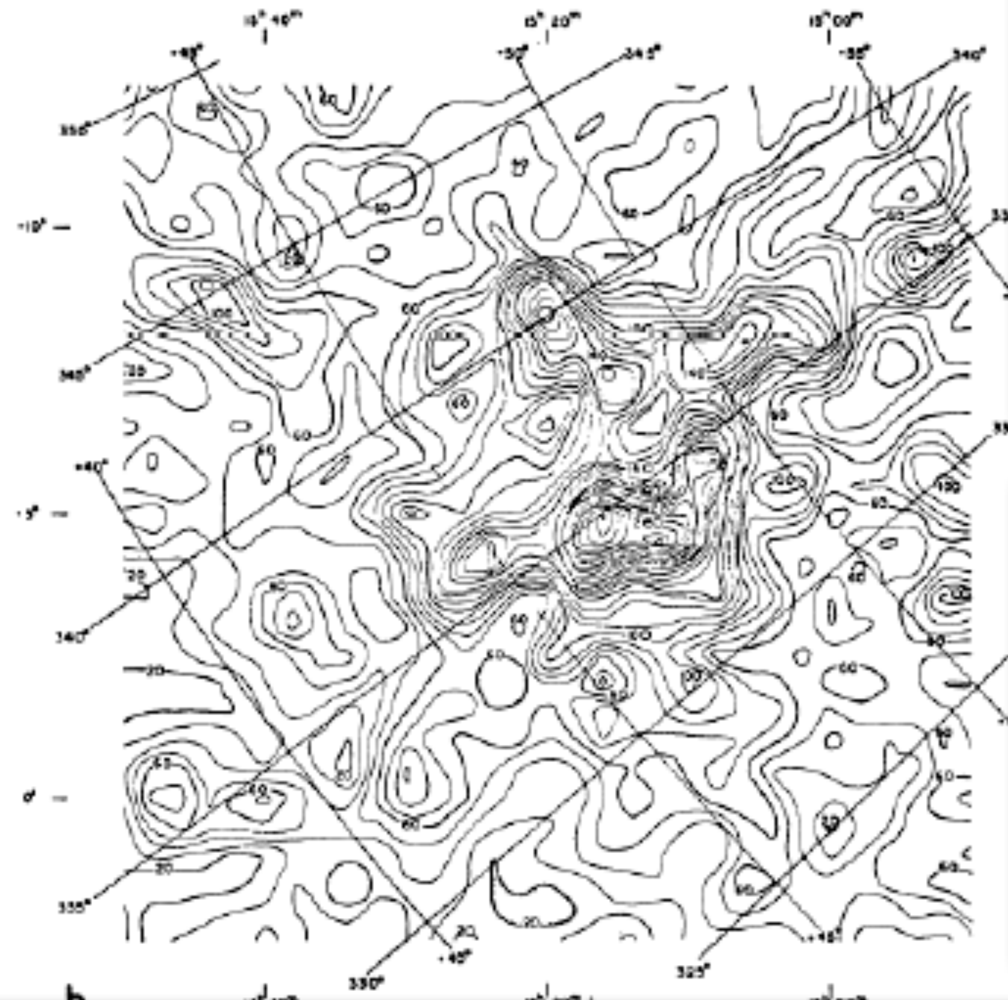
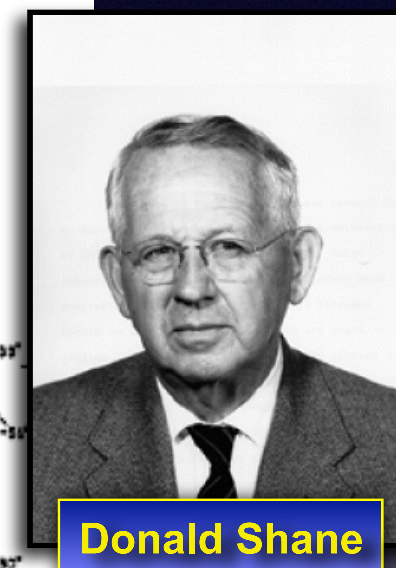
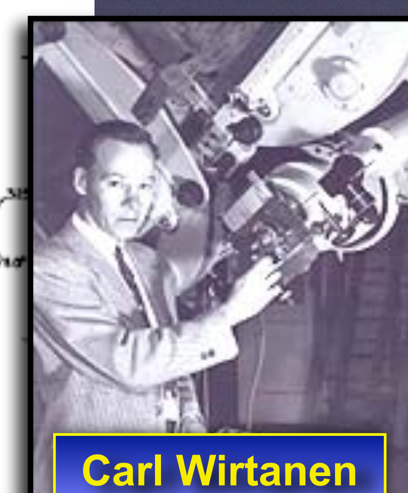


FIGURE 4

Shane-Wirtanen galaxy counts, density map

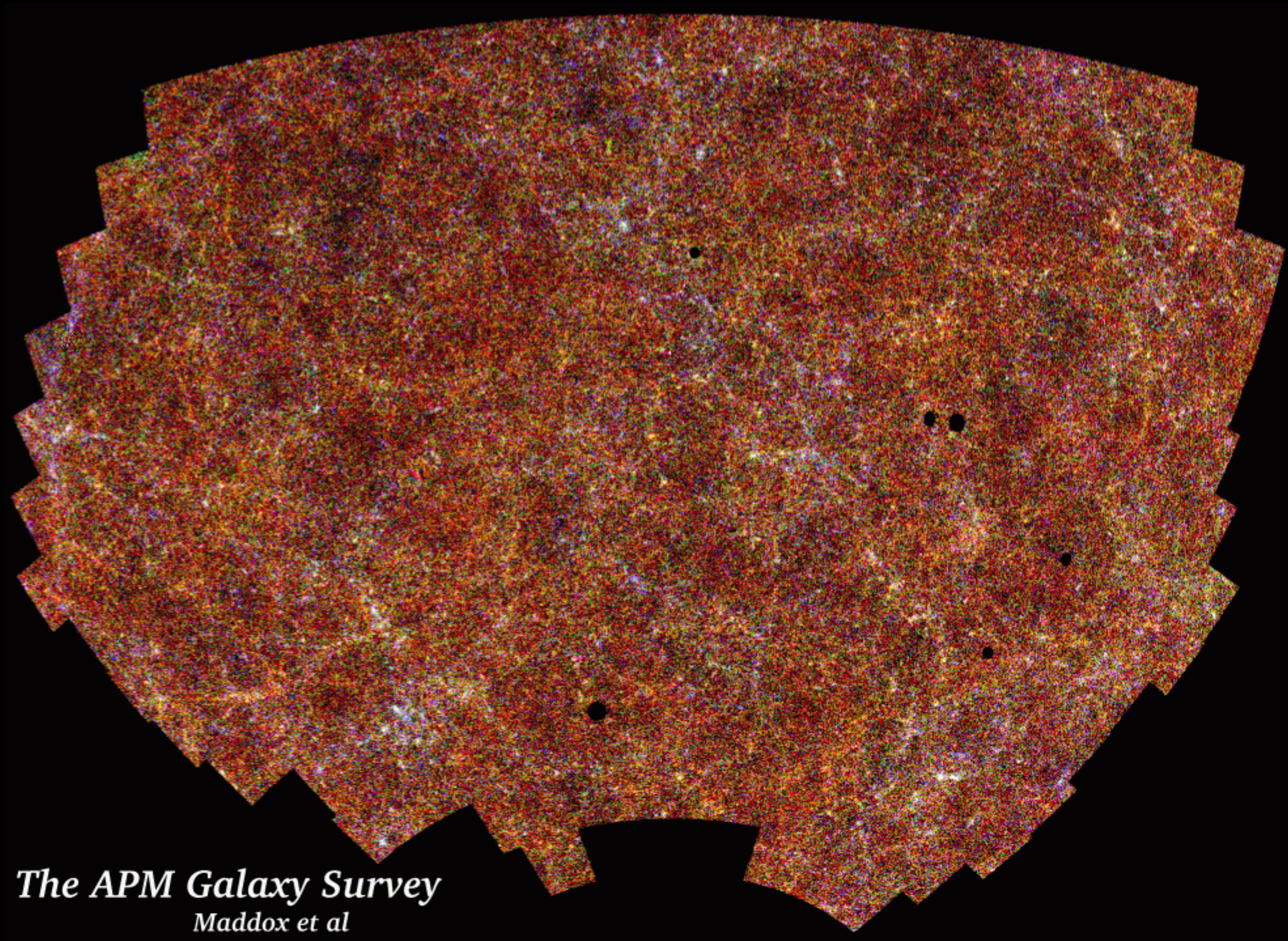






Mapping the Galaxy Distribution

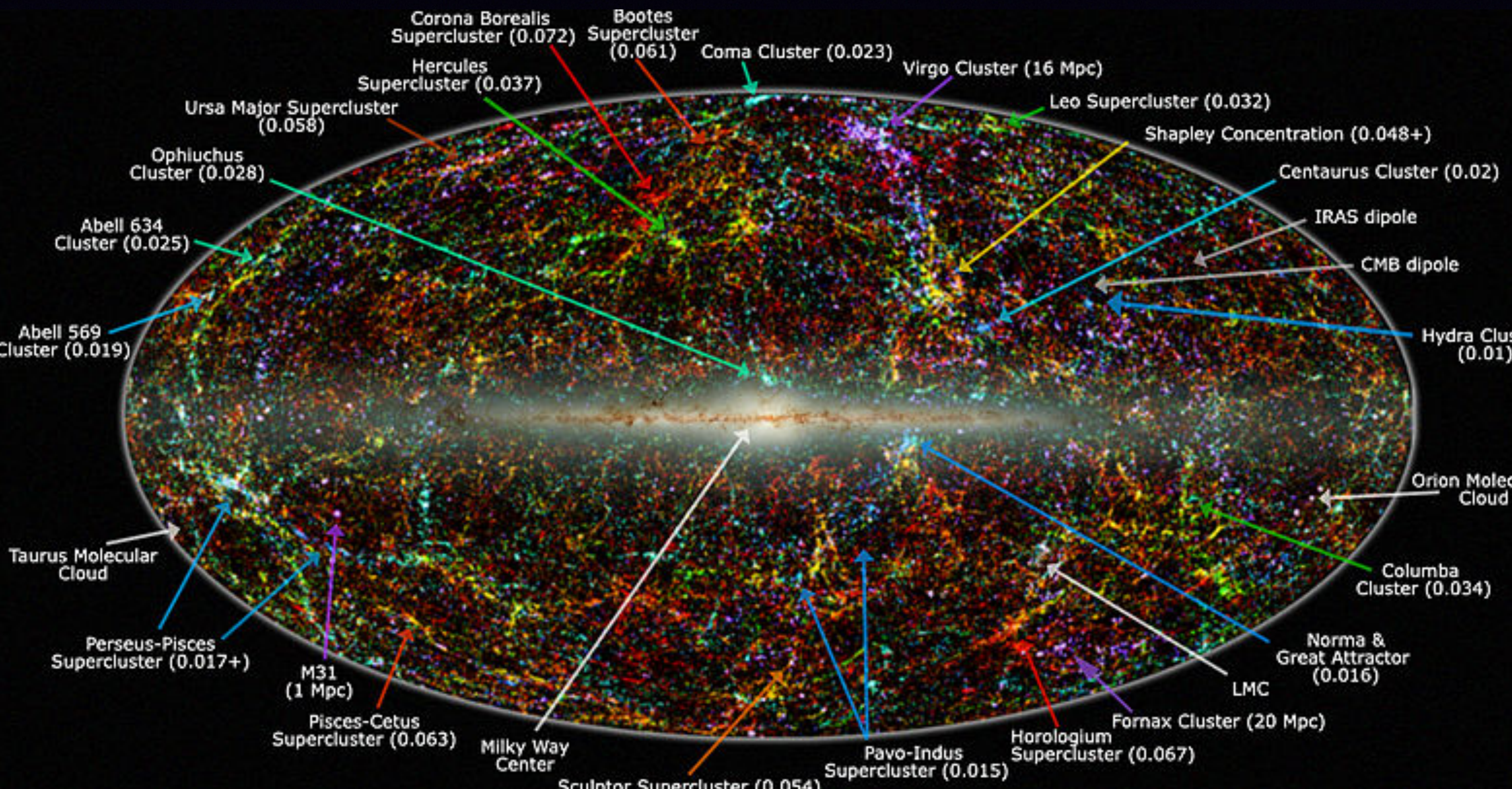
During 1990s, plate scanning machines replaced humans in identifying galaxies on photographic plates. One example is the APM Galaxy Survey, with ~2 million galaxies (Maddox et al. 1990)



The Local Universe

All sky view of the “local” Universe as mapped out by the Two-Micron All Sky Survey (2MASS). In this map galaxies have been color coded by their photometric redshift.

