Problem 1: Astrophysical fluids

In what follows, you may ignore the fact that hydrogen will be ionized above a temperature of $\sim 10^4$ K.

**a)** [3 points] Assume that the typical radius of a hydrogen atom is given by the Bohr radius $r_B = 5.29 \times 10^{-9}$ cm, compute the mean free path (assuming direct collisions) of baryonic matter in the Universe at its average, present day density. Assume that the baryonic matter is entirely made up on hydrogen and express your answer in both cm and Mpc. What is the typical mass (in Solar masses) enclosed by a sphere of radius $\lambda_{\text{mfp}}$.

**b)** [3 points] The warm phase of the interstellar medium (ISM) has a typical density of about 1 particle per cubic centimeter ($n \sim 1 \text{ cm}^{-3}$), and a typical temperature of $10^4$ K. Under the assumption that the gas mainly consists of hydrogen, what is the average density of the warm ISM in units of the average density of the Universe.

**c)** [3 points] What is the mean-free path for hydrogen atoms in the warm phase of the ISM. Express your answer in cm and in pc. What is the typical mass (in Solar masses) enclosed by a sphere of radius $\lambda_{\text{mfp}}$.

**d)** [5 points] What is the mean-free path for a hydrogen atom in the Solar interior (ignore the fact that all hydrogen inside the Sun will be ionized). Express your answer in units of the Solar radius. Give a rough estimate of the Knudsen number. Motivate your answer!
e) [5 points] Give a rough estimate for the Knudsen number of the intra-cluster gas in the outskirts of clusters. Motivate your answer.

Problem 2: Poisson brackets of angular momentum

Consider the Cartesian position vector $\vec{r} = (x, y, z)$ and the Cartesian momentum vector $\vec{p} = (p_x, p_y, p_z)$. Let $\vec{L} = \vec{r} \times \vec{p}$.

a) [4 points] Use the definition of the Poisson brackets to compute $\{L_x, L_y\}$, $\{L_y, L_z\}$ and $\{L_z, L_x\}$.

b) [4 points] Use the algebra of Poisson brackets to show that $L_z$ Poisson commutes with $|\vec{L}|^2$.

Problem 3: Harmonic Oscillator

The Hamiltonian of the harmonic oscillator is given by

$$\mathcal{H} = \frac{p^2}{2m} + \frac{1}{2} k q^2 = \frac{1}{2m} \left( p^2 + m^2 \omega^2 q^2 \right)$$

Consider the generating function

$$F(q, Q) = \frac{1}{2} m \omega q^2 \cot Q$$

a) [3 points] Using this generating function, find expressions for $p$ and $q$ as functions of the new variables $Q$ and $P$.

b) [3 points] Find an expression for the Hamiltonian as a function of the new variables $Q$ and $P$. Use this to solve the equations motion for $Q(t)$ and $P(t)$. 2
c) [3 points] Write down solutions for \( q(t) \) and \( p(t) \) as functions of \( E \) and 
\[ \omega = \sqrt{\frac{k}{m}} \]

d) [3 points] Plot or sketch phase-space trajectories in both \((q,p)\)-space and
\((Q,P)\)-space.

e) [4 points] Show that \( \{q,p\}_{Q,P} = \{q,p\}_{q,p} \).

Problem 4: Hamilton’s Characteristic Function [4 points]
Let \( W(q,p) \) be Hamilton’s characteristic function. Use the corresponding transformation rules for generating functions of this kind to how that 
\[ W = \int p_i dq_i \]

Hint: first derive an expression for \( dW/dt \) and then integrate.

Problem 5: Constants of Motion [4 points]
Consider a 1D system with Hamiltonian 
\[ \mathcal{H} = \frac{p^2}{2m} - \frac{1}{2q^2} \]

Show that \( I(t) = \frac{1}{2}pq - \mathcal{H}t \) is a constant of motion.

Problem 6: Canonical Transformatation
Consider the (unusual) Hamiltonian for a one-dimensional problem:
\[ \mathcal{H} = \omega(x^2 + 1)p \]
where $\omega$ is a constant. Consider the transformation to new phase-space variables $P = \alpha p^{1/2}$ and $Q = \beta x q^{1/2}$.

a) [4 points] Find the conditions necessary for this to be a canonical transformation

b) [4 points] find a generating function $F(x, Q)$ for this transformation.

c) [3 points] what is the Hamiltonian in the new coordinates?

(Potentially) Useful Constants

<table>
<thead>
<tr>
<th>Constant</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gravitational constant</td>
<td>$G = 6.674 \times 10^{-8}$ cm$^3$ g$^{-1}$ s$^{-2}$</td>
</tr>
<tr>
<td></td>
<td>$= 4.299 \times 10^{-9}$ Mpc $M_\odot^{-1}$ (km/s)$^2$</td>
</tr>
<tr>
<td>Proton mass $m_p$</td>
<td>$1.673 \times 10^{-24}$ g</td>
</tr>
<tr>
<td>Electron mass $m_e$</td>
<td>$9.109 \times 10^{-28}$ g</td>
</tr>
<tr>
<td>Boltzmann constant $k_B$</td>
<td>$1.38 \times 10^{-16}$ erg K$^{-1}$</td>
</tr>
<tr>
<td>electron volt $eV$</td>
<td>$1.602 \times 10^{-12}$ erg</td>
</tr>
<tr>
<td>parsec $pc$</td>
<td>$3.086 \times 10^{18}$ cm</td>
</tr>
<tr>
<td>Solar mass $M_\odot$</td>
<td>$2 \times 10^{33}$ g</td>
</tr>
<tr>
<td>Solar Radius $R_\odot$</td>
<td>$6.960 \times 10^{10}$ cm</td>
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