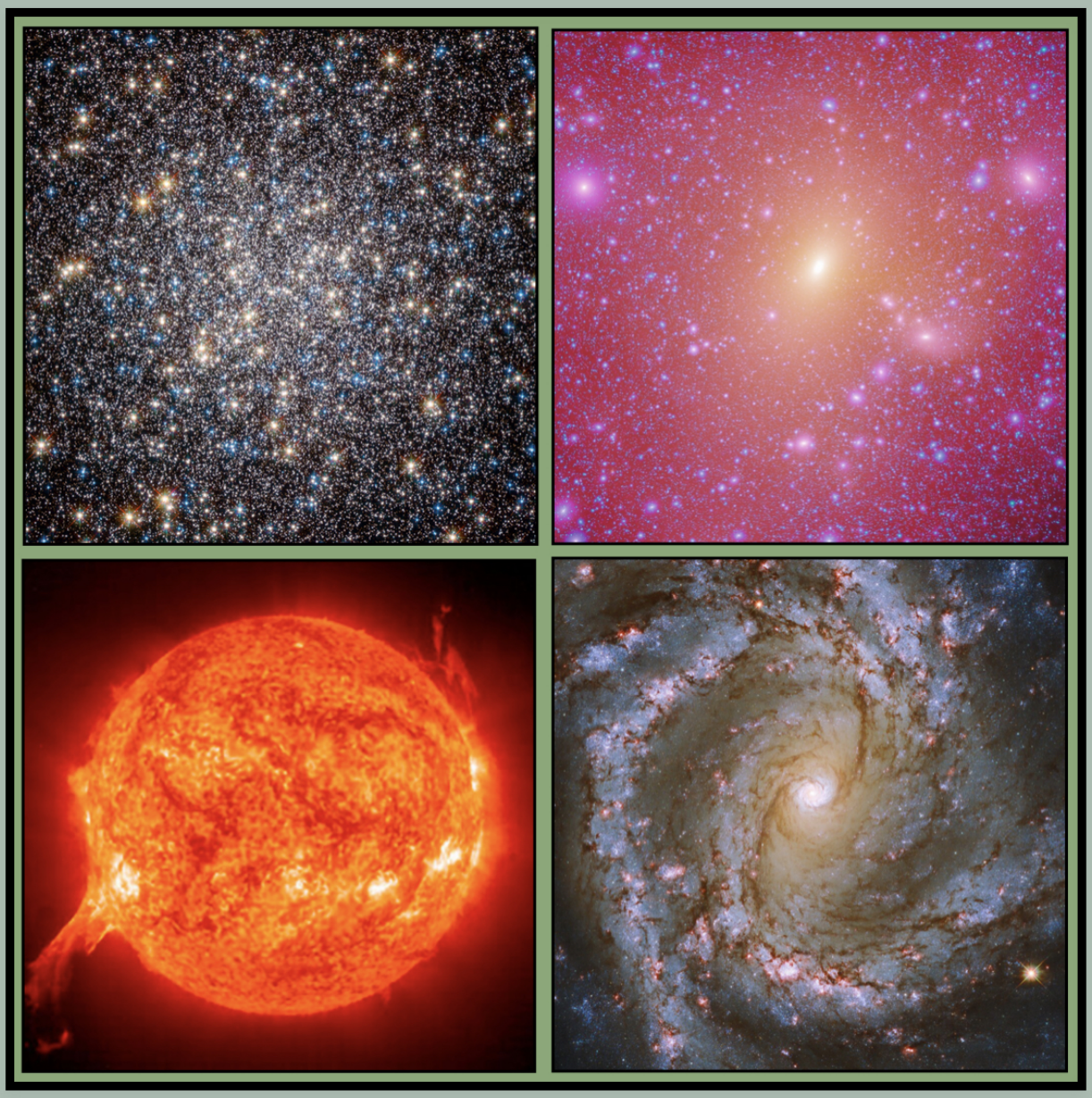
ASTR 501

Dynamics of Astrophysical Many-Body Systems

a consice treatment of fluid dynamics,
collisionless dynamics & plasma physics,
with an application to astrophysics



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These lecture notes describe the material covered during the Fall2023 semester of the course ASTR 501 at Yale University

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Main changes wrt v1.5: few typos corrected

Course Outline

This course studies the dynamics of a wide range of fluids encountered in astrophysics, ranging from dark matter halos and galaxies, to stars and gaseous planets, and from the intergalactic medium and molecular clouds to the Earth's ionosphere and the liquid oceans on Enceladus. We will see that at their core, all these fluids can be treated as Hamiltonian many-body systems (ignoring radiative processes), but that subtle differences in how the constituent particles interact with each others leads to wildly different outcomes.

After a brief introduction of theoretical foundations (Part I), in which we discuss units, length and time scales, and review classical (in particular Hamiltonian) dynamics, we start with a detailed account of kinetic theory (Part II). Starting from the Liouville theorem we use the BBGKY hierarchy to derive the Boltzmann equation, from which we derive the continuum, momentum and energy equations that describe the dynamics of many-body systems. We then examine how different types of 'collisions' between the constituent particles (short-range vs long-range, strong vs weak) lead to very different dynamics. Special attention is given to how weak, long-range interactions (characteristic of gravity) leads to diffusion in phase-space that can be described using the Fokker-Planck equation.
Next, in Part III, we study standard hydrodynamics, which applies mainly to neutral, collisional fluids. We discuss a variety of different flows; vorticity, incompressible barotropic flow, viscous flow, accretion flow, and turbulent flow, before addressing fluid instabilities and shocks.
In Part IV we briefly focus on collisionless dynamics, highlighting the subtle differences between the Jeans equations and the Navier-Stokes equations. We also give a brief account of orbit theory, and discuss the Virial theorem.
Finally, in Part V we turn our attention to plasmas. We discuss how the relevant equations (Vlasov and Lenard-Balescu) derive from kinetic theory, and then discuss plasma orbit theory, magnetohydrodynamics (MHD) and several applications.