(source: Jarrett 2004)



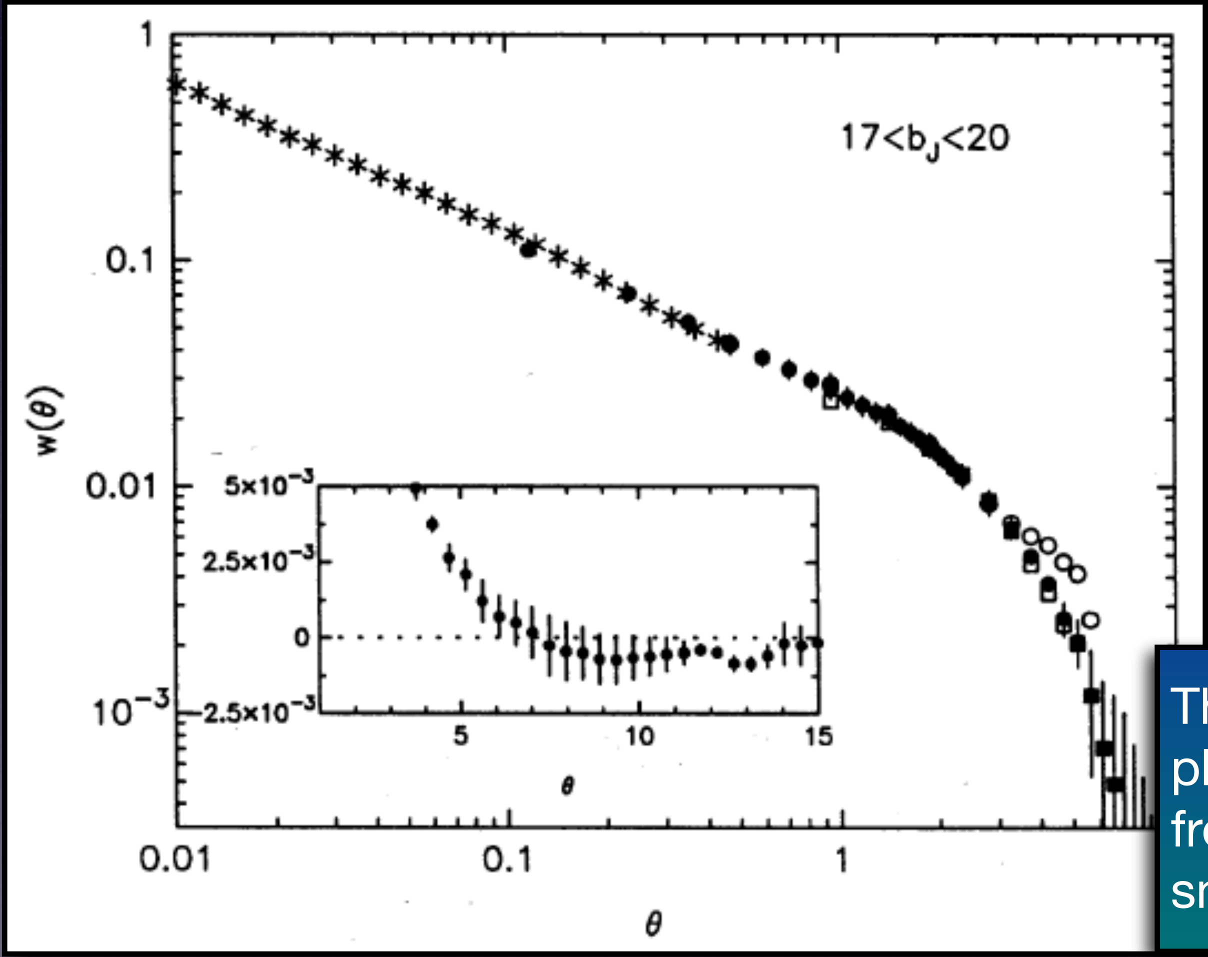
Angular Correlation Functions

Consider a sample of galaxies with $\vec{\Theta}_i = (\alpha_i, \delta_i)$ and complete down to some limiting apparent magnitude $m_{\rm lim}$:

The angular correlation function, $w(\theta)$, is defined by $\boxed{\delta\mathcal{P}_{12} = \bar{\nu}^2 \delta\Omega_1\, \delta\Omega_2\, [1 + w(\theta)]}$
where $\delta\mathcal{P}_{12}$ is the probability to find two galaxies in the infinitesimal solid angles $\delta\Omega_1$ and $\delta\Omega_2$, and $\bar{\nu}$ is the mean surface number density.

The angular correlation function is obtained using the estimator:

$$\boxed{w(\theta) = \frac{DD(\theta)\,\mathrm{d}\theta}{RR(\theta)\,\mathrm{d}\theta} - 1}$$



Source: Maddox et al. 1990, MNRAS, 242, 43

The first measurements of angular correlation function were obtained by Hauser & Peebles (1973) using the Zwicky et al. (1961-1968) and Lick (Shane & Wirtanen 1967) catalogs...

The angular correlation function of galaxies with apparent photographic magnitudes in the range $17 < b_J < 20$ obtained from the APM Galaxy Survey. Note the power-law behavior on small scales, and the dip below zero on large scales.....

Angular Correlation Functions

See MBW §6.5.3 for more details

The angular correlation function is related to the real space correlation function via a line-of-sight projection integration known as the Limber equation:

$$\boxed{w(\theta) = \int_0^\infty \mathrm{d}y\, y^4\, S^2(y) \int_{-\infty}^{\infty} \mathrm{d}x\, \xi(\sqrt{x^2 + y^2\theta^2})}$$

Limber equation

Here $S(y)$ is the survey selection function, normalized such that $\int x^2\, S(x)\, \mathrm{d}x = 1$, and defined as probability that `random' galaxy located at *y* is included in the sample

For example, for an apparent magnitude limited sample with $m < m_{\rm lim}$ we have that

$$\boxed{S(z) = \frac{\int_{L_{\rm min}(z)}^\infty \Phi(L)\, \mathrm{d}L}{\int_0^\infty \Phi(L)\, \mathrm{d}L}}$$

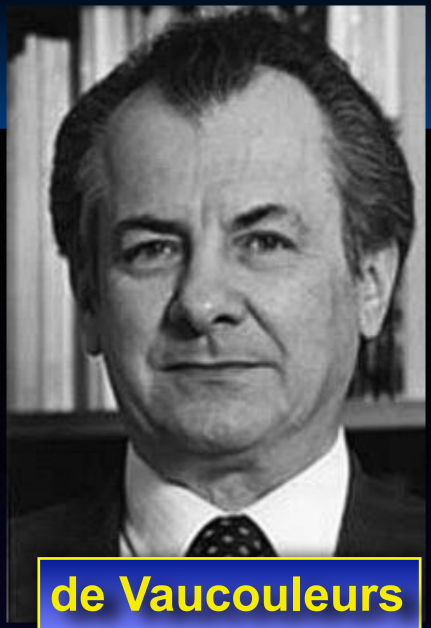
$\Phi(L)$ = luminosity function

where $L_{\rm lim}(z)$ is the luminosity of a galaxy that at z has an aparent magnitude $m_{\rm lim}$

If $\xi(r) \propto r^{-\gamma}$ then $w(\theta) \propto \theta^{1-\gamma}$ → One can infer real-space correlation function from angular correlation function...

HOWEVER; in the case of a magnitude limited sample, as is generally the case, this is only true if clustering is independent of luminosity, which is not the case (as we will see). Because of this angular correlation functions have gone out of vogue.

Galaxy Redshift Surveys



In order to properly characterize the distribution of galaxies, we need information in 3D; this is provided by galaxy redshift surveys.

First galaxy redshift surveys were constructed by Gerard de Vaucouleurs and collaborators in 1950-1970s.

The first redshift survey appropriate for measuring clustering of galaxies was the CfA survey of Huchra & Geller; This data set was used by Davis & Peebles (1983) to measure the galaxy auto-correlation function:

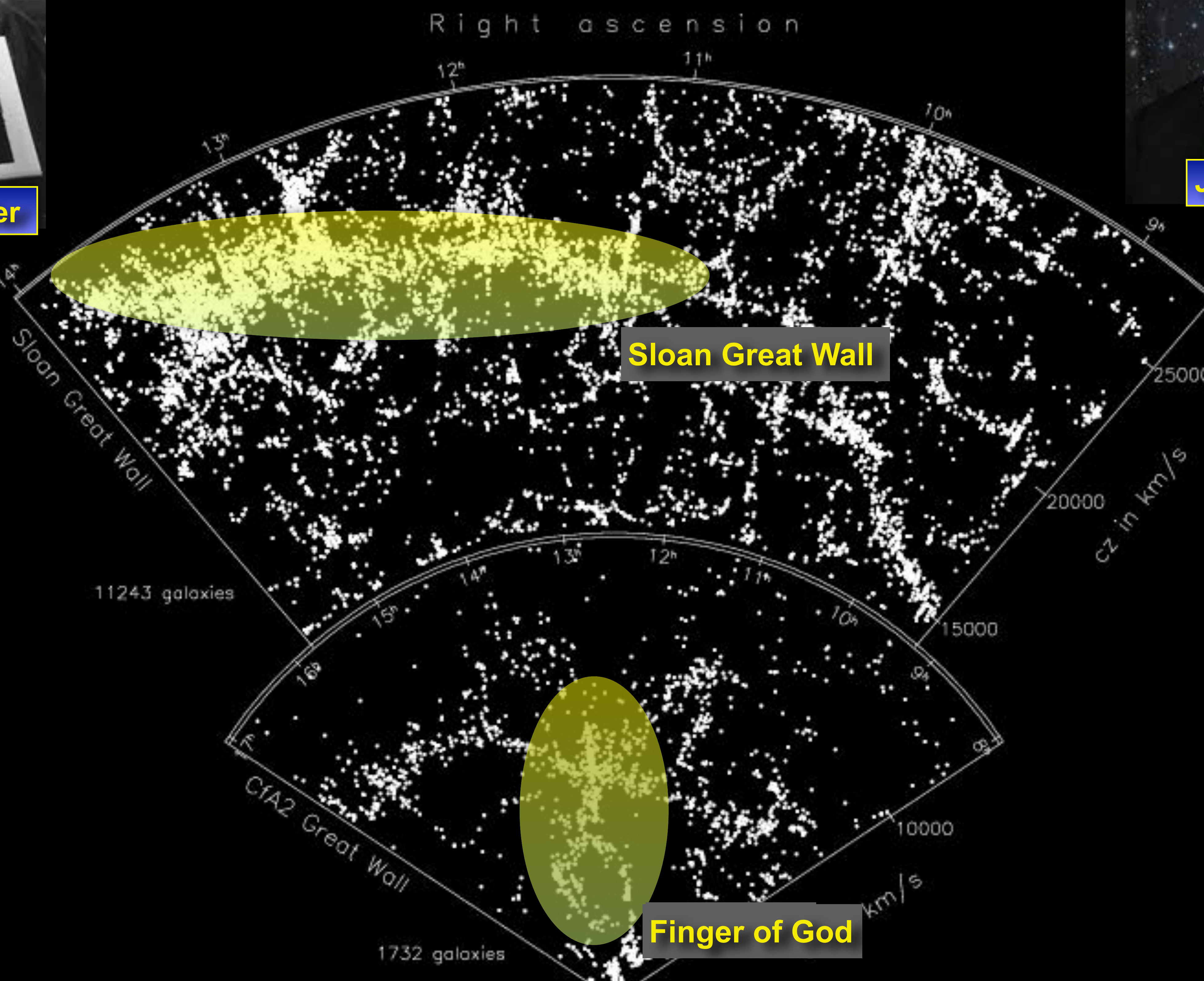
$$\xi_{\rm gg}(r) = \left(\frac{r}{r_0}\right)^{-1.8} \quad \text{with} \quad r_0 \simeq 5.4 h^{-1}\mathrm{Mpc}$$

"correlation length"

Representative Redshift Surveys		
1985	CfA	~2,500
1992	IRAS	~9,000
1995	CfA2	~20,000
1996	LCRS	~23,000
2003	2dFGRS	~250,000
2009	SDSS	~930,000

The last 30 years have seen a dramatic increase in data, culminating with the completion of the Sloan Digital Sky Survey. Currently, we have accurate measured redshifts for well over one million galaxies.

