ASTR 595: Problem Set 2

This problem set consists of 4 problems for a total of 66 points.
Due date: monday Feb 24

In this problem set, you will use the hydrodynamic equations under a variety of conditions to solve particular problems in hydrodynamics. You will work out the inner workings of a venturi meter, examine vorticity in a thin disk, determine the conditions for a star to be unstable to radial pulsations, and work determine the Bondi accretion rate of the Milky Way's black hole.

Problem 1: The Venturi Meter

The venturi meter is used to measure the flow speed in a pipe. An example is shown in Fig. 1, where the venturi meter (indicated by the dashed lines) is placed in a pipe of diameter A_1. The venturi meter itself consists of a pipe of diameter A_2, as indicated. The pipe transports an incompressible fluid of density ρ with a flow velocity u_1 (this is the quantity to be meaured). The flow velocity in the narrow pipe is u_2. The main and narrow pipes are connected via a U-shaped pipe that is filled with a fluid of density $\tilde{\rho} > \rho$. The quantity to be measured is h, the difference in the heights of the colums of the dense fluid. Throughout you may assume that the fluid flowing through the venturi meter is ideal.

a) [`3 points`] Show that, for an ideal fluid, the specific enthalpy $h \equiv \varepsilon + P/\rho$ takes on the following form:

$$h = \frac{\gamma}{\gamma - 1} \frac{P}{\rho}$$

with γ the adiabatic index.

b) [`5 points`] Assume that the venturi meter of Fig. 1 is located in the Earth's gravitational field, with the direction of gravitational acceleration perpendicular to the flow direction. Express the pressure difference at points 1 and 2 in terms of u_1, u_2, and ρ. You can assume that the flow is laminar.

c) [`5 points`] Derive an expression for u_1 as function of h, the densities ρ and $\tilde{\rho}$, and the areas A_1 and A_2.

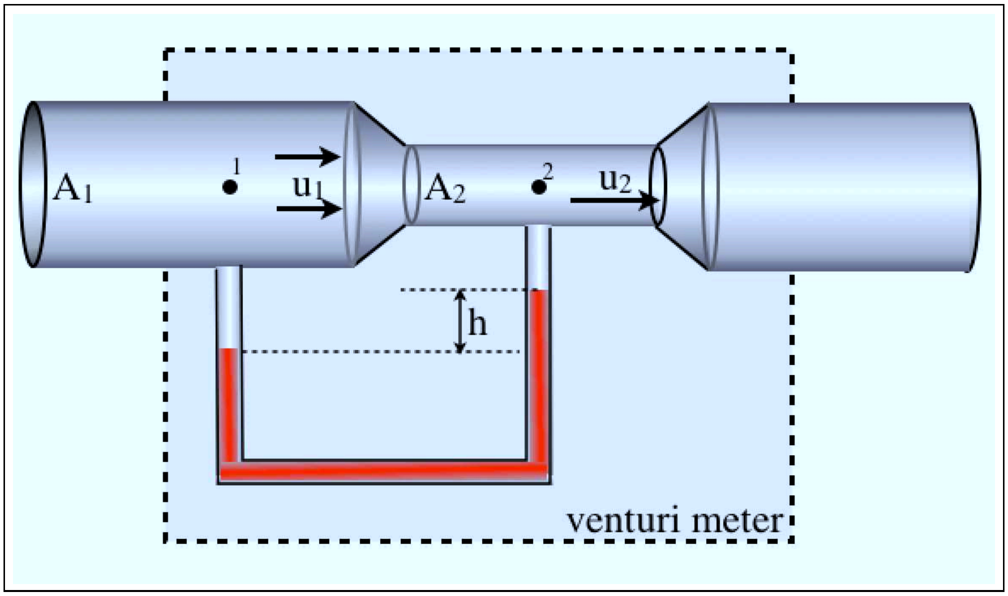

Figure 1: Illustration of a venturi meter.

Problem 2: Vorticity in a thin disk [`8 points`]

Consider an infinitessimally thin disk of fluid in rotation around the disk's (vertical) symmetry axis. It is given that the flow is symmetric around the same symmetry axis, and that the circulation around any curve C on the disk is zero, as long as the curve does not enclose the symmetry point of the disk, $R = 0$. Derive an expression for the velocity field $\vec{u}(\vec{x})$, i.e., what are the various components of $\vec{u}$ as function of location in the disk? Clearly explain your steps in this derivation.