It is assumed that the student is familiar with vector calculus, with curvi-linear coordinate systems, and with differential equations. A brief overview of these topics (and more) is provided in the various Appendices at the end of these Lecture Notes.

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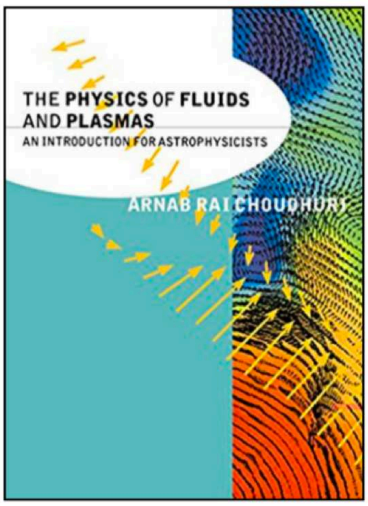
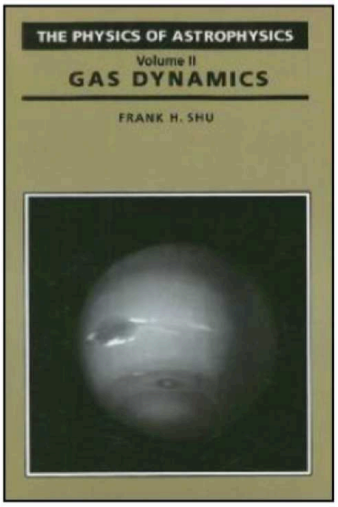
LITERATURE

The material covered and presented in these lecture notes has relied heavily on a number of excellent textbooks listed below.

- **Theoretical Astrophysics**
  by M. Bartelmann (ISBN-978-3-527-41004-0)
- **Modern Classical Physics**
  by K. Thorne & R. Blandford (ISBN-978-0-691-15902-7)
- **Classical Mechanics (3rd edition)**
  by H. Goldstein, C. Poole & J. safko (ISBN-978-0-201-65702-9)
- **Galactic Dynamics**
  by J. Binney & S. Tremaine (ISBN-978-0-691-13027-9)
- **Dynamics and Evolution of Galactic Nuclei**
  by D. Merritt (ISBN-978-0-15860-0)
- **The Physics of Fluids and Plasmas**
  by A. Choudhuri (ISBN-0-521-55543)
- **The Physics of Astrophysics–II. Gas Dynamics**
  by F. Shu (ISBN-0-935702-65-2)
- **Modern Fluid Dynamics for Physics and Astrophysics**
  by O. Regev, O. Umurhan & P. Yecko (ISBN-978-1-4939-3163-7)
- **Principles of Astrophysical Fluid Dynamics**
  by C. Clarke & B.Carswell (ISBN-978-0-470-01306-9)
- **Introduction to Plasma Theory**
  by D. Nicholson (ISBN-978-0-894-6467705)
- **Introduction to Modern Magnetohydrodynamics**
  by S. Galtier (ISBN-978-1-316-69247-9)
- **Galaxy Formation & Evolution**
  by H. Mo, F. van den Bosch & S. White (ISBN-978-0-521-85793-2)


THE PHYSICS OF FLUIDS AND PLASMAS
AN INTRODUCTION FOR ASTROPHYSICISTS
ARNAB RAI CHOUDHURY


THE PHYSICS OF ASTROPHYSICS
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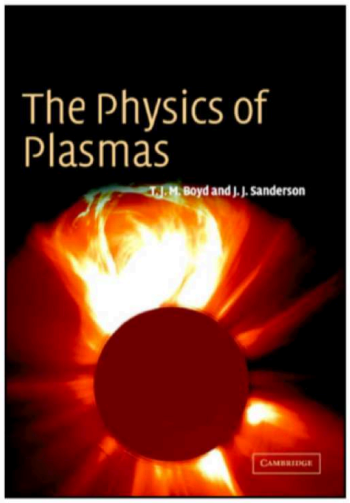
FRANK H. SHU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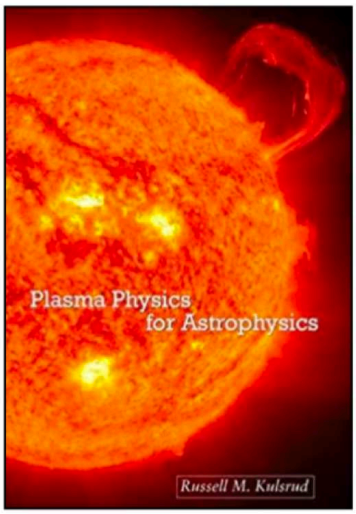
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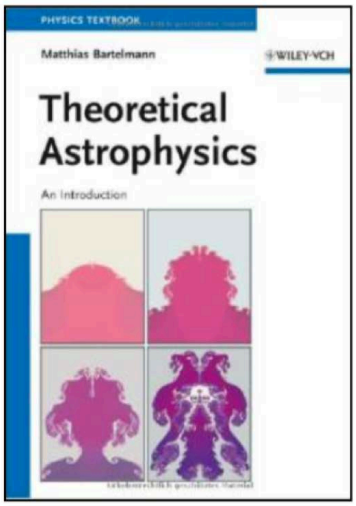
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Theoretical Astrophysics
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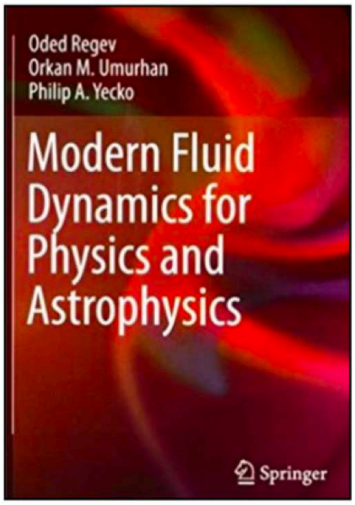
Oded Regev
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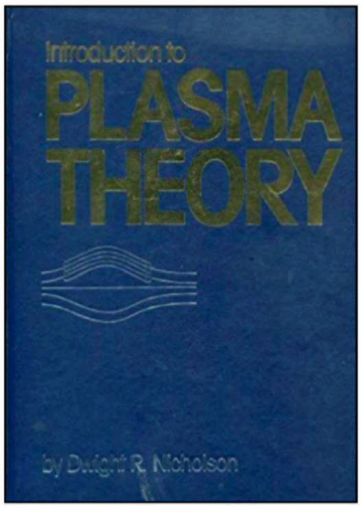
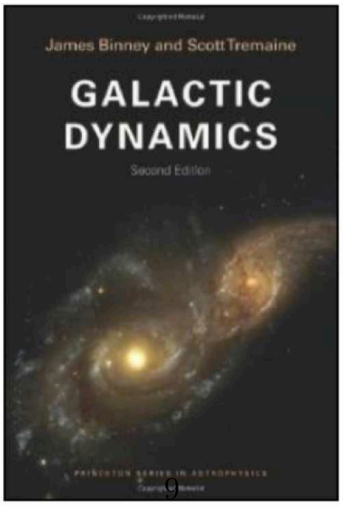
Modern Fluid Dynamics for Physics and Astrophysics
Springer