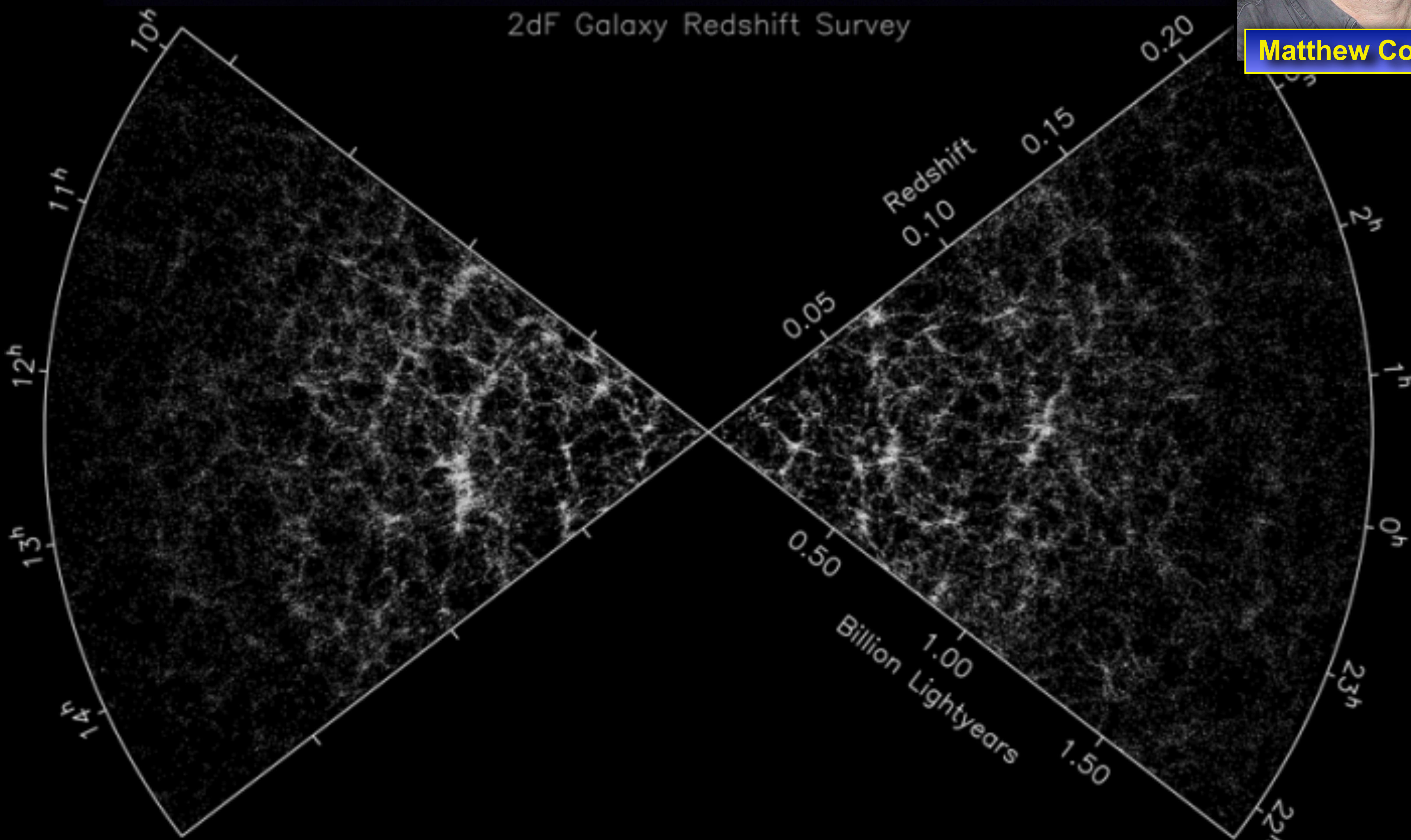
The CfA Redshift Survey



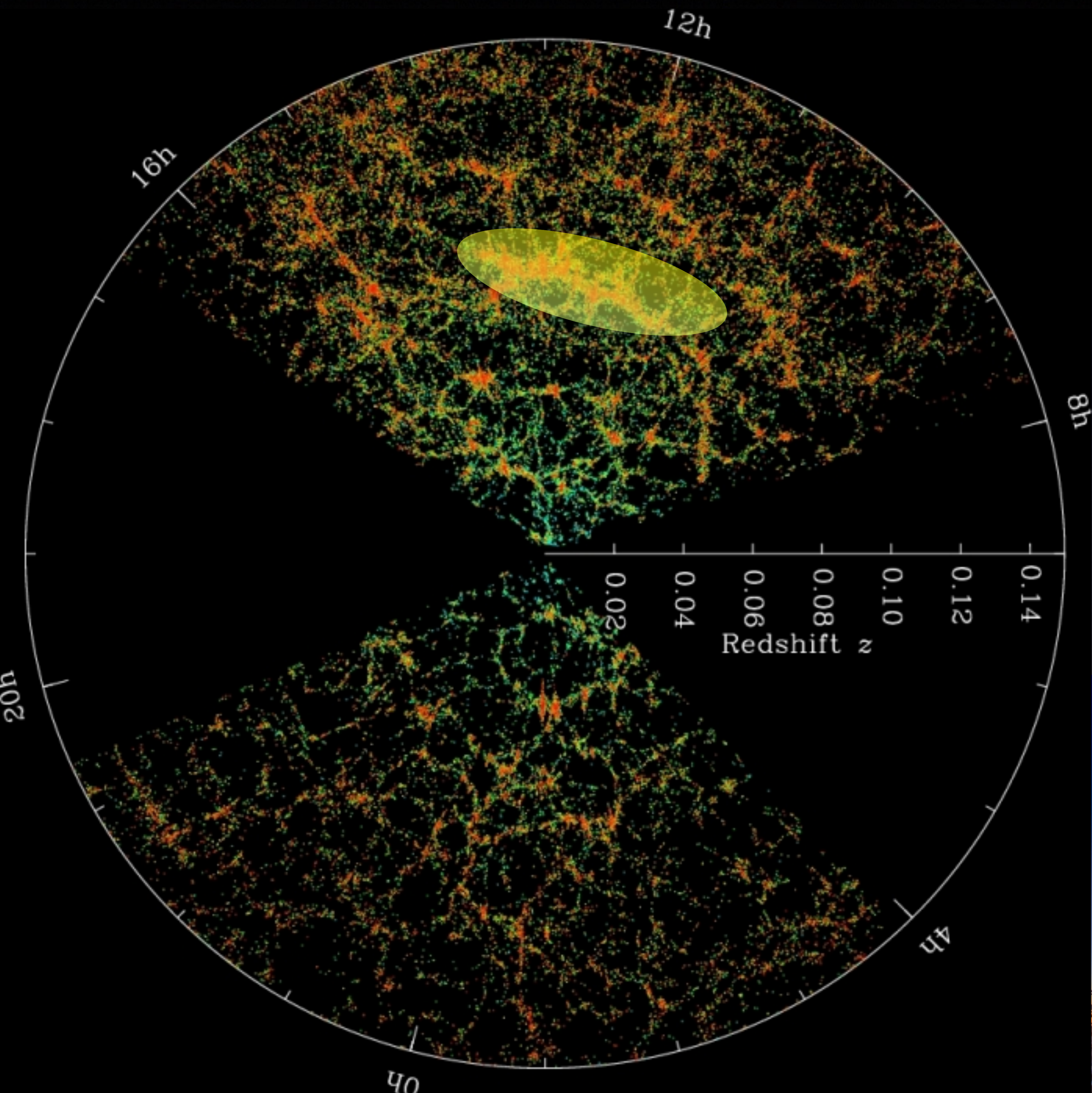
The 2dFGRS

Using the APM Galaxy Survey as input source catalogue, Colless et al. (2001) constructed the Two-Degree Field Galaxy Redshift Survey (2dFGRS), containing redshifts for ~220.000 galaxies, covering ~1500 sq. deg. on the Southern Sky


Matthew Colless



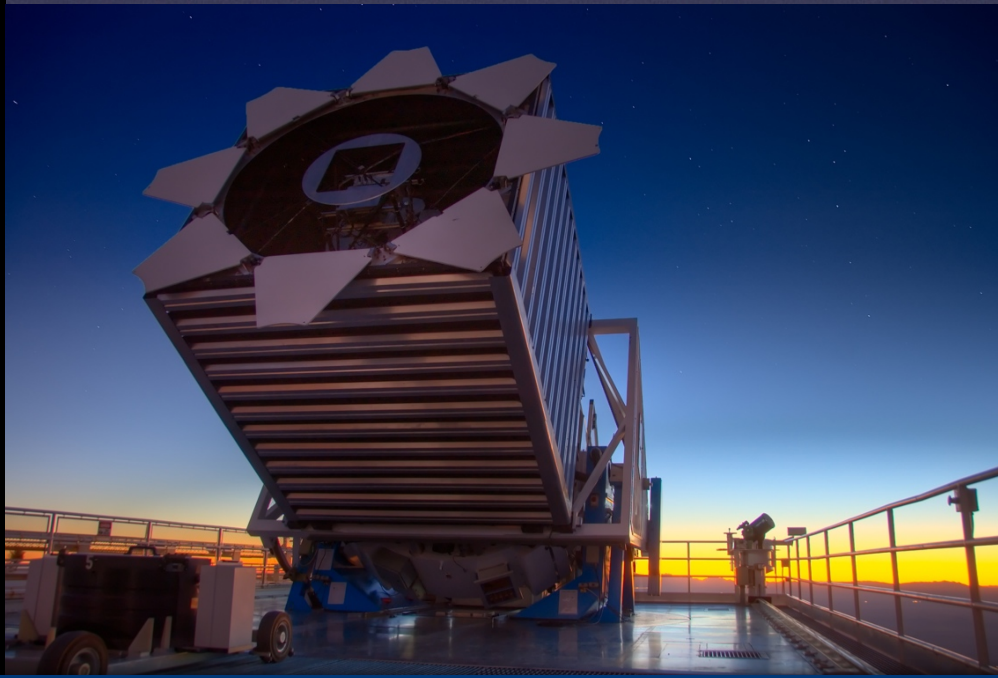
The Sloan Digital Sky Survey



At present, the largest galaxy redshift survey is the Sloan Digital Sky Survey (SDSS).

Using the dedicated 2.5m telescope at Apache Point Observatory, it imaged more than 8000 sq. deg. of sky in five passbands (ugriz), and obtained spectra for 930,000 galaxies and 120,000 quasars.

For more info: www.sdss.org



Redshift Space Distortions

A galaxy redshift survey consists of a large number of 3D `positions' $(\alpha_i, \delta_i, z_i)$

Define: $\boxed{\vec{s}_i = \left(\frac{c\, z_i}{H_0}\right) \hat{r}_i = \left(\frac{v_i}{H_0}\right) \hat{r}_i}$ v_i = radial velocity

with $\hat{r}_i$ the unit direction vector in the direction (α_i, δ_i)

We call $s_i \equiv |\vec{s}_i| = v_i/H_0$ the redshift distance of galaxy i

Recall: $\boxed{v = v_{\rm exp} + v_{\rm pec} = H_0\, l(z) + v_{\rm pec}}$

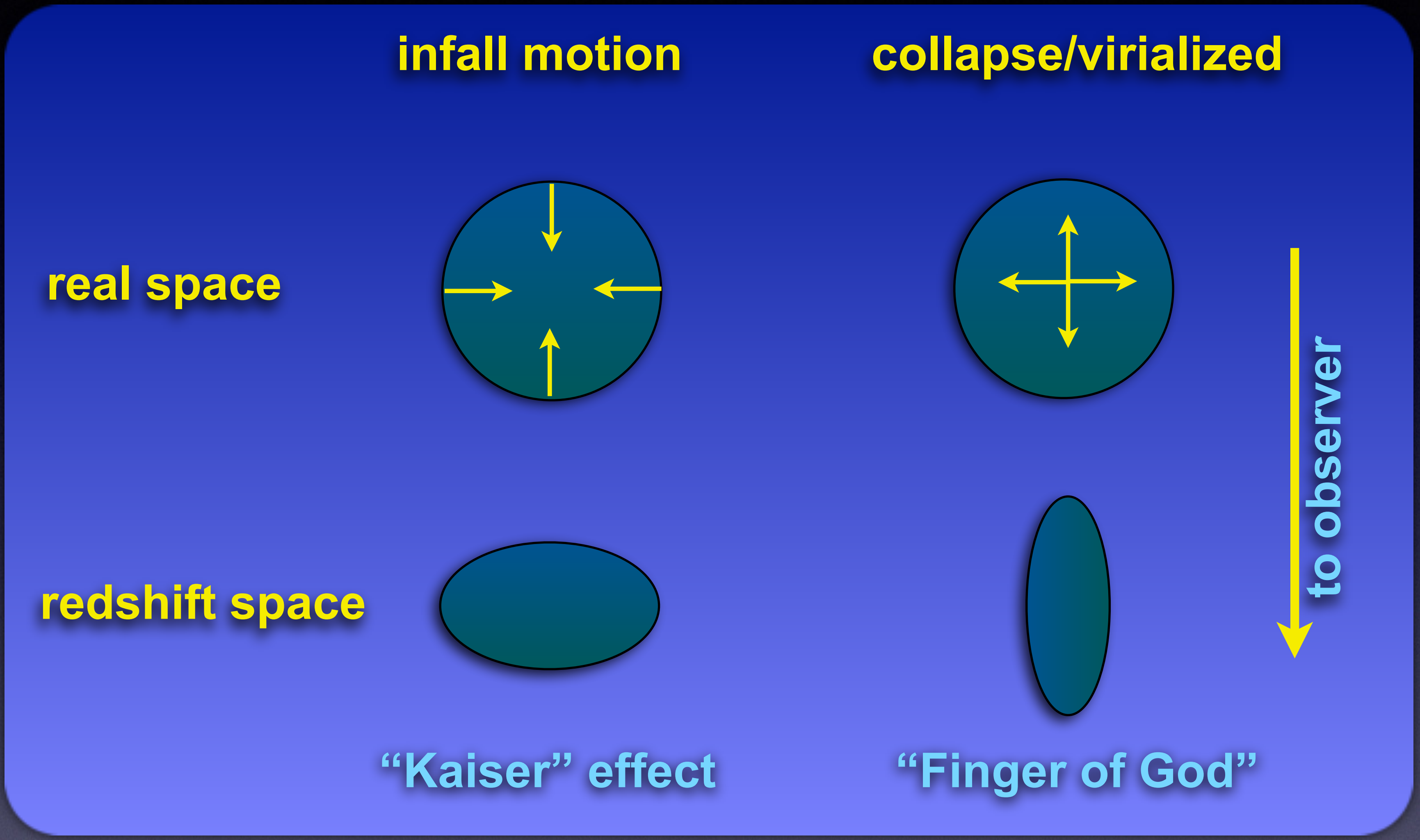
with $l(z)$ the proper distance to the galaxy

Due to peculiar velocities, the redshift distances available from a galaxy redshift survey deviate from the true, proper distances. This results in redshift space distortions in the clustering measurements.

