ASTR 595: Problem Set 1

This problem set consists of 4 problems for a total of 46 points.
Due date: monday Jan 27

In this problem set, you will make order-of-magnitude estimates for the densities, masses, and velocities encountered in astrophysical fluids. You will also experiment with converting between Eulerian and Lagrangian, and between vector and index forms of the hydrodynamic equations.

Problem 1: Astrophysical fluids

In what follows, you may ignore the fact that hydrogen will be ionized above a temperature of $\sim 10^4$ K.

a) [`3 points`] Assume that the typical radius of a hydrogen atom is given by the Bohr radius $r_{\rm B} = 5.29 \times 10^{-9}$ cm, compute the mean free path of baryonic matter in the Universe (at its average, present day density). Assume that the baryonic matter is an ideal gas and Express your answer in both cm and Mpc. What is the typical mass (in Solar masses) enclosed by a sphere of radius $\lambda_{\rm mfp}$.

b) [`3 points`] The warm phase of the interstellar medium (ISM) has a typical density of about 1 particle per cubic centimeter ($n \sim 1{\rm cm}^{-3}$). and a typical temperature of 10^4 K. Under the assumption that the gas mainly consists of hydrogen, what is the average density of the warm ISM in units of the average density of the Universe.

c) [`3 points`] What is the mean-free path for hydrogren atoms in the warm phase of the ISM. Express your answer in cm and in pc. What is the typical mass (in Solar masses) enclosed by a sphere of radius $\lambda_{\rm mfp}$.

d) [`3 points`] What is the mean-free path for a hydrogen atom in the Solar interior (ignore the fact that all hydrogren inside the Sun will be ionized). Express your answer in units of the Solar radius. Give a rough estimate of the Knudsen number. Motivate your answer!

e) [`5 points`] Give a rough estimate for the Knudsen number of the intracluster gas in the outskirts of clusters. Motivate your answer.

Problem 2: Ideal Gas

An ideal gas has an equation of state $P = n\,k_{\rm B}\,T$, and the microscopic velocities of the particles follow a Maxwell-Boltzmann distribution:

$$f(\vec{v})\mathrm{d}^3\vec{v} = \left(\frac{m}{2\pi k_{\rm B}T}\right)^{3/2} \exp\left[-\frac{mv^2}{2k_{\rm B}T}\right] \mathrm{d}^3\vec{v}$$

a) [`3 points`] What is the typical pressure of the warm phase of the ISM (i.e., the phase at $T \sim 10^4\,\mathrm{K}$). Express your answer in cgs units as well as in atmospheres (atm).

b) [`4 points`] What is the mean microscopic velocity, $\langle v\rangle$, of HI atoms at $T = 10^4$ K? Express your answer in km/s.

c) [`6 points`] What is the kinetic energy of an HI atom moving at this mean speed? Express your answer in eV. What fraction of the atoms in this gas can ionize each other?

d) [`4 points`] What is the typical time between collisions for an individual, neutral HI atom at $T = 10^4$ K and the average (baryonic) density of the Universe. How does this compare to the free-fall time of gas at this density, which is defined as $t_{\rm ff} = (3\pi/32G\rho)^{-1/2}$

Problem 3: Navier-Stokes; from index form to vector form [`6 points`]

The Navier-Stoker equation in index form, as derived in class, is given by

$$\rho\frac{\mathrm{d}u_i}{\mathrm{d}t} = -\frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\frac{\partial u_k}{\partial x_k}\right)\right] + \frac{\partial}{\partial x_i}\left(\eta\frac{\partial u_k}{\partial x_k}\right) - \rho\frac{\partial \Phi}{\partial x_i}$$

Show clearly, step-by-step and using text were needed, that this can be written in vector form as

$$\rho \frac{\mathrm{d}\vec{u}}{\mathrm{d}t} = -\nabla P + \mu \nabla^2 \vec{u} + \left(\eta + \frac{1}{3}\mu\right) \nabla(\nabla \cdot \vec{u}) - \rho \nabla \Phi$$

Problem 4: The Energy equation for an ideal fluid [`6 points`]

In class we wrote down the energy equation of an ideal fluid as

$$\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} = -\frac{P}{\rho} \nabla \cdot \vec{u} - \frac{\mathcal{L}}{\rho}$$

We also wrote down the energy equation (again for an ideal fluid) in conservative form as

$$\frac{\partial E}{\partial t} + \nabla \left[(E + P)\vec{u}\right] = \frac{\partial \Phi}{\partial t} - \mathcal{L}$$

Show that these are equivalent.

(Potentially) Useful Constants

Gravitional constant	G	=	$6.674 \times 10^{-8}\ \mathrm{cm}^3\,\mathrm{g}^{-1}\,\mathrm{s}^{-2}$
		=	$4.299 \times 10^{-9}\ \mathrm{Mpc}\, M_\odot^{-1}\, (\mathrm{km/s})^2$
Proton mass	m_p	=	$1.673 \times^{-24}$ g
Electron mass	m_e	=	9.109×10^{-28}g
Boltzmann constant	k_B	=	$1.38 \times 10^{-16}\ \mathrm{erg\,K}^{-1}$
electron volt	eV	=	1.602×10^{-12} erg
parsec	pc	=	3.086×10^{18} cm
Solar mass	$M_\odot$	=	2×10^{33}g
Solar Radius	$R_\odot$	=	6.960×10^{10} cm