Introduction to
PLASMA THEORY
by Dwight R. Nicholson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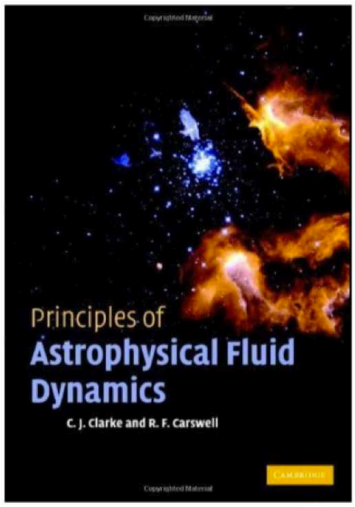
Principles of
Astrophysical Fluid Dynamics
C. J. Clarke and R. F. Carswell
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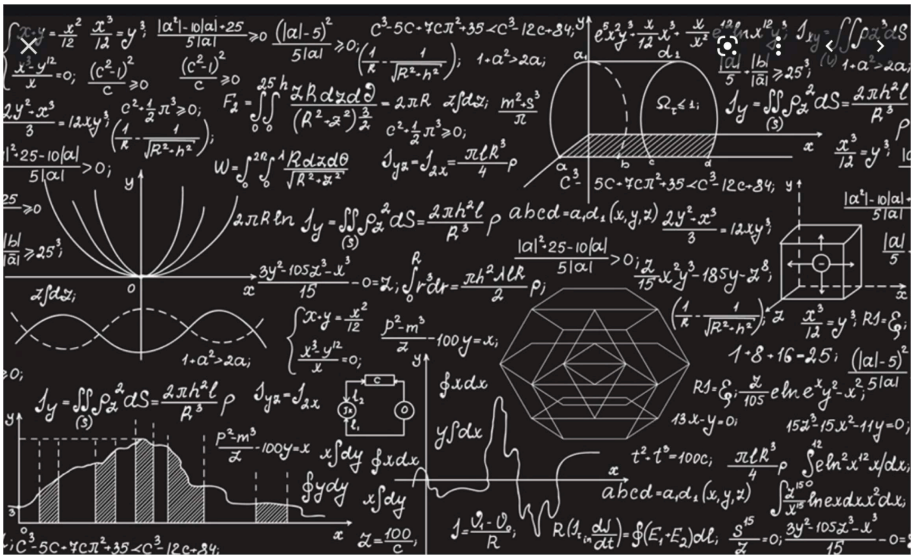
Part I: Theoretical Foundations



In this introductory part of the course, we define our unit system, we introduce the most relevant time and length scales, and we give a broad outline of collisional relaxation. We also review classical mechanics, with an emphasis on the Hamiltonian formulation, canonical transformations, Poisson brackets, and the Hamilton-Jacobi equation.

The material covered in this part is described in more detail in the following excellent textbooks:

- *Classical Mechanics* by J.Taylor
- *Classical Mechanics* by H.Goldstein, C.Poole & J.Safko
- *Modern Classical Physics* by K.Thorne & R. Blandford

CHAPTER 0

Units

Throughout these lecture notes, we adopt Gaussian cgs units. Length are measured in centimeter (cm), masses in grams (g), and time in seconds (s). Temperature will be expressed in Kelvin (K). When convenient, we will occasionally convert these to astrophysical units, such as parsecs (pc), Solar masses $M_\odot$, and/or Gigayear (Gyr), but cgs units will be the standard. The derived units of force, energy and power are listed in the Table below.

The unit of charge is the electrostatic unit (esu), which is equivalent to the stat-coulomb (stC), which is chosen such that the Coulomb force between two charges q separated by a distance r is

$$F_{\rm Coulomb} = \frac{q^2}{r^2}$$

Note that with this choice the dielectric constant of the vacuum, ε_0, is dimensionless and equal to unity. Consequently, electric and magnetic fields are defined to have the same unit. Their unit is such that the electrostatic force caused by an electric field E on a charge q is

$$F_{\rm electric} = q\,E$$

This implies that charge, and electric/magnetic fields have the units listed in the Table below. Note that E^2 and B^2 now have the units of energy density!