Although these redshift space distortions complicate the interpretation of the clustering, they also contain useful information. After all, the peculiar velocities are induced by the cosmic matter distribution.

Redshift Space Distortions

Since peculiar velocities only cause distortions along the line-of-sight, they introduce anisotropies in the observed correlation function:



- On large scales, peculiar velocities reflect the (linear) infall motions towards overdensities, causing a circle in real space to appear `squashed' in redshift space. This is often called the Kaiser effect
- On small, peculiar velocities reflect the (non-linear) virialized motion of galaxies inside their host haloes, causing a circle in real space to appear `stretched' in redshift space. This is often called the Finger-of-God effect

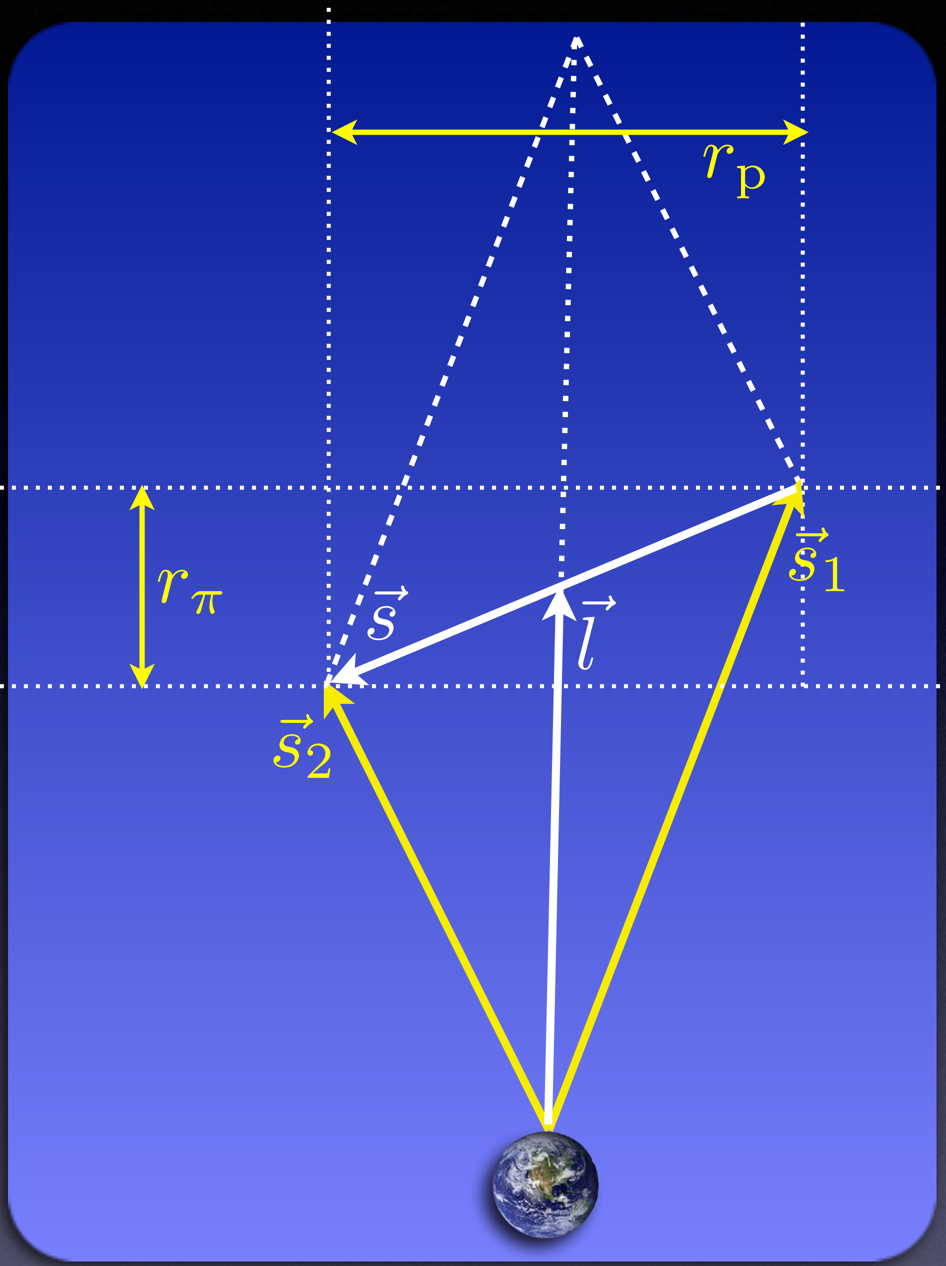
Redshift Space Distortions

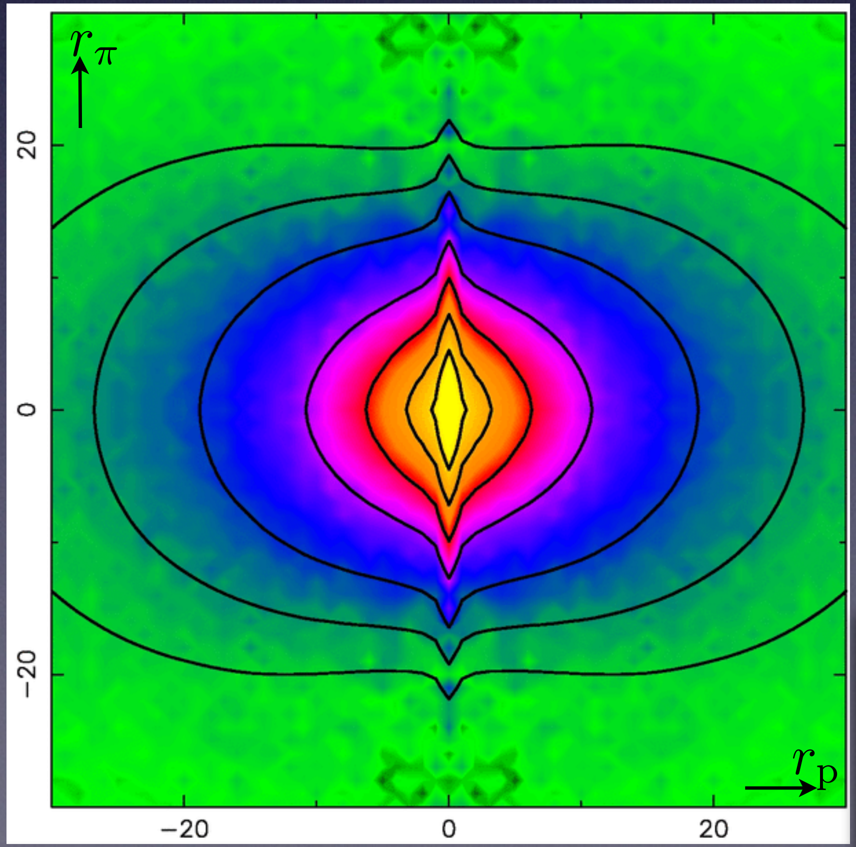
One expresses the distance between two galaxies in their components perpendicular, $r_{\rm p}$, and parallel, r_π, to the line-of-sight, which are defined as

$$r_\pi \equiv \frac{\vec{s}\cdot\vec{l}}{|\vec{l}|} \quad \text{and} \quad r_{\rm p} \equiv \sqrt{s^2 - r_\pi^2}$$

Here $\vec{s} \equiv \vec{s}_1 - \vec{s}_2$ and $\vec{l} \equiv \frac{1}{2}(\vec{s}_1 + \vec{s}_2)$ (see diagram)

These are used to measure the two-dimensional two-point correlation function $\xi(r_{\rm p}, r_\pi)$, which is anisotropic.

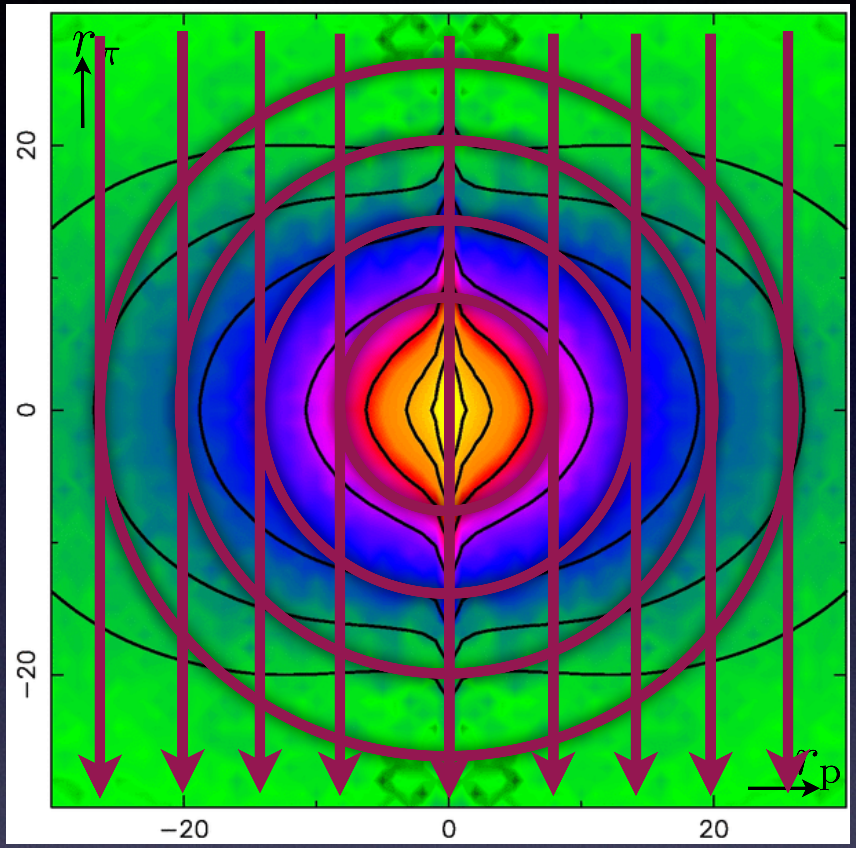




The two-point correlation function $\xi(r_{\rm p}, r_\pi)$ obtained from the 2dFGRS by Hawkins et al. (2003). Note the anisotropies due to Finger-of-God and Kaiser effect

The Projected Correlation Function

From the two-dimensional, two-point correlation function $\xi(r_{\rm p}, r_\pi)$ one can construct several one-dimensional two-point correlation function:



1) The redshift space correlation function $\xi(s)$

here $s = \sqrt{r_{\rm p}^2 + r_\pi^2}$ is simply the redshift space distance between two galaxies

2) The projected correlation function $w_{\rm p}(r_{\rm p})$

$$\boxed{w_{\rm p}(r_{\rm p}) = \int_{-\infty}^{\infty} \xi(r_{\rm p}, r_\pi)\, {\rm d}r_\pi}$$

Since redshift space distortions only affect r_π, the projected correlation function is unaffected by redshift space distortions. Hence, it is identical to a simple projection of the real-space correlation function, which is given by an Abel transform:

$$\boxed{w_{\rm p}(r_{\rm p}) = 2\int_{r_{\rm p}}^{\infty} \xi(r)\, \frac{r\,{\rm d}r}{(r^2 - r_{\rm p}^2)^{1/2}}}$$

NOTE: $w_{\rm p}(r_{\rm p})$ has the units of length → one typically plots $w_{\rm p}(r_{\rm p})/r_{\rm p}$

