**ASTR 595: Problem Set 3**

**This problem set consists of 2 problems for a total of 60 points.**
**Due date: friday Apr 3 noon, (via e-mail)**

*In this problem set, you will write computer programs, in a language of your choice, to integrate the 1D linear advection equation and the heat equation. You will suplement this with stability analysis to develop understanding of the behaviour of the Finite Difference Approximations (FDA) used.*

**Problem 1: The 1D Linear Advection Equation**

Write a computer program to numerically integrate the 1D linear advection equation. Call the program 'LinearAdvection', so we can refer to it at times. Set up a linear 1D grid $x_i$ using 100 cells that cover the domain $x \in [0, 1]$. Use a constant advection speed $v = 1.0$ (arbitrary units). As input the program should read the Courant parameter, $\alpha_c \equiv v\Delta t/\Delta x$, from which the time-step to be used is to be determined. In addition, the program should allow for the following FDA update schemes:

1. Forward-Time Backward-Space (FTBS) upwind scheme
2. Lax-Friedrich (LX) scheme
3. Lax-Wendroff (LW) scheme
4. Beam-Warming (BW) scheme

The initial conditions (IC) to be considered in this problem set are the same as those used in the letures notes, i.e., $u(x, 0) = 1.0$ for $0.4 \leq x \leq 0.6$ and $u(x, 0) = 0.0$ for all other $x$.

**a)** [`8 points`] Integrate the ICs forward in time for a time $T = 1.0$ using periodic boundary conditions and a Courant parameter $\alpha_c = 0.1$ for all four schemes listed above. Show the results and check that they reproduce those shown in Fig. 30 of the lecture notes (Chapter 17).

**b)** [`8 points`] Now use the FTCS, LF, LW and BW schemes to integrate for $T = 3.0$ using $\alpha_c = 0.1$, 0.8 and 1.1. Show the results and discuss the outcome (explain the dependencies on $\alpha_\mathrm{c}$). For the latter, make use of stability analysis and/or modified equations as discussed in Chapter 18.

**c)** [`4 points`] The second-order backwards space finite difference scheme approximates the first-order spatial derivative as

$$u_i' = \frac{3u_i - 4u_{i-1} + u_{i-2}}{2\Delta x}$$

Use Taylor series to derive an expression for the truncation error of this scheme.

**d)** [`5 points`] Implement this second-order backward space difference scheme in LinearAdvection and integrate for $T = 0.2$. Experiment with different values of $\alpha_\mathrm{c}$ and compare this to the results obtained using the first-order accuracte FTBS scheme. Discuss the results.

**e)** [`5 points`] Implement this fourth-order finite difference scheme which approximates the first order spatial derivative as

$$u_i' = \frac{-u_{i+1} + 8u_{i+1} - 8u_{i-1} + u_{i-2}}{12\Delta x}$$

and compare its outcomes at $T = 0.2$ to that of the second-order FTCS scheme for different values of $\alpha_\mathrm{c}$. Discuss the results.

**Problem 2: The Heat Equation**

Consider the linear heat conduction equation in 1D given by

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2}$$

with $\kappa > 0$ a constant conduction coefficient. For this problem we will set $\kappa = 1$. Using a second-order accurate FDA scheme we obtain the following update formula for this equation:

$$u_i^{n+1} = u_i^n + \eta(u_{i+1}^n - 2u_i^n + u_{i-1}^n)$$

with $\eta \equiv \kappa \Delta t / (\Delta x)^2$.

**a)** [`6 points`] Use a von Neumann stability analysis to compute the amplification factor

$$A_{\rm FDA} \equiv \frac{U_k^{n+1}}{U_k^n}$$

Here $U_k^n$ is the Fourier component of the mode with wavenumber $k$, which is related to the mesh function $u_i^n$ as

$$u_i^n = U_k^n \mathrm{e}^{ikx_i}$$

Similarly,

$$u_{i+1}^n = U_k^n \mathrm{e}^{ik(x_i+\Delta x)}$$

etc. Note that because the heat equation is linear there will be no mode coupling, and we can thus consider the response of a single mode. Based on the expression for $A_{\rm FDA}$ derived, discuss the stability criterion for this FDA scheme.

**b)** [`4 points`] Substitute $u(x) = U_k \mathrm{e}^{ikx}$ in the heat equation and show that the amplification factor inferred directly from the PDE is given by

$$A_{\rm PDE} = \mathrm{e}^{-\kappa\, k^2\, \Delta t} = \mathrm{e}^{-\eta (k\Delta x)^2}$$

**c)** [`6 points`] Plot $A_{\rm FDA}$ and $A_{\rm PDE}$ as a function of $k\Delta x$ (from 0 to $\pi$). Show such plots for three different values of $\eta$: 0.25, 0.5 and 1.5. Discuss when and where the PDE and FDA are stable. What kind of perturbations (numerical noise) will go unstable first; those on large or small scales? Explain your answer.

**d)** [`8 points`] Write a program 'HeatEquation' to integrate the heat equation using the FDA scheme listed above starting from the same ICs as in problem 1. Use a spatial mesh of 64 grid points covering the domain $x \in [0, 1]$. Adopt boundary conditions such that $u(x) = 0$ for $x < 0$ and $x > 1$. Show the results at $T = 0.01$ and $T = 0.05$ using $\eta = 0.4$ and $\eta = 0.501$. Discuss the results in light of the stability analysis performed above.

**e)** [`6 points`] Now consider the following update formula for the heat equation, which is based on a *time-implicit* FDA:

$$u_i^{n+1} = u_i^n + \eta(u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1})$$

Note that this differs from the time-explicit FDA used above in that the spatial second-order derivative is now evaluated at time $t^{n+1}$ rather than $t^n$. Use the same von Neumann stability analysis as under **(a)** to compute the amplification factor $A_{\rm FDA}$ corresponding to this time-implicit FDA. Plot $A_{\rm FDA}$ and $A_{\rm PDE}$ as functions of $k\Delta x$ (from 0 to $\pi$) for $\eta = 0.25$, 0.5 and 10.0. Discuss the implications of your findings for stability <u>and</u> accuracy of this implicit scheme.