Quantity	cgs unit	in cgs	SI unit	cgs to SI conversion
Force	dyn	$\mathrm{g\,cm\,s^{-2}}$	Newton	$1\mathrm{dyn} = 10^{-5}\,\mathrm{N}$
Energy	erg	$\mathrm{g\,cm^2\,s^{-2}}$	Joule	$1\mathrm{erg} = 10^{-7}\,\mathrm{J}$
Power	$\mathrm{erg\,s^{-1}}$	$\mathrm{g\,cm^2\,s^{-3}}$	Watt	$1\mathrm{erg/s} = 10^{-7}\,\mathrm{W}$
Charge	esu	$\mathrm{g^{1/2}\,cm^{3/2}\,s^{-1}}$	Coulomb	$1\mathrm{esu} = 3.336\times10^{-10}\,\mathrm{C}$
EM Field	Gauss	$\mathrm{g^{1/2}\,cm^{-1/2}\,s^{-1}}$	Tesla	$1\mathrm{G} = 10^{-4}\,\mathrm{T}$

CHAPTER 1

Many-Body Systems in Astrophysics

This chapter and the next are introductory chapters in which you will encounter many terms that will be explained and discussed in more detail in the remainder of these lecture notes. Students are advised to revisit these chapters several times during the course to advance their understanding of the big picture.

With the exception of rocky planets and asteroids, all objects in the Universe can be characterized as some kind of **fluid** consisting of large numbers of constituent particles. Examples are stars (balls of plasma), galaxies and globular clusters (collections of stars), dark matter halos (agglomerates of dark matter particles), as well as molecular clouds, planet atmospheres, the ISM, ICM, CGM and IGM (gases or plasmas that span a wide range in densities and temperatures). This course studies the dynamics of all these systems.

In general, a fluid is a substance that can flow, has no fixed shape, and offers little resistance to an external stress. It's constituent particles can move 'freely' past one another, and, in the absence of self-gravity, a fluid will take on the shape of its 'container'. Also, stress forces cause a fluid to change its shape **at a steady rate**. For comparison, in a solid the particles are largely locked in place, the shape of the solid is self-imposed, and when acted upon by a stress force its shape is deformed by a fixed amount.

In this course we shall distinguish and discuss three different types of fluids:

- **Collisional fluids** in which the particle interactions are short-range, such that two-body collisions among them are well separated in both space and time. Examples of such fluids are liquids and (neutral) gases.
- **Collisionless fluids** in which the particle interactions are long-range (typically gravity). Examples of these are Cold Dark Matter halos and the stellar components of galaxies.
- **Plasmas** in which (a large fraction of) the particles are ionized and subject to long-range Coulomb interactions. As we will see, plasmas can be both collisionless and collisional at the same time.

In what follows, we use 'particles' to refer to the constituent particles of the sytem in question, and we use the terms 'interaction', 'encounter' and 'collision' without distinction, unless specifically mentioned otherwise.

Dynamical Treatments of Fluids

Our goal in this course is to develop dynamical models for each of these three types of fluids. A dynamical model consists of two ingredients: a description of the **state** of the system, and a set of **equations** that describe how that state evolves with time.

At the most fundamental level, one can argue that ultimately each and every fluid is made up of elementary particles that need to be described using quantum (field) theory. At this most basic level the state of the fluid is described in terms of the N-particle wave function $\psi(\vec{x}_1, \vec{x_2}, ..., \vec{x}_N)$, which evolves in time according to the **Schrödinger equation**. Since N is typically extremely large, this fluid-model is obviously utterly unfeasible. Fortunately, it is also unneccesary. According to what is known as **Ehrenfest's theorem**, a system of N *quantum* particles can be treated as a system of N *classical* particles if the characteristic separation between the particles is large compared to the **de Broglie wavelength**

$$\lambda_{\rm dB} = \frac{h}{p} \simeq \frac{h}{\sqrt{m k_{\rm B} T}}$$

Here h is the Planck constant, $p = m\,v$ is the particle's momentum, and m is the particle mass. This de Broglie wavelength indicates the 'characteristic' size of the wave-packet that according to quantum mechanics describes the particle, and is typically very small. Except for extremely dense fluids such as white dwarfs and neutron stars, or 'exotic' types of dark matter (i.e., 'fuzzy dark matter'), the de Broglie wavelength is always much smaller than the mean particle separation, and classical, Newtonian mechanics suffices. In this course we therefore focus exclusively on classical fluids.

Classically, the state of a fluid is described by the $6N$ phase-space coordinates (position and momentum of each particle) of the particles. As we will discuss in Part I, such a classical, Newtonian system of N particles can be described by a **Hamiltonian**, and the corresponding equations of motions. Clearly, when N is very large it is unfeasible to solve the 6N equations of motion for all the positions and momenta of all particles. However, in some cases we can 'downsample' the number of particles (while increasing their mass such that the total mass of the fluid remains the same), and solve the equations of motion using large computers. This is basically how astronomers use **N-body simulations** to model collisionless fluids such as galaxies or

dark matter halos. Plasma physicists also use N-body simulations called **Particle-in-Cell (PIC) simulations** in which each N-body particle represents an actual individual charged particle (electron or ion). As we will discuss in Part V, certain aspects of plasmas can only be addressed using this approach.

Another way to model fluids is by using **kinetic theory**, which we discuss in detail in Part II of these lectures. In kinetic theory the state of the fluid is described by the (1-particle) distribution function (DF) $f(\vec{x}, \vec{p}, t)$, which describes the number density of particles in 6-dimensional 'phase-space' $(\vec{x}, \vec{p})$, i.e., how many particles are there with positions in the 3D volume $\vec{x} + \mathrm{d}\vec{x}$ and momenta in the 3D volume $\vec{p} + \mathrm{d}\vec{p}$. The equations that describe how $f(\vec{x}, \vec{p}, t)$ evolves with time depends on the type of fluid. The evolution of collisional fluids, like a neutral gas or liquid, is described by the **Boltzmann equation**. If the fluid is collisionless (to good approximation), its evolution is described by the **Collisionless Boltzmann Equation (CBE)**, which is also known as the **Vlasov equation**. If collisions due to long-range forces cannot be ignored, the evolution of the DF can be described by a **Fokker-Planck equation** or the **Lenard-Balescu equation**.