The Projected Correlation Function

The Abel integral can be inverted to give

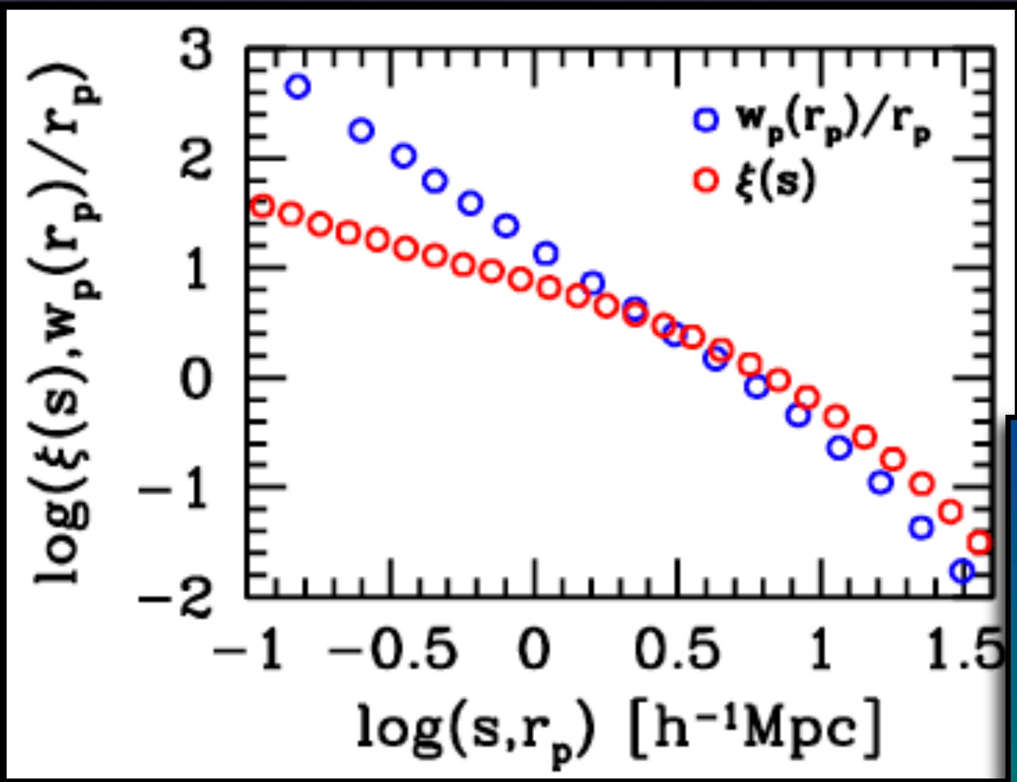
$$\xi(r) = -\frac{1}{\pi}\int_r^\infty \frac{\mathrm{d}w_\mathrm{p}}{\mathrm{d}r_\mathrm{p}} \frac{\mathrm{d}r_\mathrm{p}}{(r_\mathrm{p}^2 - r^2)^{1/2}}$$

One can infer the real-space correlation function from the projected correlation function

In particular, if the projected correlation function is a power-law, $w_\mathrm{p}(r_\mathrm{p}) = A\, r_\mathrm{p}^{1-\gamma}$ then the real-space correlation function is also a power law, $\xi(r) = (r/r_0)^{-\gamma}$, with

$$r_0^\gamma = \frac{A\,\Gamma(\gamma/2)}{\Gamma(1/2)\,\Gamma([\gamma-1]/2)}$$

$\Gamma(x)$ = Gamma function



This figure show both the projected and the redshift-space correlation functions obtained from the 2dFGRS. Note how the redshift space correlation function overestimates the correlation power on large scales due to Kaiser effect, and underestimates the power on small scales due to Finger-of-God effect.

Modeling Redshift Space Distortions

Let $n(\vec{r})$ denote the number density of galaxies in real space
and $n^{(s)}(\vec{s})$ denote the number density of galaxies in redshift space

Conservation of particle number implies that $n^{(s)}(\vec{s})\,\mathrm{d}^3\vec{s} = n(\vec{r})\,\mathrm{d}^3\vec{r}$ and thus

$$\boxed{1 + \delta^{(s)}(\vec{s}) = [1 + \delta(\vec{r})\left|\left|\frac{\mathrm{d}\vec{s}}{\mathrm{d}\vec{r}}\right|\right|^{-1}}$$

Using that $\vec{s} = \vec{r} + v_r\,\hat{r}$ one can show that $\delta^{(s)}_{\vec{k}} = \left(1 + \beta\,\mu^2_{\vec{k}}\right)\,\delta_{\vec{k}}$ (see MBW §6.3.1 for derivation)

Here we have defined the parameter $\boxed{\beta = \frac{1}{b}\,\frac{\mathrm{d}\ln D}{\mathrm{d}\ln a}}$

with $D(a)$ the linear growth rate, b the bias of the galaxies in consideration, and $\mu_{\vec{k}}$ the cosine of the angle between $\vec{k}$ and the line-of-sight.

Since the linear growth rate is a function of the matter density, this is often written as

$$\boxed{\beta = \frac{f(\Omega_\mathrm{m})}{b} \simeq \frac{\Omega_\mathrm{m}^{0.6}}{b}}$$

Modeling Redshift Space Distortions

We have that the power spectrum in redshift space is related to that in real-space according to

$$\boxed{P^{(s)}(\vec{k}) = \left[1 + \beta\,\mu_{\vec{k}}^2\right]^2 \; P(k)}$$

Note that $P^{(s)}(\vec{k})$ is anisotropic, while $P(\vec{k}) = P(k)$ is not.

Expanding $P^{(s)}(\vec{k})$ in harmonics of $\mu_{\vec{k}}$ we can write that

see MBW §6.3.1 for derivation

$$\xi_{\rm lin}(r_{\rm p}, r_\pi) = \xi_0(s)\,\mathcal{P}_0(\mu) + \xi_2(s)\,\mathcal{P}_2(\mu) + \xi_4(s)\,\mathcal{P}_4(\mu)$$

monopole $$\xi_0(s) = \left(1 + \frac{2}{3}\beta + \frac{1}{5}\beta^2\right)\,\xi(s)$$

quadrupole $$\xi_2(s) = \left(\frac{4}{3}\beta + \frac{4}{7}\beta^2\right)\,\left[\xi(s) - \frac{3}{s^3}J_3(s)\right]$$

$$J_l(s) = \int_0^s \xi(x)\,x^{l-1}\,{\rm d}x$$

hexadecapole $$\xi_4(s) = \frac{8}{35}\beta^2\,\left[\xi(s) + \frac{15}{2s^3}J_3(s) - \frac{35}{2s^5}J_5(s)\right]$$

Here $\mu = r_\pi / s$ is the cosine of the angle between $\vec{s}$ and the line-of-sight, $s = (r_{\rm p}^2 + r_\pi^2)^{1/2}$, and $\mathcal{P}_l(x)$ is the $l^{\rm th}$ order Legendre polynomial.

We have made it explicit that this equation is only valid in the linear regime...

Modeling Redshift Space Distortions

Given a value for β and the real-space correlation function, $\xi(r)$, which can be obtained from $\xi(r_\mathrm{p}, r_\pi)$ via the projected correlation function, $w_\mathrm{p}(r_\mathrm{p})$, the above equation yields a model for $\xi(r_\mathrm{p}, r_\pi)$ on linear scales that takes proper account of the coupling between the density and velocity fields.

Note that this model only accounts for linear motions, i.e., the Kaiser effect.

To model the non-linear virialized motions of galaxies one can convolve $\xi_\mathrm{lin}(r_\mathrm{p}, r_\pi)$ with the distribution function of pairwise peculiar velocities, $f(v_{12}|r)$:

$$\boxed{1 + \xi(r_\mathrm{p}, r_\pi) = \int_{-\infty}^{\infty} [1 + \xi_\mathrm{lin}(r_\mathrm{p}, r_\pi)] \; f(v_{12}|r) \, \mathrm{d}v_{12}}$$

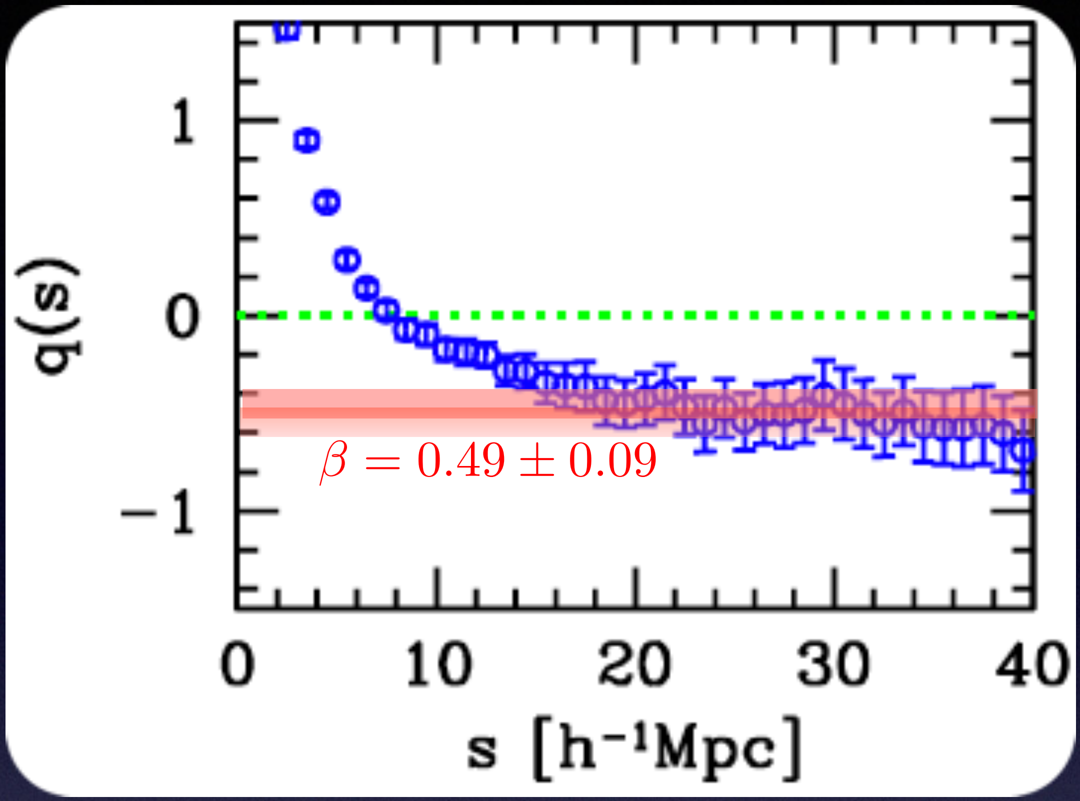
Unfortunately, the form of $f(v_{12}|r)$ is not known a priori...

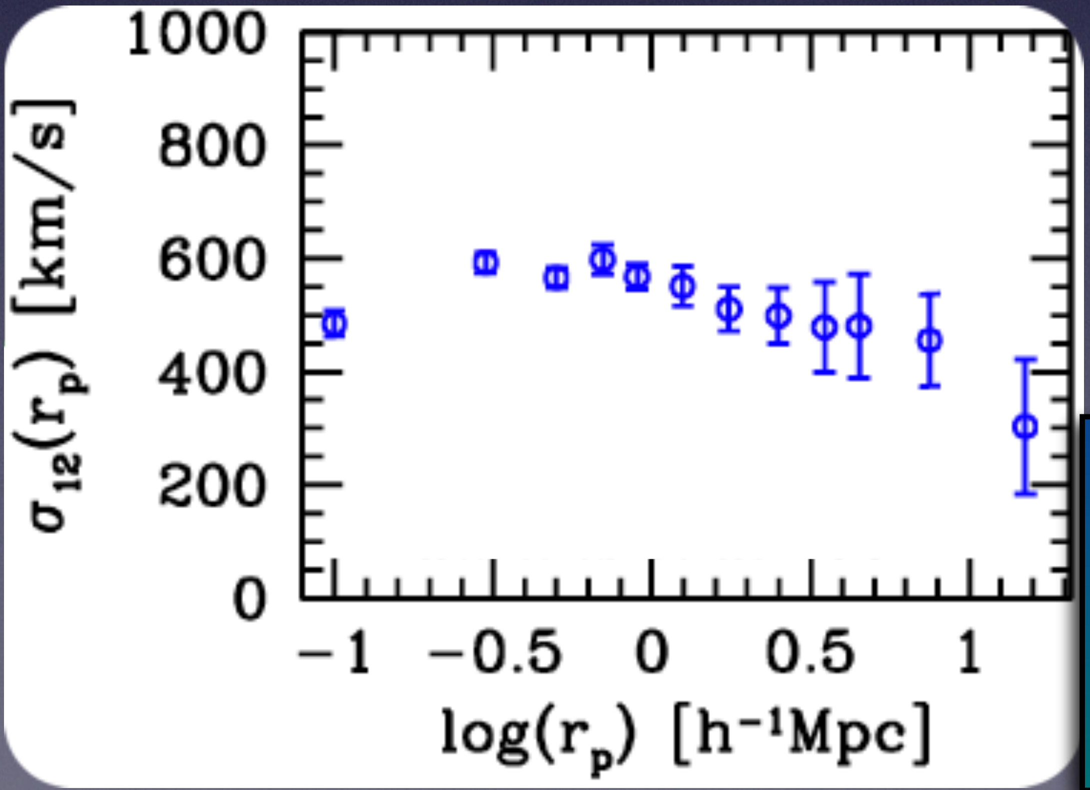
Based on theoretical considerations one often adopts an exponential form

$$\boxed{f(v_{12}|r) = \frac{1}{\sqrt{2}\,\sigma_{12}(r)} \exp\left[-\frac{\sqrt{2}\,|v_{12}|}{\sigma_{12}(r)}\right]}$$