Problem 3: Purely Radial Stellar Oscillations

Consider a spherical, barotropic star for which $P = K\rho^\gamma$. The goal is to derive conditions for γ under which the star is stable to radial oscillations. Suppose the star is *uniformly* expanded from an initial equilibrium configuration such that the position of a fluid element (or mass shell) changes from r_0 to $r_0(1+\delta)$. Throughout we shall assume that δ is small, such that we can use perturbation theory. From the Euler equation (i.e., ignoring viscosity) we can write down the acceleration of a fluid element at a distance r from the center of the star as

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r} - \frac{GM(r)}{r^2}$$

where v is the radial component of the velocity (we are considering purely radial motions here) and $M(r)$ is the mass enclosed within radius r.

a) [`6 points`] Use Taylor series expansion to show that, to linear order, the density of the perturbed mass shell obeys $\rho = \rho_0(1 - 3\delta)$.

b) [`5 points`] Using the same strategy, given a similar expression for P in terms of P_0, δ, and γ.

c) [`6 points`] Substitute the expressions for r, ρ and P in the expression for the radial acceleration, keeping only terms up to linear order, and derive for what values of γ the star will be stable to radial oscillations. Note: assume that the initial configuration was one of equilibrium, so that

$$-\frac{1}{\rho_0}\frac{\mathrm{d}P_0}{\mathrm{d}r_0} - \frac{GM(r_0)}{r_0^2} = 0$$

and <u>explain your answer</u>!

Problem 4: Bondi Accretion

Consider a homogeneous, ideal fluid of density ρ_∞ and with sound speed c_∞. If a spherical body of mass M (i.e., a star or a black hole) is placed in this medium (at $\vec{r}=0$), the gravity due to M will induce a flow towards it. Consequently, the mass M will start to accrete matter from its ambient medium. In what follows we assume that the flow that develops is steady and isothermal (i.e., it is assumed that the heating associated with the compression of the fluid is radiated away, such that the temperature of the accreting fluid remains constant).

a) [`4 points`] Show that the pressure potential $h = \int \mathrm{d}P/\rho$ for the isothermal accretion flow is equal to $c_\infty^2 \ln(\rho/\rho_\infty)$, where ρ is the density in the flow at the radius where h is evaluated.

b) [`4 points`] Use the continuity equation to show that $r^2\,\rho(r)\,u(r)$ is independent of the distance r from M, and show that this implies that

$$\frac{\mathrm{d}\ln\rho}{\mathrm{d}r} = -\frac{2}{r} - \frac{\mathrm{d}\ln u}{\mathrm{d}r}$$

Here $u(r) = |\vec{u}(\vec{r})|$ is the velocity of the accretion flow.

The above implies that the mass accretion rate $\dot{M} = 4\pi r^2 \rho u$ is also constant with radius (and, under the assumption of an infinite supply of ambient medium) and time.

c) [`5 points`] Starting from the momentum equation for the accretion flow in Eulerian form, show that

$$(u^2 - c_\mathrm{s}^2)\frac{\mathrm{d}\ln u}{\mathrm{d}r} = \frac{2c_\mathrm{s}^2}{r}\left[1 - \frac{GM}{2c_\mathrm{s}^2 r}\right]$$

`Hint:` Use the results derived under **b)**.

d) [`3 points`] The radius at which the fluid flow transits from being subsonic to being supersonic is called the sonic radius, r_s. Derive an expression for r_s is terms of M and c_s.

e) [`4 points`] Use the fact that in a steady, barotropic flow Bernoulli's constant is conserved along particle paths to show that

$$\rho_{\rm s} \equiv \rho(r_{\rm s}) = \rho_\infty {\rm e}^{1.5}$$

`Hint:` use that $\lim_{r\to\infty} u = 0$.

f) [`2 points`] Write down an expression for the accretion rate onto M as a function of ρ_∞, c_∞ and M.

g) [`6 points`] The center of the Milky Way harbors a black hole of mass $M \simeq 10^6 M_\odot$. Assuming the (infinite reservoir of) ambient gas has a number density $n = 1\,{\rm cm}^{-3}$, a mean molecular weight $\mu = 1$ (i.e., the mean particle mass is equal to the mass of a proton), and a temperature $T = 10^7$K, how long does it take for the black hole to double its mass due to steady, isothermal Bondi accretion of gas from its ambient medium? Show your derivation in detail and express your answer in units of the Hubble time $t_{\rm H} \simeq 10^{10}$ year, which is roughly the age of the Universe.

(Potentially) Useful Constants

Gravitional constant	G	$=$	$6.674 \times 10^{-8}\,{\rm cm}^3\,{\rm g}^{-1}\,{\rm s}^{-2}$
		$=$	$4.299 \times 10^{-9}\,{\rm Mpc}\,M_\odot^{-1}\,({\rm km/s})^2$
Proton mass	$m_{\rm p}$	$=$	$1.673\,\times^{-24}\,{\rm g}$
Electron mass	$m_{\rm e}$	$=$	$9.109 \times 10^{-28}{\rm g}$
Boltzmann constant	$k_{\rm B}$	$=$	$1.38 \times 10^{-16}\,{\rm erg\,K}^{-1}$
electron volt	eV	$=$	$1.602 \times 10^{-12}\,{\rm erg}$
parsec	pc	$=$	$3.086 \times 10^{18}\,{\rm cm}$
Solar mass	$M_\odot$	$=$	$2 \times 10^{33}{\rm g}$
Solar Radius	$R_\odot$	$=$	$6.960 \times 10^{10}\,{\rm cm}$