Finally, fluids can also be modelled as a **continuum**. This means we ignore that fluids are made up of constituent particles, and rather describe the fluid with continuous fields. For a collisional fluid, its state is fully described by four fields: the density $\rho(\vec{x})$, the (bulk) velocity $\vec{u}(\vec{x})$, the pressure $P(\vec{x})$, and the internal, specific energy $\varepsilon(\vec{x})$ or, equivalently, the temperature $T(\vec{x})$. The equations that describe the time-evolution of $\rho(\vec{x})$, $\vec{u}(\vec{x})$, and $\varepsilon(\vec{x})$ are called the **continuity equation**, the **momentum equations**, which for a collisional fluid are known as the **Navier-Stokes equations**, and the **energy equation**, respectively. Collectively, we refer to these as the **hydrodynamic equations** or **fluid equations**. For an ideal (or perfect) fluid (i.e., no viscosity and/or conductivity), the Navier-Stokes equations reduce to what are known as the **Euler equations**. Magnetohydrodynamics (MHD) is a continuum model used to describe magnetized plasmas, which basically combines the fluid equations with Maxwell's equations. The fluid equations are also sometimes used to model collisionless fluids, in which case the equivalent of the Euler equation is known as the **Jeans equation**.

The goal of ASTR 501 is to acquaint you with each of these different approaches, thereby developing a deep understanding of the intricate differences between the various types of fluids, which ultimately arise from the different forces by which the particles interact with each other. To this end, we start with a brief primer on Hamiltonian dynamics, which is the starting point of kinetic theory. After a detailed derivation of the Boltzmann and Fokker-Planck equations, we then show how the

macroscopic continuum equations ultimately derive from kinetic theory by taking moments of the Boltzmann equation.

Collisions: The main difference between the different types of fluids listed above is the way their constituent particles interact.

In general, we can envision an interaction between two particles, 1 and 2, as depicted in Fig. 7. The interaction is characterized by an encounter speed $v = |\vec{v}_1 - \vec{v}_2|$ and an impact parameter b. The outcome of the interaction is a deflection angle θ. Note that we consider **classical physics** throughout (i.e., we assume that the interparticle separation $\lambda_{\rm int} \gg \lambda_{\rm dB}$). We also assume that **all collisions are elastic** (i.e., no dissipation of energy or momentum in the collision). The latter implies that we can describe our system using **Hamiltonian dynamics**, which is the only assumption we will make in deriving our **kinetic theory** in Part II.

In reality, off course, collisions between electrons and atoms or ions are not always elastic. For example, electrons can recombine with ions, a process that will release a photon (recombination radiation). Alternatively, collisions can excite bound electrons. If this is followed by spontaneous decay, a photon is emitted rendering the collision dissipative. Although these processes are obviously important if we want to understand the radiation we receive from astrophysical systems, typically the time scales of radiative losses ('cooling') are much larger than most of the dynamical times that are of interest to us. Hence, we do not make large errors by ignoring these radiative losses.

Interactions that only cause a non-negligible deflection, θ, if the impact parameter b is smaller than the typical interparticle separation are called **short-range interactions**. An example is the **vanderWaals force**, which is the dominant interaction force between molecules and atoms in a neutral gas or liquid. The short-range character of this force is reflected by the fact that for separations that are large compared to the sizes of the particles (atoms), $F_{\rm vdW} \propto r^{-7}$. Hence, if the gas is sufficiently dilute, then the mean-free path of a particle is much larger than the mean particle separation and the collisions are well separated in both time and space. In other words, the particles only have interactions with a single other particle at a time, and in between these collisions the particles move (roughly) in a straight line (unless the mean free path is large, and the particles move in an external potential).

The dominant force between particles in galaxies and cold dark matter halos is **gravity**, which is a **long-range force** (i.e., $F_{\rm grav} \propto r^{-2}$). Hence, each particle feels the gravitational force from all other particles; in other words, each particle

undergoes *simultaneous* collisions with all other particles. If N is sufficiently large, than the net effect of all these collisions is the same as if N where infinite, which corresponds to the case where the matter is not made of discrete particles, but is a continuous 'fluid'. In such a continuous medium there are no 'collisions'; rather, the particles move on smooth orbits governed by the smooth potential associated with the smooth mass distribution. In other words, in a system governed by long-range interactions, if N is sufficiently large the system can be approximated as being **collisionless**.

As we will see, it is useful to split the interactions in **strong interactions**, which are typically defined as interactions for which the deflection angle $\theta > 90^{\rm o}$, and **weak interactions**, which have $\theta < 90^{\rm o}$. Or, in terms of the impact parameter, strong and weak interactions have $b < b_{90}$ and $b > b_{90}$, respectively, where b_{90} is defined as the impact parameter for which $\theta = 90^{\rm o}$. As we will see, the net impact of strong interactions is always subdominant to that of the (many more) weak interactions, and one does not make a big error by simply ignoring the former (which is occasionally done).

It is important to realize that a 'collisionless' system is not truly collisionless; it is only approximately collisionless. Under certain circumstances (i.e., N is not very large, or one particle is much more massive than the other particles), collisional effects do play a role, and these are always dominated by the weak (large-b) encounters. As we will see, the net effect of many weak encounters is two-fold. On the one hand, the collisions act to reduce the motion of the particle in question (friction). On the other hand the many weak interactions cause the particle in question to undergo **diffusion** in both configuration space and momentum space. These effects are described by the **Fokker-Planck equation** which we discuss in Part II of these lecture notes.