By fitting the above model for $\xi(r_\mathrm{p}, r_\pi)$ to the data, one can constrain both β as well as the peculiar pairwise velocity dispersion, $\sigma_{12}(r)$

Modeling Redshift Space Distortions





The best way to measure $\beta = f(\Omega_{\rm m})/b$ is via the quadrupole-to-monopole ratio

$$q(s) \equiv \frac{\xi_2(s)}{(3/s^3)\int_0^s \xi_0(s')\, s'^2\, {\rm d}s' - \xi_0(s)}$$

where $\xi_l(s) = \frac{2l+1}{2}\int_{-1}^{+1} \xi(r_{\rm p}, r_\pi)\, \mathcal{P}_l(\mu)\, {\rm d}\mu$

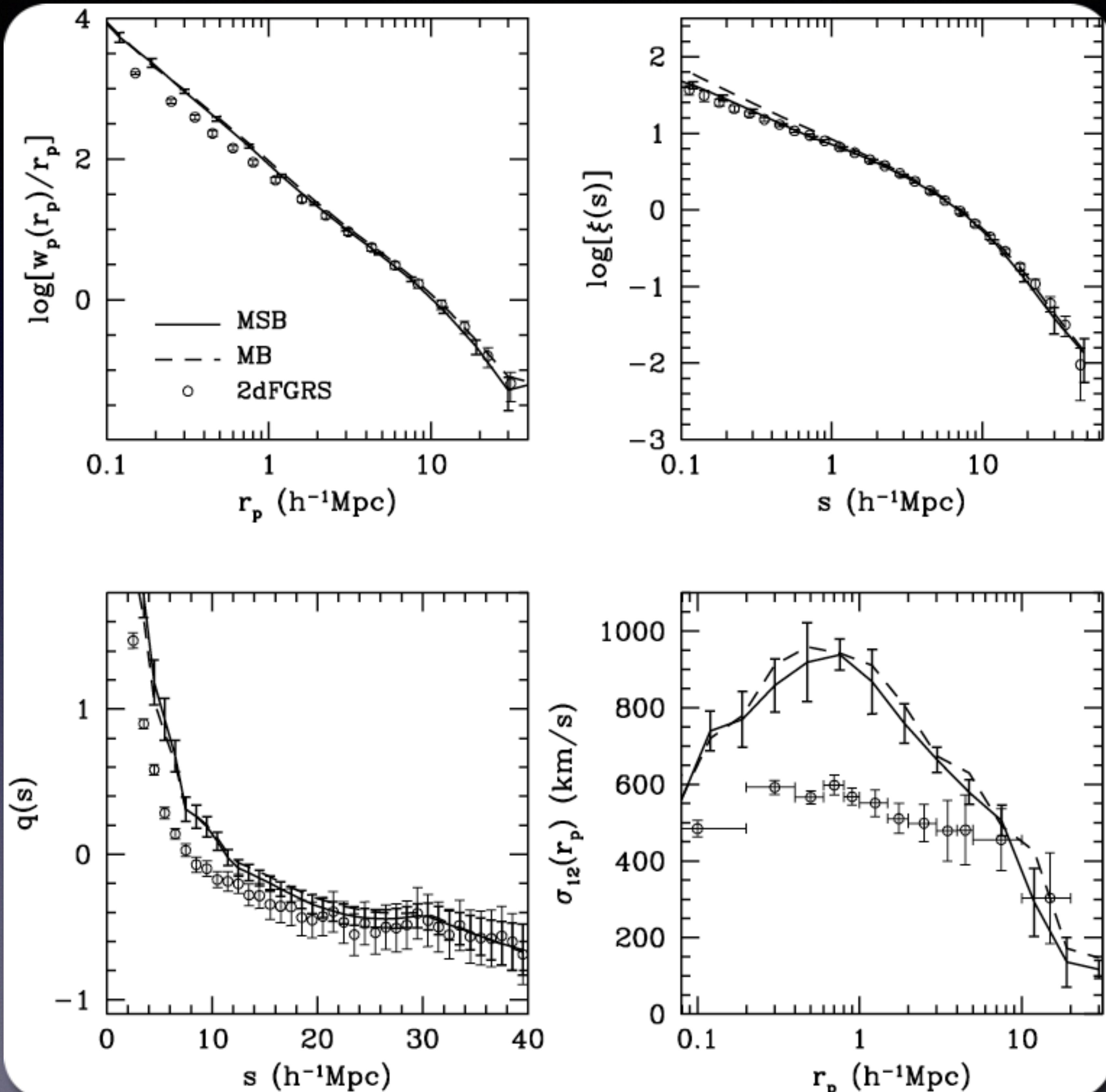
In the linear regime, one has that

$$q(s) = \frac{-(4/3)\beta - (4/7)\beta^2}{1 + (2/3)\beta + (1/5)\beta^2}$$

β follows directly from asymptotic value of $q(s)$

Figures show quadrupole-to-monopole ratio and pairwise velocity dispersions obtained from 2dFGRS by Hawkins et al. (2003). The former indicates that $\beta = 0.49 \pm 0.09$ while the latter shows that galaxies separated by ~1 Mpc/h (10 Mpc/h) have a 1D pairwise speed of ~600 (500) km/s

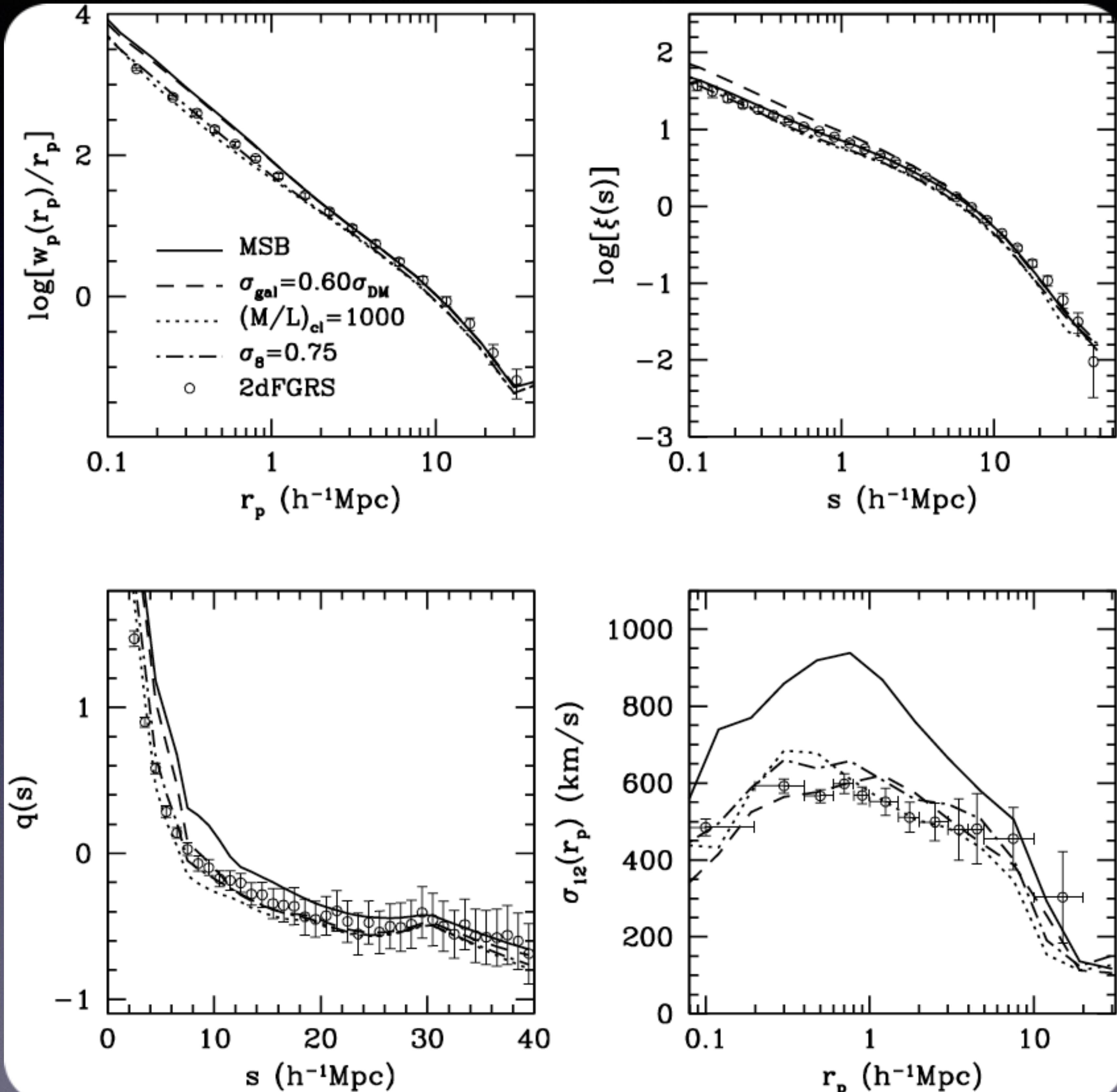
Modeling Redshift Space Distortions



In 2004, Yang et al. (2004) used a Conditional Luminosity Function (CLF) to populate dark matter haloes in a ΛCDM simulation for the WMAP1 cosmology. These were used to construct mock versions of the 2dFGRS, from which clustering was measured.

A comparison with clustering data from the true 2dFGRS from Hawkins et al. (2003) revealed problems with clustering on small scales and with the pairwise velocity dispersions....

Modeling Redshift Space Distortions

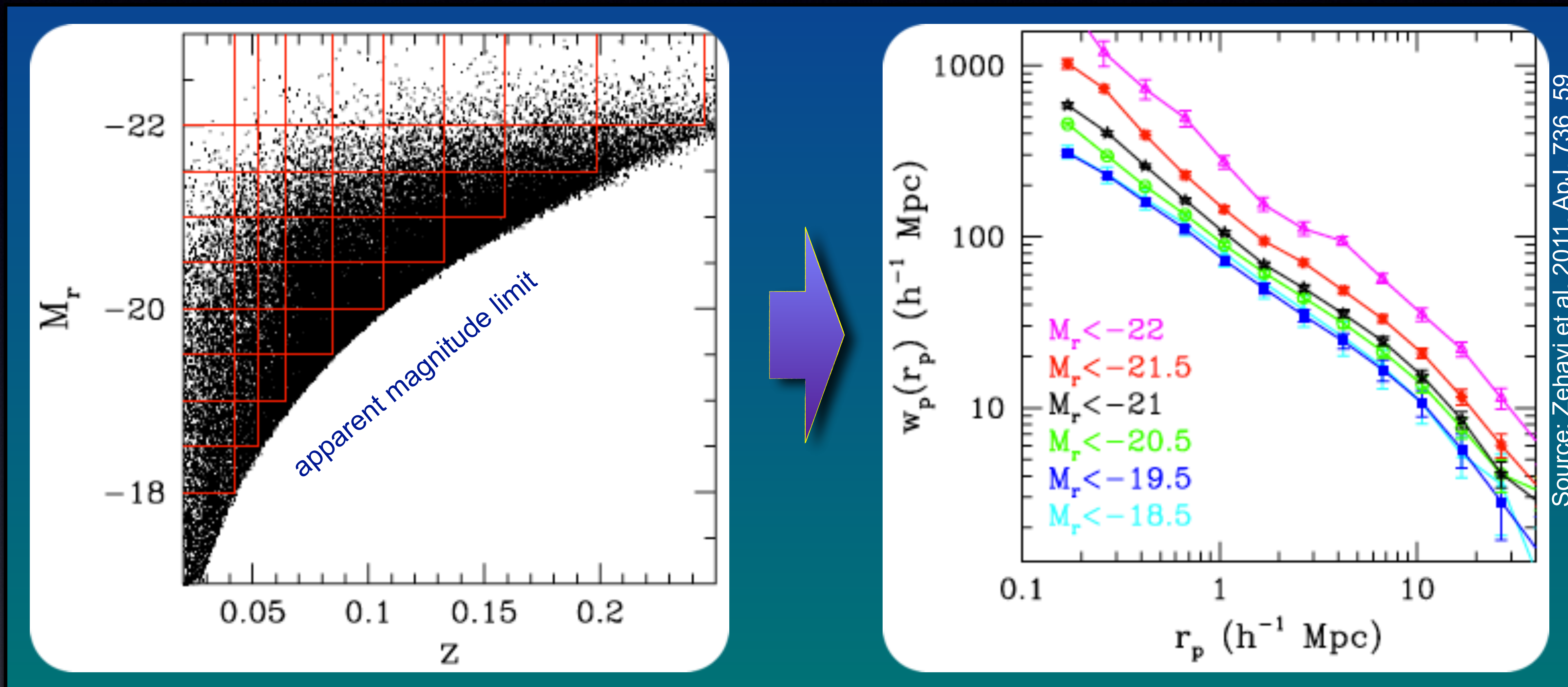


Yang et al. (2004) argued that this implied either (i) that clusters have a mass-to-light ratio (in the b_J-band) of ~1000 $M_\odot/L_\odot$ (~3x higher than what several methods suggested), or (ii) that $\sigma_8 \simeq 0.75$, rather than the then favored 0.9 .

