Problem 1: The 1D Linear Advection Equation

Write a computer program to numerically integrate the 1D linear advection equation. Call the program ‘LinearAdvection’, so we can refer to it at times. Set up a linear 1D grid \( x_i \) using 100 cells that cover the domain \( x \in [0, 1] \). Use a constant advection speed \( v = 1.0 \) (arbitrary units). As input the program should read the Courant parameter, \( \alpha_c \equiv v \Delta t / \Delta x \), from which the time-step to be used is to be determined. In addition, the program should allow for the following FDA update schemes:

1. Forward-Time Backward-Space (FTBS) upwind scheme
2. Lax-Friedrich (LX) scheme
3. Lax-Wendroff (LW) scheme
4. Beam-Warming (BW) scheme

The initial conditions (IC) to be considered in this problem set are the same as those used in the lectures notes, i.e., \( u(x, 0) = 1.0 \) for \( 0.4 \leq x \leq 0.6 \) and \( u(x, 0) = 0.0 \) for all other \( x \).

a) [8 points] Integrate the ICs forward in time for a time \( T = 1.0 \) using periodic boundary conditions and a Courant parameter \( \alpha_c = 0.1 \) for all four schemes listed above. Show the results and check that they reproduce those shown in Fig. 30 of the lecture notes (Chapter 17).

b) [8 points] Now use the FTCS, LF, LW and BW schemes to integrate for \( T = 3.0 \) using \( \alpha_c = 0.1, 0.8 \) and 1.1. Show the results and discuss
the outcome (explain the dependencies on $\alpha_c$). For the latter, make use of 

stability analysis and/or modified equations as discussed in Chapter 18.

c) [4 points] The second-order backwards space finite difference scheme approximates the first-order spatial derivative as

$$u_i' = \frac{3u_i - 4u_{i-1} + u_{i-2}}{2\Delta x}$$

Use Taylor series to derive an expression for the truncation error of this scheme.

d) [5 points] Implement this second-order backward space difference scheme in LinearAdvection and integrate for $T = 0.2$. Experiment with different values of $\alpha_c$ and compare this to the results obtained using the first-order accurate FTBS scheme. Discuss the results.

e) [5 points] Implement this fourth-order finite difference scheme which approximates the first order spatial derivative as

$$u_i' = \frac{-u_{i+1} + 8u_{i+1} - 8u_{i-1} + u_{i-1}}{12\Delta x}$$

and compare its outcomes at $T = 0.2$ to that of the second-order FTCS scheme for different values of $\alpha_c$. Discuss the results.

**Problem 2: The Heat Equation**

Consider the linear heat conduction equation in 1D given by

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2}$$

with $\kappa > 0$ a constant conduction coefficient. For this problem we will set $\kappa = 1$. Using a second-order accurate FDA scheme we obtain the following update formula for this equation:

$$u_i^{n+1} = u_i^n + \eta(u_{i+1}^n - 2u_i^n + u_{i-1}^n)$$
with \( \eta \equiv \kappa \Delta t / (\Delta x)^2 \).

**a)** [6 points] Use a von Neumann stability analysis to compute the amplification factor

\[
A_{FDA} \equiv \frac{U^{n+1}_k}{U^n_k}
\]

Here \( U^n_k \) is the Fourier component of the mode with wavenumber \( k \), which is related to the mesh function \( u^n_i \) as

\[ u^n_i = U^n_k e^{ikx_i} \]

Similarly,

\[ u^{n+1}_i = U^n_k e^{ik(x_i + \Delta x)} \]

etc. Note that because the heat equation is linear there will be no mode coupling, and we can thus consider the response of a single mode. Based on the expression for \( A_{FDA} \) derived, discuss the stability criterion for this FDA scheme.

**b)** [4 points] Substitute \( u(x) = U_k e^{ikx} \) in the heat equation and show that the amplification factor inferred directly from the PDE is given by

\[
A_{PDE} = e^{-\kappa k^2 \Delta t} = e^{-\eta (k\Delta x)^2}
\]

c) [6 points] Plot \( A_{FDA} \) and \( A_{PDE} \) as a function of \( k\Delta x \) (from 0 to \( \pi \)). Show such plots for three different values of \( \eta \): 0.25, 0.5 and 1.5. Discuss when and where the PDE and FDA are stable. What kind of perturbations (numerical noise) will go unstable first; those on large or small scales? Explain your answer.

d) [8 points] Write a program ‘HeatEquation’ to integrate the heat equation using the FDA scheme listed above starting from the same ICs as in problem 1. Use a spatial mesh of 64 grid points covering the domain \( x \in [0, 1] \). Adopt boundary conditions such that \( u(x) = 0 \) for \( x < 0 \) and \( x > 1 \). Show the results at \( T = 0.01 \) and \( T = 0.05 \) using \( \eta = 0.4 \) and \( \eta = 0.501 \). Discuss the results in light of the stability analysis performed above.
e) [6 points] Now consider the following update formula for the heat equation, which is based on a time-implicit FDA:

\[ u_{n+1}^i = u_i^n + \eta(u_{n+1}^{i+1} - 2u_i^{n+1} + u_{n+1}^{i-1}) \]

Note that this differs from the time-explicit FDA used above in that the spatial second-order derivative is now evaluated at time \( t_{n+1} \) rather than \( t_n \). Use the same von Neumann stability analysis as under (a) to compute the amplification factor \( A_{FDA} \) corresponding to this time-implicit FDA. Plot \( A_{FDA} \) and \( A_{PDE} \) as functions of \( k\Delta x \) (from 0 to \( \pi \)) for \( \eta = 0.25, 0.5 \) and 10.0. Discuss the implications of your findings for stability and accuracy of this implicit scheme.