ASTR 501: Problem Set 2

This problem set consists of 5 problems for a total of 45 points.
Due date: Thursday Sep 21

In this problem set, you will put to practice what you learned in class about Hamiltonian dynamics.

Problem 1: Poisson brackets of angular momentum

Consider the Cartesian position vector $\vec{r} = (x, y, z)$ and the Cartesian momentum vector $\vec{p} = (p_x, p_y, p_z)$. Let $\vec{L} = \vec{r} \times \vec{p}$.

a) [`4 points`] Use the definition of the Poisson brackets to compute $\{L_x, L_y\}$, $\{L_y, L_z\}$ and $\{L_z, L_x\}$.

b) [`4 points`] Use the algebra of Poisson brackets to show that L_z Poisson commutes with $|\vec{L}|^2$.

Problem 2: Harmonic Oscillator

The Hamiltonian of the harmonic oscillator is given by

$$\mathcal{H} = \frac{p^2}{2m} + \frac{1}{2}\, k\, q^2 = \frac{1}{2m}\left(p^2 + m^2 w^2 q^2\right)$$

Consider the generating function

$$F(q, Q) = \frac{1}{2}\, m\, \omega\, q^2 \, \mathrm{cot} Q$$

a) [`3 points`] Using this generating function, find expressions for p and q as functions of the new variables Q and P.

b) [`4 points`] Find an expression for the Hamiltonian as a function of the new variables Q and P. Use this to solve the equations motion for $Q(t)$ and $P(t)$.

c) [`3 points`] Write down solutions for $q(t)$ and $p(t)$ as functions of E and $\omega = \sqrt{k/m}$

d) [`3 points`] Plot or sketch phase-space trajectors in both (q, p)-space and (Q, P)-space

e) [`4 points`] Show that $\{q, p\}_{Q,P} = \{q, p\}_{q,p}$.

Problem 3: Hamilton's Characteristic Function [`4 points`]

Let $W(\vec{q}, \vec{P})$ be Hamilton's characteristic function. Use the corresponding transformation rules for generating functions of this kind to show that

$$W = \int p_i \mathrm{d}q_i$$

Hint: first derive an expression for $\mathrm{d}W/\mathrm{d}t$ and then integrate.

Problem 4: Constants of Motion [`5 points`]

Consider a 1D system with Hamiltonian

$$\mathcal{H} = \frac{p^2}{2m} - \frac{1}{2q^2}$$

Show that $I(t) = \frac{1}{2}pq - \mathcal{H}t$ is a constant of motion.

Problem 5: Canonical Transformation

Consider the (unusual) Hamiltonian for a one-dimensional problem:

$$\mathcal{H} = \omega(x^2 + 1)p$$

where ω is a constant. Consider the transformation to new phase-space variables $P = \alpha p^{1/2}$ and $Q = \beta x p^{1/2}$.

a) [`4 points`] Find the conditions necessary for this to be a canonical transformation

b) [`4 points`] find a generating function $F(x, Q)$ for this transformation.

c) [`3 points`] what is the Hamiltonian in the new coordinates?