One year later the 3rd year data release from the WMAP mission largely confirmed that $\sigma_8 \simeq 0.75$.

Luminosity Dependence of Clustering

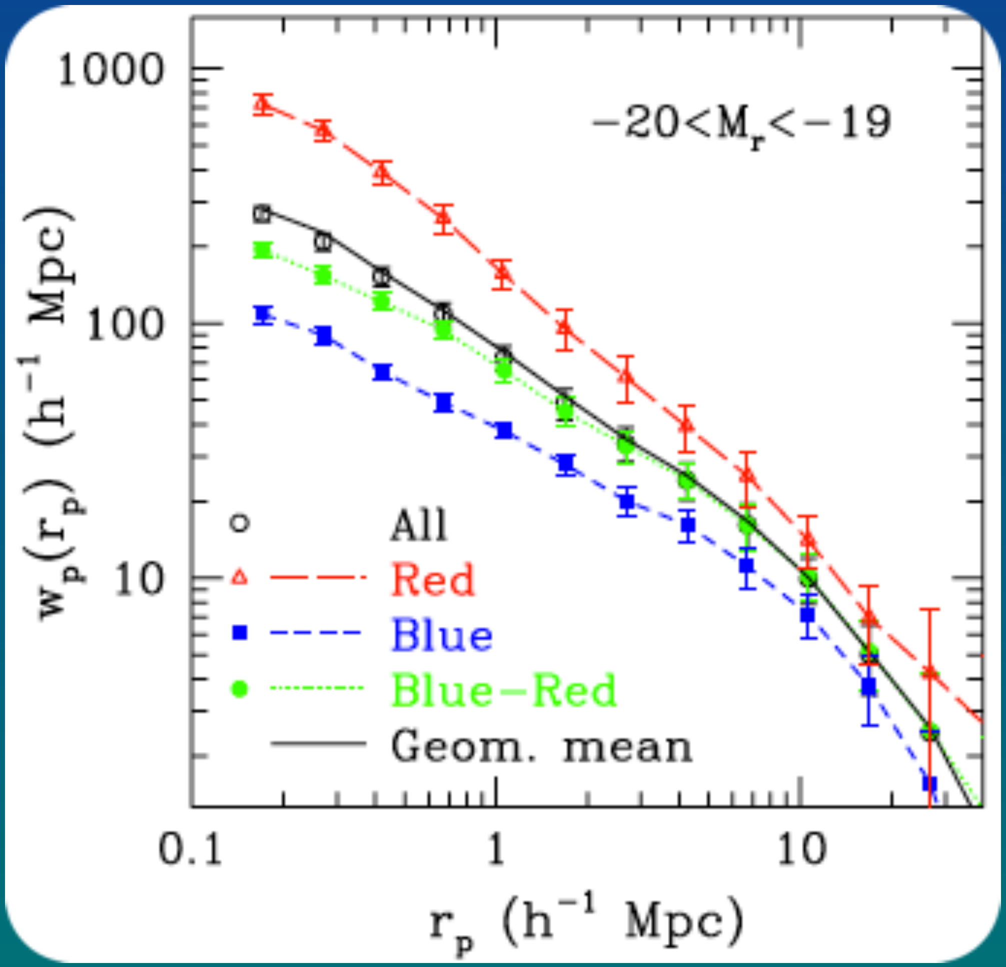
Using volume limited samples selected from the SDSS, Zehavi et al. (2011) measured the projected correlation functions for different luminosity bins.

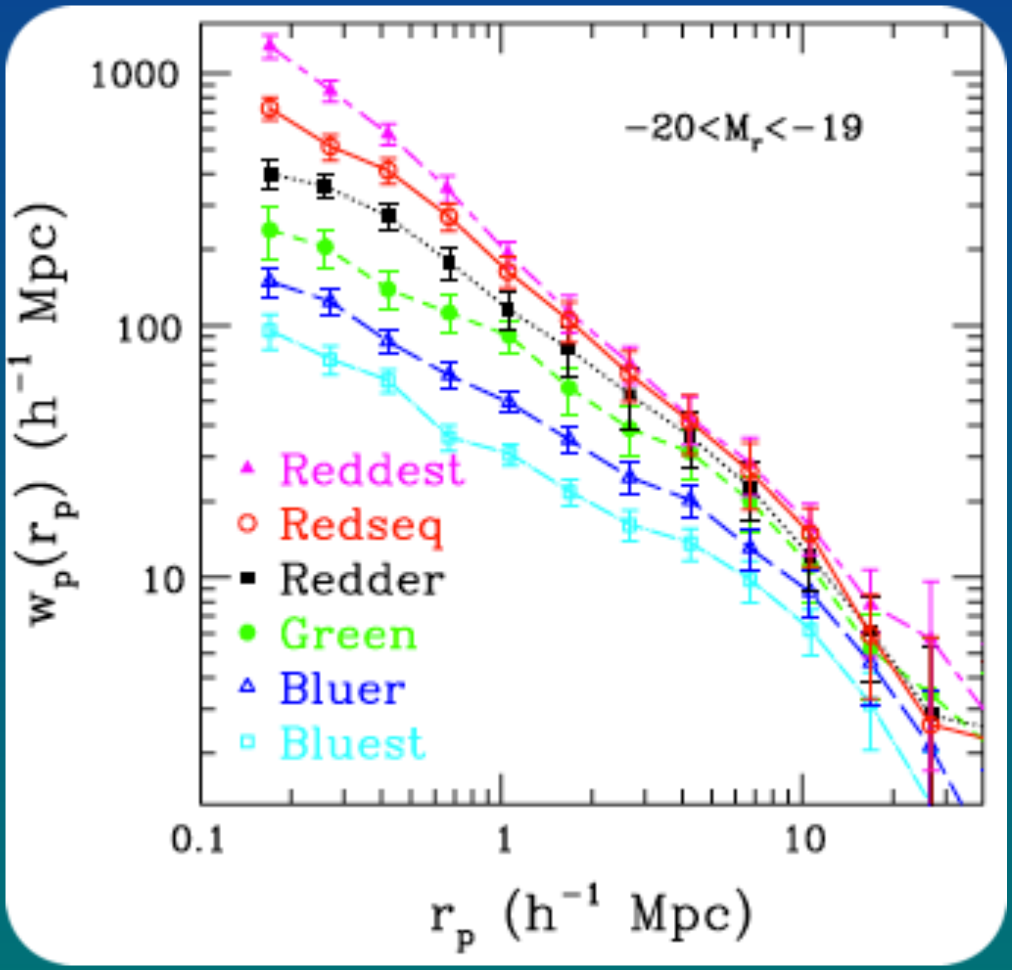


More massive galaxies are more strongly clustered.

Color Dependence of Clustering

Zehavi et al. (2011) also split the different luminosity bins in red and blue subsamples, and computed their projected correlation functions..



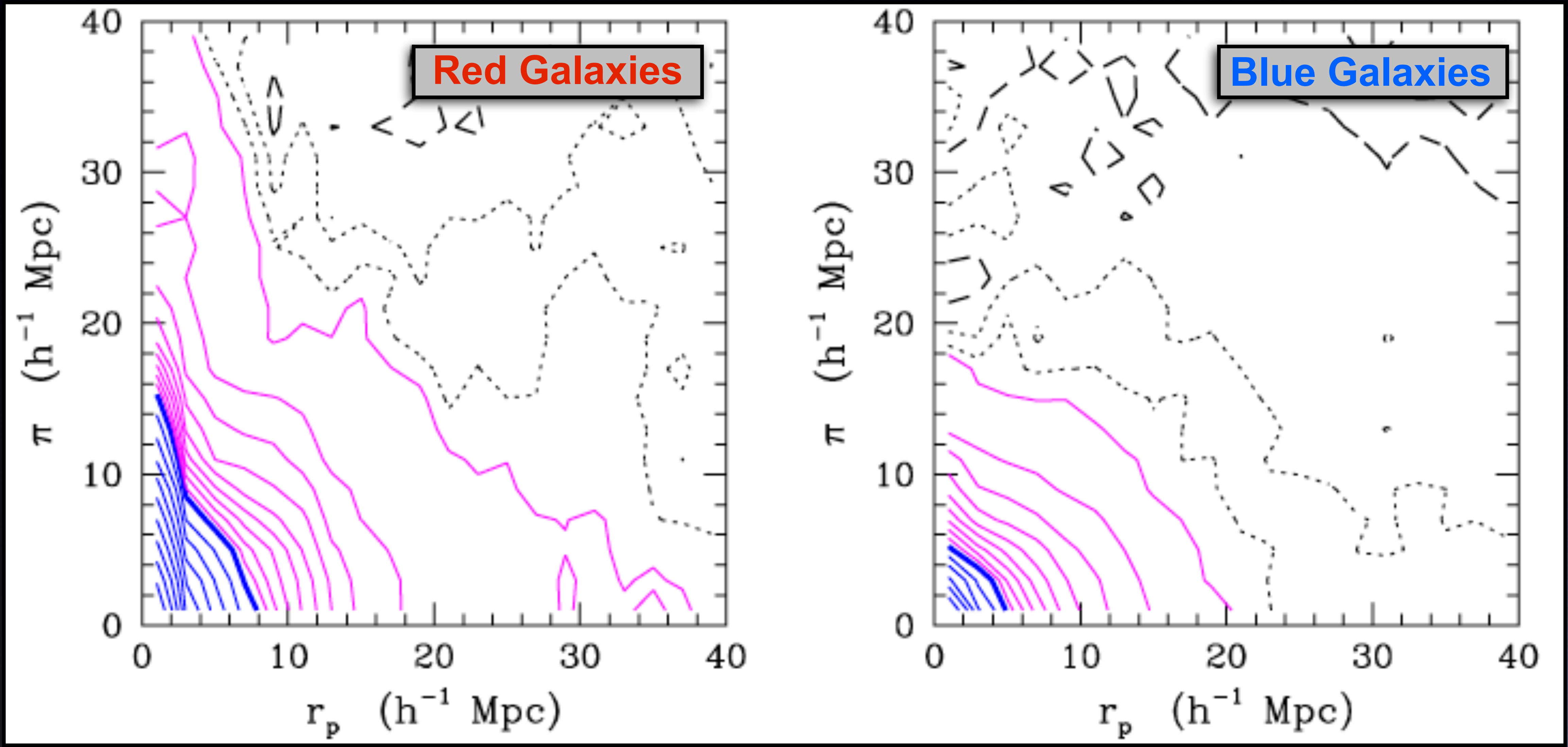


Source: Zehavi et al. 2011, ApJ, 736, 59

Redder galaxies are more strongly clustered...

Color Dependence of Clustering

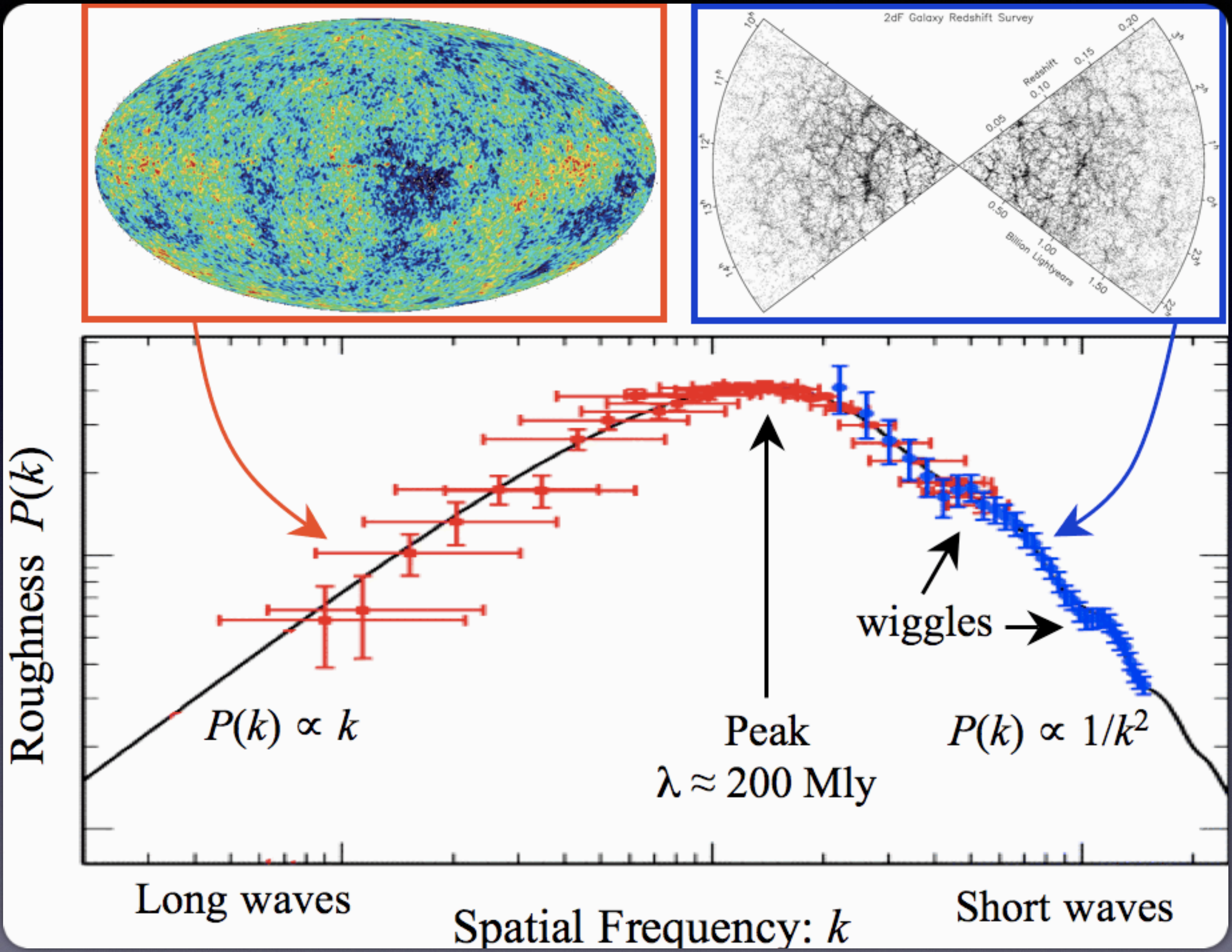
Zehavi et al. (2011) also split the different luminosity bins in red and blue subsamples, and computed their projected correlation functions..