The time scale on which two-body interactions cause the system to evolve is called the **two-body relaxation time**. For 'collisionless' systems, like a galaxy, it is typically much larger than any other time scale of interest. For a collisional gas, on the other hand, it is of the order of the time scale between individual collisions.

Finally, let's consider a **plasma**, which is a fluid in which the constituent particles are electrically charged, such that the dominant interparticle force is the **Coulomb force**. Similar to gravity, the Coulomb force is a long range force scaling as r^{-2}. However, because there are both positive and negative electric charges, the Coulomb force due to a charged particle inside a plasma is typically screened beyond what is known as the **Debye length**, $\lambda_{\rm D}$. The total number of particles within the Debye sphere, and thus the total number of particles that undergo simultaneous interactions, is $N_{\rm D} \sim n\lambda_{\rm D}^3$

If $N_{\rm D} \ll 1$ then charged particles have interactions with a single, individual particle at a time, and these collisions are well separated in time and space; hence the system basically behaves as a neutral fluid. Such a system is generally not considered a plasma despite the fact that its particles carry electric charge. In fact, as we will see, in this limit the typical kinetic energy of a charged particle is smaller than the potential energy due to its nearest neighbor, and there is a strong tendency for the electrons and ions to recombine, thus converting the plasma into a neutral fluid.

If $N_{\rm D} \gg 1$, then the charged particles undergo many simultaneous Coulomb interactions. This is known as the plasma limit. In this limit the **plasma frequency**, $\omega_{\rm p}$, which is the natural frequency with which the electrons in a plasma oscillate relative to the ions, is much larger than the collision frequency, which is defined here as the inverse of the two-body relaxation time. In other words, a plasma can undergo many oscillations (with the plasma frequency) before it undergoes significant damping due to two-body interactions, and the plasma can thus be considered collisionless. As we will see in Part V, the two-body relaxation time for a plasma can be expressed in terms of the plasma oscillation time $\tau_{\rm p} = 2\pi/\omega_{\rm p}$ as follows:

$$\tau_{\rm relax} \simeq \frac{N_{\rm D}}{\ln N_{\rm D}}\,\tau_{\rm p} \simeq \frac{N_{\rm D}}{\ln N_{\rm D}}\,10^{-4}\ {\rm s}\,\left(\frac{n_{\rm e}}{\rm cm^{-3}}\right)^{-1/2}$$

Compared to astrophysical time scales this relaxation time is extremely short. For example, even for a plasma with $\Lambda \sim n\lambda_{\rm D}^3 = 10^{10}$, the relaxation time for a plasma at ISM densities ($n \sim 1\,{\rm cm^{-3}}$) is only about 12 hours, much shorter than any relevant (hydro)-dynamical time. Hence, although plasmas can often be considered collisionless fluids, on astrophysical time scales plasmas are collisionally relaxed, and thus well described by a Maxwell-Boltzmann distribution.

Thus, we see that depending on the time-scale of the phenomena of interest, we can treat plasmas as either collisionless (short time scales, $\tau \ll \tau_{\rm relax}$) or collisional (long time scales, $\tau \gg \tau_{\rm relax}$). For example, when studying how **Landau damping** causes the 'dissipation' of **Langmuir waves** (which are non-propagating plasma waves in which perturbations in the electrical field cause oscillating separation of electrons and ions), the plasma can be treated as collisionless. On the other hand, when considering phenomena with long time scales, the plasma can be treated as collisional (i.e., having relaxed). In that case we don't need to treat electrons and ions separately. Rather, we treat the plasma as a single fluid. Note, though, that in this **one-fluid model** of plasma physics we need to account for the effect of collisions. As we will see, their main effect is to transfer momentum between electrons and ions, which in turn manifests as an electrical current. This is the area of **magnetohydrodynamics** (MHD).

Collective behavior: Systems in which the two-body interactions are governed by a long-range force (i.e., gravitational N-body systems and plasmas) are subject to collective effects. Whenever a local change/perturbation has an instant (ignoring retardation effects) impact on large scales, affecting many other particles, we speak of **collective behavior**. Adding a charge to a plasma immediately affects all particles in its Debye length; hence, **Debye shielding** is an example of collective behavior. Other examples are **plasma oscillations** (i.e., Langmuir waves), their dissipation due to **Landau damping**, and **violent relaxation**, which causes a relaxation of gravitational N-body systems due to large, coherent fluctuations in the overall gravitational potential of the system. Warps, spiral arms and bars in disk galaxies are also manifestations of collective behavior. Note that collisional fluids governed by short-range interactions do not show collective behavior. Here, instead, the impact of a local change has to diffuse outwards over time due to two-body collisions. Even acoustic waves (sound waves) that simultaneously impact many particles are not collective in nature; rather they are propagated by *local* fluctuations in pressure, which is ultimately related to local two-body collisions among the particles.

Particle Trajectories: In a **collisional fluid**, such as a neutral gas or liquid, the collisions among the particles are **short range**, **well-separated in space-and time**, and causing **large deflections**. In between the collisions, the particles travel in straight lines. Hence, particle trajectories are random walks, and the collisions drive the system towards a Maxwell-Boltzmann distribution on a very short time scale.

In a **gravitational N body system**, the interactions are **long-range**, with each particle feeling the gravitational force from all other particles. To good approximation the system is collisionless and the particles move along **orbits** in a smooth potential conserving their **integrals of motion** (see Chapter 4). Since individual orbits do not interact with each other, they can be considered 'building blocks'; i.e., one can equally well think of a galaxy as a collection of orbits rather than a collection of particles. However, due to the discreteness of the particles, the system is not perfectly collisionless and two-body collisions cause particle trajectories to deviate (slighty) from smooth orbits. Since weak interactions dominate over strong interactions, these deviations manifest as a **diffusion in phase-space** (which is described by the Fokker-Planck equation) that ultimately results in two-body relaxation.

