ASTR 501: Problem Set 3

This problem set consists of 5 problems for a total of 40 points.
Due date: Tue Oct 17

Problem 1: Gravitational encounter [`10 points`]

Fig. 1 depicts a gravitational encounter between two stars (blue dots) of equal mass m. The impact parameter of the encounter is b, and the encounter speed is v. Show that, in the limit of a weak encounter (which means that the deflection angle α is small), the velocity of each star in the direction perpendicular to the encounter (i.e., along the direction of b) changes by an amount

$$\Delta v_\perp = \frac{2\,G\,m}{b\,v}$$

You may assume that α is negligbly small, such that the stars move along straight lines (indicated by the thin, grey, dotted lines).

Problem 2: Lagrangian Flow

Consider a fluid who's flow is given by the following vector field

$$\vec{u}(\vec{x},t) = (2xy, 4z^2t, 8x - 2yz)$$

a) [`4 points`] What is $\mathrm{d}\vec{u}/\mathrm{d}t$ at $\vec{x} = (1,2,3)$ at $t = 2$?

b) [`3 points`] When and where is the flow incompressible?

Problem 3: The Stress Tensor

Consider a fluid in a 2-dimensional, Cartesian coordinate system (x_1, x_2), with stress tensor

$$\sigma_{ij} = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}$$

and let $\hat{n}$ and $\hat{t}$ be the unit normal and unit tangent vectors of a surface S, for which $\hat{n}$ is rotated by angle θ with respect to the x_1 axis.

a) [`3 points`] Express $\hat{n}$ and $\hat{t}$ in terms of θ, i.e., what are n_1, n_2, t_1 and t_2 in $\hat{n} = (n_1, n_2)$ and $\hat{t} = (t_1, t_2)$?

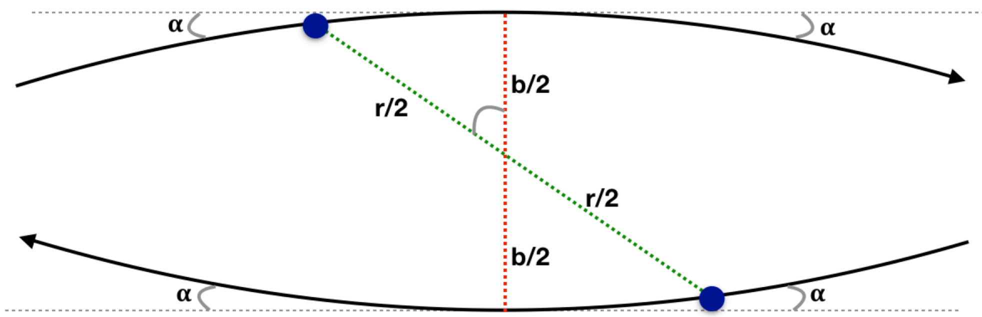

Figure 1: Illustration of an encounter between two (point) masses.

b) [`4 points`] Show that the normal stress, $\Sigma_{\rm n}$, can be written as

$$\Sigma_{\rm n} = \sigma_{11}\,\cos^2\theta + \sigma_{22}\,\sin^2\theta + \sigma_{12}\,\sin 2\theta$$

and derive a similar expression for the shear stress, $\Sigma_{\rm t}$.

c) [`4 points`] Consider the case $\sigma_{11} = \sigma_{12} = \sigma_{21} = 0$. Under what angle θ is the shear stress on S maximal? For what angle θ does the shear on S vanish? What are the normal stresses in both cases?

Problem 4: The Energy equation for an ideal fluid [`6 points`]
In class we wrote down the energy equation of an ideal fluid as

$$\frac{{\rm d}\varepsilon}{{\rm d}t} = -\frac{P}{\rho}\nabla\cdot\vec{u} - \frac{\mathcal{L}}{\rho}$$

We also wrote down the energy equation (again for an ideal fluid) in conservative form as

$$\frac{\partial E}{\partial t} + \nabla\left[(E+P)\vec{u}\right] = \rho\frac{\partial\Phi}{\partial t} - \mathcal{L}$$

Show that these are equivalent.

Problem 5: Momentum Equations in Conservation Law Form [`6 points`]
The continuity equation in Eulerian vector form is given by

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{u}) = 0$$

This equation is said to be in 'conservation law form', which is a form that is particularly useful for solving numerically.
The Euler equation describing conservation of momentum in Eulerian vector form is

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\, \vec{u} = -\frac{\nabla P}{\rho} - \nabla \Phi$$

which is not in conservation law form. Show that, for an inviscid fluid, and in the absence of gravity, this equation can be put in the following conservation law form

$$\frac{\partial (\rho \vec{u})}{\partial t} + \nabla \cdot \mathbf{\Pi} = 0$$

where $\Pi_{ij} = \rho\, u_i\, u_j - \sigma_{ij}$ is the momentum flux density tensor.