ASTR 501: Problem Set 4

This problem set consists of 5 problems for a total of 52 points.
Due date: Mon Apr 18

Problem 1: The Jeans Mass

a) [`4 points`] Give an estimate for the Jeans length (in meters) and Jeans mass (in kg) for the air in the Astr 501 class room (room temperature). Also, compute the ratio between the phase velocity and group velocity for sound wave in this room with a frequency of 1 KHz.

b)[`2 points`] Derive the Jeans length (in parsec) and Jeans mass (in Solar masses) for an ideal fluid with a temperature of $T = 10^4\mathrm{K}$, and a density equal to the critical density for closure, which is $\rho_{\rm crit} = 1.4 \times 10^{11}\,\mathrm{M}_\odot/\,\mathrm{Mpc}^3$. You may assume that the fluid is composed entirely of ionized hydrogen.

c)[`3 points`] Collisionless systems also obey the Jeans criterion, even though they do not obey an equation of state. The Jeans length for a collisionless system is the same as for a collisional fluid, but with the sound speed $c_{\rm s}$ replaced by the 1D velocity dispersion σ. Derive an expression for the Jeans length of a spherically symmetric, isotropic galaxy in units of the characteristic radius of the galaxy (`Hint:` use the Virial theorem).

Problem 2: Schwarzschild Criterion [`6 points`]

The specific entropy is given by $s = CP/\rho^\gamma$, where C is some constant. Starting from

$$\frac{\rho}{\gamma P}\frac{\mathrm{d}P}{\mathrm{d}z} > \frac{\mathrm{d}\rho}{\mathrm{d}z}$$

show that this Schwarzschild criterion for convective stability can be written as $\mathrm{d}s/\mathrm{d}z > 0$.

Problem 3: Purely Radial Stellar Oscillations

Consider a spherical, barotropic star for which $P = K\rho^\gamma$. The goal is to derive conditions for γ under which the star is stable to radial oscillations. Suppose the star is *uniformly* expanded from an initial equilibrium configuration such that the position of a fluid element (or mass shell) changes from r_0 to $r_0(1+\delta)$. Throughout we shall assume that δ is small, such that we can use perturbation theory. From the Euler equation (i.e., ignoring viscosity) we can write down the acceleration of a fluid element at a distance r from the center of the star as

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r} - \frac{GM(r)}{r^2}$$

where v is the radial component of the velocity (we are considering purely radial motions here) and $M(r)$ is the mass enclosed within radius r.

a) [`6 points`] Use Taylor series expansion to show that, to linear order, the density of the perturbed mass shell obeys $\rho = \rho_0(1 - 3\delta)$.

b) [`5 points`] Using the same strategy, given a similar expression for P in terms of P_0, δ, and γ.

c) [`6 points`] Substitute the expressions for r, ρ and P in the expression for the radial acceleration, keeping only terms up to linear order, and derive for what values of γ the star will be stable to radial oscillations. Note: assume that the initial configuration was one of equilibrium, so that

$$-\frac{1}{\rho_0}\frac{\mathrm{d}P_0}{\mathrm{d}r_0} - \frac{GM(r_0)}{r_0^2} = 0$$

and <u>explain your answer</u>!

Problem 4: The Rayleigh-Taylor instability

Following a supernova explosion, a shell of matter with density $\rho_{\rm s}$ is plowing into the ISM with a speed $u_{\rm s} = \mathcal{M}\, c_{\rm s,ISM}$, where $\mathcal{M}$ is the Mach number and $c_{\rm s,ISM}$ is the sound speed of the ISM. The shell has a thickness d and the sound speed of the dense shell material is $c_{\rm s,shell}$.

a) [`4 points`] The shell material experiences a ram pressure $P = \rho_{\rm ISM} u_{\rm s}^2$, which causes it to decelerate. Show that the magnitude of the deceleration is given by

$$a = \frac{\rho_{\rm ISM}\, c_{\rm s,ISM}^2\, \mathcal{M}^2}{\rho_{\rm s}\, d}$$

`HINT:` use that pressure is force per unit area.

The shell is subject to Rayleigh-Taylor (RT) instability which obeys the following dispersion relation

$$\omega = \pm ik\sqrt{\frac{a}{k}\, A}$$

where

$$A = \frac{\rho_{\rm s} - \rho_{\rm ISM}}{\rho_{\rm s} + \rho_{\rm ISM}}$$

is called the Atwood number.

b) [`6 points`] Derive an expression for the growth rate of RT perturbations (i.e., the timescale on which the perturbations will grow) with a wavenumber $k = 1/d$. Expresss your answer in terms of the overdensity $\delta = (\rho_{\rm s}/\rho_{\rm ISM}) - 1$ and the sound crossing time of the shell $\tau_{\rm s} \equiv d/c_{\rm s,shell}$. You may assume that at the interface there is pressure equilibrium between the shell and the ISM.

Problem 5: The Virial Temperature

Virialized dark matter haloes are often defined as having a radius $r_{\rm vir}$, called the virial radius, that encloses an average density of 200 times the critical density $\rho_{\rm crit} = 1.36 \times 10^{11} M_\odot/{\rm Mpc}^3$. The latter is the density for which the Universe as a whole is 'flat' (i.e., has Euclidian geometry). The circular velocity at the virial radius is called the virial velocity and is denoted by $V_{\rm vir}$. Throughout you may assume that halos are spherically symmetric.

a) [`3 points`] Derive expressions for $r_{\rm vir}$ and $V_{\rm vir}$ as functions of the halo's mass M, and compute $r_{\rm vir}$ (in kpc) and $V_{\rm vir}$ (in km/s) for a halo of mass $M = 10^{12} M_\odot$ (roughly the mass of the Milky Way halo).

b) [`5 points`] When gas is accreted by a dark matter halo, it experiences an accretion shock, which converts its infall motion into thermal motion. Derive an expression for the temperature of this shocked gas after it falls into a halo of mass M. Assume that the gas comes from infinity where it has zero velocity, and it is accelerated by the gravity of the halo, until it hits the halo's virial shock at a radius $r_{\rm vir}$. You may approximate the potential of the halo by a point mass, i.e., $\Phi(r) = -GM(r)/r$. Ignore radiative losses, and express your answer in terms of the virial velocity.

c) [`2 points`] Determine the virial temperature for a halo of $M = 10^{12} M_\odot$ in Kelvin. Assume that the gas is made of pure, fully ionized hydrogen.

(Potentially) Useful Constants

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\begin{aligned}
G &= 6.674 \times 10^{-8}\,{\rm cm^3\,g^{-1}\,s^{-2}} \\
&= 4.299 \times 10^{-9}\,{\rm Mpc}\,M_\odot^{-1}\,({\rm km/s})^2 \\
m_{\rm p} &= 1.673\, \times^{-24}\,{\rm g} \\
k_{\rm B} &= 1.38 \times 10^{-16}\,{\rm erg\,K^{-1}} \\
M_\odot &= 2 \times 10^{33}{\rm g}
\end{aligned}
$$