Finally, in a **plasma**, the interactions are also **long-range**, governed by the **Coulomb force** between the charges, or rather by the **Lorentz force** when also accounting for the presence of magnetic fields. Charged particles will gyrate the magnetic field

lines (with the cyclotron frequency). Due to gradients and curvature in the magnetic field, the guiding center of this helical motion typically will drift and can even experience reflection (see Chapter 26), such that particle trajectories in a plasma can be extremely complicated. Similar to a gravitational N-body system, two-body interactions, dominated by many weak interactions, cause a diffusion that drives the velocity distribution towards a Maxwell-Boltzmann distribution.

Examples of Many-Body Systems (Fluids) in Astrophysics:

- **Stars:** stars are spheres of plasma in **hydrostatic equilibrium** (i.e., gravitational force is balanced by pressure gradients). Densities and temperatures in a given star cover many orders of magnitude. To good approximation, its **equation of state** is that of an **ideal gas**.

- **Giant (gaseous) planets:** Gaseous planets are large spheres of gas, albeit with a rocky core. Contrary to stars, though, the gas is neutral and typically so dense and cold that it can no longer be described with the **equation of state** of an **ideal gas**.

- **Planet atmospheres:** The atmospheres of planets are stratified, gaseous fluids retained by the planet's gravity. They are an example of non-self-gravitating fluids. The Earth's **ionosphere**, which extends from roughly 50 to 600 miles above the Earth's surface is a plasma, having been ionized by the incident UV radiation from the Sun. Another plasma that makes up the Earth's atmosphere are the charged particles that are trapped in the Earth's **magnetosphere** and which give rise to the Van Allen radiation belts. The magnetosphere crucially shields us from another plasma, namely the **Solar wind**.

- **White Dwarfs & Neutron stars:** These objects (stellar remnants) can be described as fluids with a **degenerate equation of state**.

- **Proto-planetary disks:** the dense disks of gas and dust surrounding newly formed stars out of which planetary systems form. Proto-planetary disks are complicated, but fascinating astrophysical environments governed by gravity, hydrodynamics and radiation. Understanding this interplay of processes is crucial for understanding planet formation.

- **Inter-Stellar Medium (ISM):** The gas in between the stars in a galaxy. The ISM is typically extremely complicated, and roughly has a **three-phase**

structure: it consists of a dense, cold ($\sim$ 10K) molecular phase, a warm ($\sim 10^4$K) phase, and a dilute, hot ($\sim 10^6$K) phase. All these phases are in rough pressure equilibrium with each other. Stars form out of the dense molecular phase, while the hot phase is (shock) heated by supernova explosions. The reason for this three phase medium is associated with the various cooling mechanisms. At high temperature when all gas is ionized, the main cooling channel is **Bremsstrahlung** (acceleration of free electrons by positively charged ions). At low temperatures ($< 10^4$K), the main cooling channel is molecular cooling (or cooling through hyperfine transitions in metals).

- **Inter-Galactic Medium (IGM):** The gas in between galaxies. This gas is typically very, very dilute (low density). It is continuously 'exposed' to adiabatic cooling due to the expansion of the Universe, but also is heated by radiation from stars (galaxies) and AGN (active galactic nuclei). The latter, called **'reionization'**, assures that the temperature of the IGM is $\sim 10^4$K.

- **Intra-Cluster Medium (ICM):** The hot gas in clusters of galaxies. This is gas (or rather a plasma) that has been shock heated when it fell into the cluster; typically gas passes through an **accretion shock** when it falls into a massive dark matter halo, converting its infall velocity into thermal motion. Material that enters the ICM through cold filaments can avoid this accretion shock, though. The cold filaments penetrating the hot ICM triggers **Kelvin-Helmholtz instabilities** which can stir turbulence in the ICM.

- **Circum-Galactic Medium (CGM):** The hot halo gas in halos less massive than a cluster. The CGM is the interplay between infalling gas that was recently accreted and outflowing gas being expelled from the central galaxy due to feedback processes. The CGM is often characterized as the reservoir for future star formation. It is extremely difficult to observe in emission, and most of what we know from it derives from absorption line spectroscopy using background sources.

- **Accretion disks:** Accretion disks are gaseous, viscous disks in which the viscosity (enhanced due to turbulence and/or magnetic fields) causes a net rate of radial inflow towards the center of the disk, while angular momentum is being transported outwards.

- **Galaxies (stellar component):** the stellar component of galaxies is characterized as a collisionless fluid; to very, very good approximation, two stars in a galaxy will never *directly* collide with other. The term collisionless is somewhat

of a misnomer; formally speaking the system is extremely collisional; each star has simultaneous 'collisions' (i.e., interactions) with all other stars. Yet, the outcome is such that the system behaves as if it is entirely collisionless. The impact of (two-body) collisions only manifest on very long timescale.

- **Globular Clusters:** Similar to galaxies, these are large collections of stars. Unlike galaxies, though, GCs do not reside in their own dark matter halo. In addition, due to the smaller number of stars, they are more collisional than galaxies, which results in a rich dynamics.

- **Dark matter halos:** Depending on the nature of dark matter, it can be a collisionless fluid, as in standard cold dark matter (CDM), it can be a collisional fluid, as in self-interaction dark matter (SIDM), or it can behave as a quantum-mechanical Bose-Einstein condensate, as in fuzzy dark matter (FDM).

CHAPTER 2

Characteristic Time & Length Scales

Before examining the types of interactions one encounters in the various fluids described in the previous chapter, we first introduce a number of important length- and time-scales.