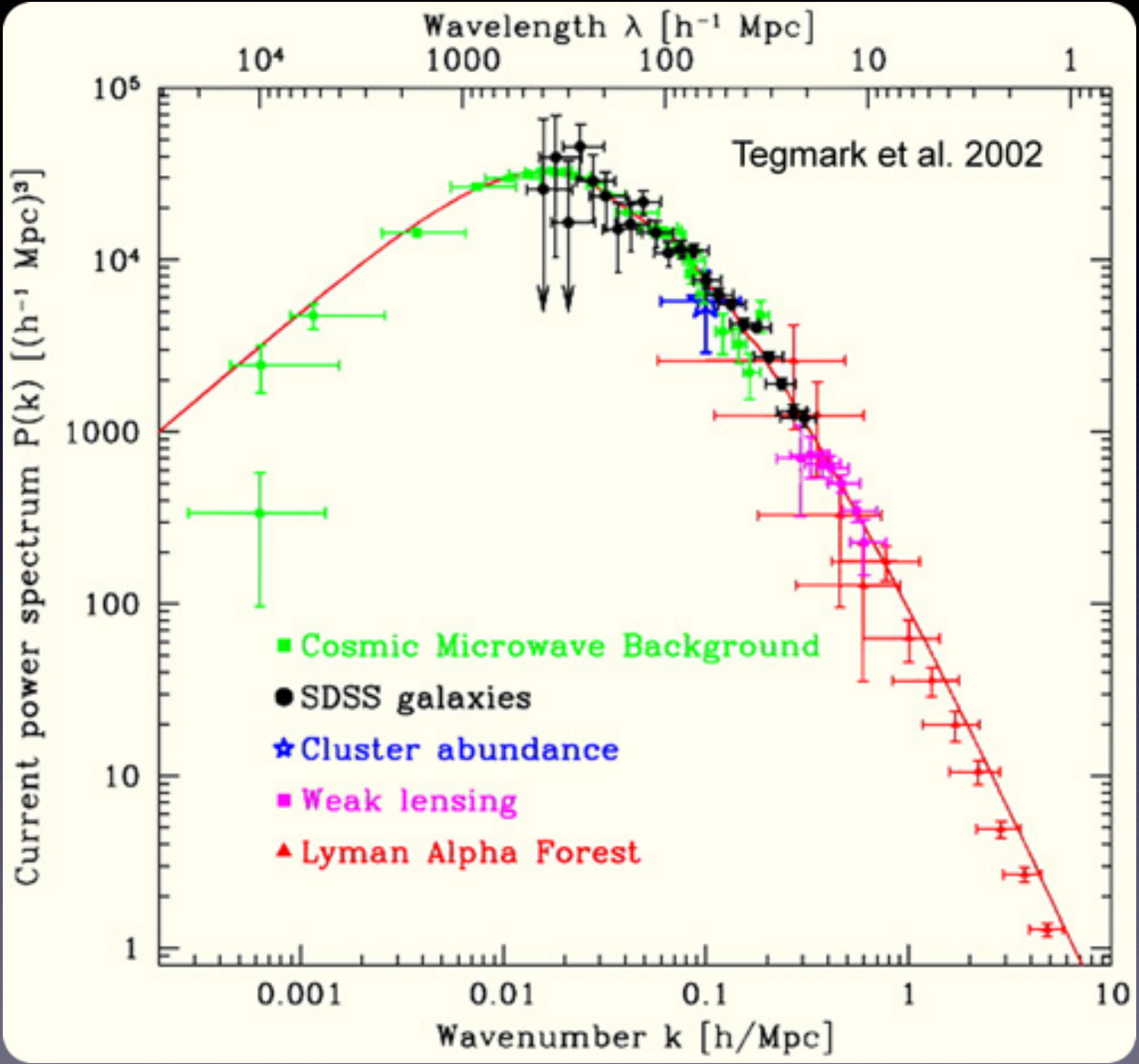
Source: Zehavi et al. 2011, ApJ, 736, 59

Redder galaxies also show more pronounced fingers of God..

The Galaxy Power Spectrum



The Matter Power Spectrum

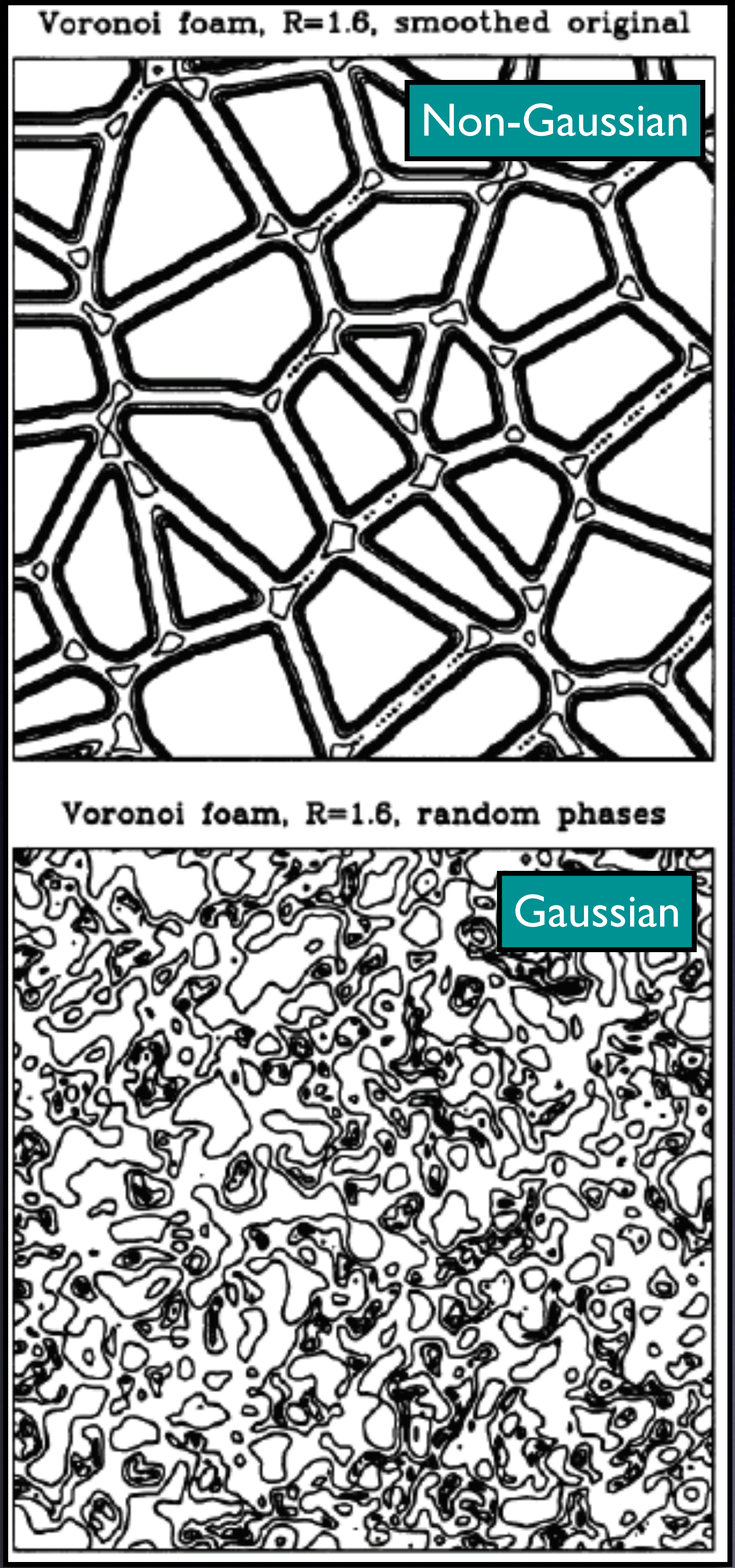


Phase Information

RECALL: once structure formation has gone non-linear, the power spectrum no longer suffices to completely describe the cosmological density field.

In particular, the power spectrum alone does not capture the phase information: the coherence of cosmic structures such as pancakes, filaments, voids etc.

This is illustrated in the figure to the right, which shows two density distributions that have identical power spectra, but very different phases for the corresponding modes....As is evident the eye is very sensitive to phase information....



These two images have identical power spectra, by construction!

Lecture 12

SUMMARY

Summary: key words & important facts

Key words	
reduced/irreducible corr fnc	projected correlation function
Poisson sampling	Redshift space distortions
Wiener-Khinchin theorem	Kaiser effect
Limber equation	Finger-of-God effect

- The reduced (or irreducible) correlation functions express the part of the n-point correlation functions that cannot be obtained from lower-order correlation functions
- For a Gaussian random field, all connected moments (=reduced correlation functions) of $n > 2$ are equal to zero (i.e., $\zeta = \eta = 0$).

  → One can use ζ and η to test whether the density field is Gaussian or not...
- If galaxy formation is a Poisson sampling of the density field, then all n-point correlation functions of the galaxy distribution are identical to those of the matter distribution

  This is not the case though; galaxies are biased tracers of the mass distribution
- On large (linear) scales, redshift space distortions (RSDs) depend on linear growth rate. On small (non-linear) scales, RSDs reveal FoG indicative of virial motion within halos
- Redder and more massive/luminous galaxies are more strongly clustered

Summary: key equations & expressions

n-point correlation function

$$\xi^{(n)} \equiv \langle \delta_1\, \delta_2 \ldots \delta_n \rangle$$

n-point irreducible correlation function

$$\xi^{(n)}_{\rm red} \equiv \langle \delta_1\, \delta_2 \ldots \delta_n \rangle_{\rm c}$$

$$\xi(r) = \frac{DD(r)\,\Delta r}{RR(r)\,\Delta r} - 1$$ 2-pt function (discrete)

$$w(\theta) = \frac{DD(\theta)\,{\rm d}\theta}{RR(\theta)\,{\rm d}\theta} - 1$$ angular 2-pt (discrete)

$$P(k) \equiv V\,\langle |\delta_{\vec{k}}|^2 \rangle = P_{\rm gg}(k) + \frac{1}{\bar{n}}$$ power spectrum (discrete)

projected correlation function

$$w_{\rm p}(r_{\rm p}) = \int_{-\infty}^{\infty} \xi(r_{\rm p}, r_\pi)\,{\rm d}r_\pi = 2\int_{r_{\rm p}}^{\infty} \xi(r)\,\frac{r\,{\rm d}r}{(r^2 - r_{\rm p}^2)^{1/2}}$$

$$\xi(r) = -\frac{1}{\pi}\int_r^{\infty} \frac{{\rm d}w_{\rm p}}{{\rm d}r_{\rm p}}\,\frac{{\rm d}r_{\rm p}}{(r_{\rm p}^2 - r^2)^{1/2}}$$

Limber equation

$$w(\theta) = \int_0^{\infty} {\rm d}y\, y^4\, S^2(y) \int_{-\infty}^{\infty} {\rm d}x\, \xi(\sqrt{x^2 + y^2\theta^2})$$

redshift space distortions

$$P^{(s)}(\vec{k}) = \left[1 + \beta\,\mu_{\vec{k}}^2\right]^2\, P(k) \qquad \beta = \frac{1}{b}\,\frac{{\rm d}\ln D}{{\rm d}\ln a} = \frac{f(\Omega_{\rm m})}{b} \simeq \frac{\Omega_{\rm m}^{0.6}}{b}$$

$$1 + \xi(r_{\rm p}, r_\pi) = \int_{-\infty}^{\infty} [1 + \xi_{\rm lin}(r_{\rm p}, r_\pi)]\; f(v_{12}|r)\,{\rm d}v_{12}$$