- **Hubble radius:** $\lambda_{\rm H} \simeq c/H_0 \simeq 3000 h^{-1}$ Mpc. This is a rough estimate of the present-day size of the Universe (within the horizon).
- **System size:** R. The characteristic size of an astrophysical many-body system.
- **Jeans length:** $\lambda_{\rm J} = c_{\rm s}\sqrt{\pi/G\rho}$, where $c_{\rm s}$ is the sound speed. Objects larger than the Jeans length are unstable to gravitational collapse.
- **Debye length:** $\lambda_{\rm D} \sim \sqrt{k_{\rm B}T_{\rm e}/4\pi n_{\rm e}e^2}$ (Plasma only). Here $k_{\rm B}$ is the Boltzmann constant, and $T_{\rm e}$, $n_{\rm e}$ and e are the temperature, number density and electrical charge of the electrons. The Debye length is a measure of how far the electrostatic charge of a particle carries before it is shielded by the roughly equal numbers of positive and negative charges.
- **Mean-free path:** $\lambda_{\rm mfp} = (n\,\sigma)^{-1}$, where n and σ are the number density and (effective) collisional cross section, respectively. The mean-free path is the typical distance a particle travels in between two collisions.
- **Interparticle separation:** $\lambda_{\rm int} = n^{-1/3} \sim R/N^{1/3}$. The mean distance between two particles.
- **de Broglie wavelength:** $\lambda_{\rm dB} = h/(m\,v)$, where h is Planck's constant and v is the (non-relativistic) velocity of the particle. The de Broglie wavelength is the quantum-mechanical size of the wave-packet corresponding to the particle in question.
- **Larmor radius:** $\lambda_{\rm L} = m\,v_\perp\,c/|q|\,B$, is the radius of the circular motion of a charged particle in the presence of a uniform magnetic field. Here $v_\perp$ is the velocity of the charged particle projected perpendicular to the magnetic field, whose amplitude is B, m and q are the mass and electric charge of the particle,

and c is the speed of light. The Larmor radius is sometimes referred to as the gyro-radius.

A few general comments regarding these length scales: As long as $\lambda_{\rm int} \gg \lambda_{\rm dB}$ we can ignore quantum mechanical effects (interference) and treat the particles as well-separated, classical particles. Throughout these lecture notes, this will be the regime we are considering. Examples of astrophysical fluids that do NOT meet this criterion are **white dwarfs** and **neutron stars**, which are held up against gravity by degeneracy pressure from electrons and neutrons, respectively, and **Fuzzy Dark Matter** (also known as Scalar Field Dark Matter or Wave Dark Matter), which consists of ultra-light (typically $\sim 10^{-22}$ eV) axions for which $\lambda_{\rm dB}$ is astrophysically large.

A system that has undergone gravitational collapse and has virialized, has $R \sim \lambda_{\rm J}$. To see this, we can write the equivalent of the Jeans length for a collisionless gravitational system as $\lambda_{\rm J} = \sigma_{\rm v}\sqrt{\pi/G\rho}$, i.e., by replacing the sound speed, $c_{\rm s}$, with a characteristic velocity dispersion, $\sigma_{\rm v}$. Using that a virialized system obeys $2K + W = 0$, which implies that $M\sigma_{\rm v}^2 \sim GM^2/R$ (see Part IV for details), and that $\rho \sim M/R^3$, it is easy to see that indeed $\lambda_{\rm J} \sim R$.

If $\lambda_{\rm mfp} \gg \lambda_{\rm int}$ then the particles experience collisions that are well separated in both time and space. This is the situation we typically encounter in (dilute) gases, and will be the working hypothesis when describing liquids and neutral gases (i.e., hydrodynamics).

For a collisionless system the mean-free path is ill-defined. However, it is useful to approximate the collisional cross section in a gravitational N-body system as $\sigma = \pi b_{90}^2$, with b_{90} the impact parameter for which the collision causes a deflection by a 90-degree angle (see Chapter 1). With this definition, and using the Virial Theorem (see Chapter 22), one obtains that $\lambda_{\rm mfp} \simeq NR$, i.e., much, much larger than the size of the system, as expected.

If we use $\langle v \rangle$ to denote the typical velocity of a constituent particle, then we can associate a characteristic time scale, $\tau = \lambda/\langle v \rangle$, with each of the characteristic length scales, λ, mentioned above. Here we highlight a few of the more important ones:

- **Hubble time:** $\tau_{\rm H} = \lambda_{\rm H}/c = H_0^{-1} \sim 9.78 h^{-1}$ Gyr. A rough estimate for the age of the Universe.
- **Crossing time:** $\tau_{\rm c} = R/\langle v \rangle$. This is the typical time needed to cross a system in the absence of collisions.

- **Sound crossing time:** $\tau_{\rm s} = R/c_{\rm s}$. This is the typical time on which a sound wave crosses the system. It is the fastest time scale on which a hydro-dynamical system can respond to changes.

- **Collision time:** $\tau_{\rm coll} = \lambda_{\rm mfp}/\langle v \rangle = (n\,\sigma\,\langle v \rangle)^{-1}$. This is the time scale in between collisions. For a collisionless fluid, using that $\sigma = \pi b_{90}^2$, we obtain that $\tau_{\rm coll} \sim N\tau_{\rm c}$ which is roughly equal to the two-body relaxation time (see below).

- **Plasma oscillation time:** $\tau_{\rm p} = \sqrt{\pi m_{\rm e}/n_{\rm e}e^2} \sim \lambda_{\rm D}/\langle v \rangle$. This is the characteristic time scale on which a charged plasma reacts to a charge imbalance.

- **Gyration period:** $\tau_{\rm gyro} = 2\pi\lambda_{\rm L}/v_{\perp}$. The period with with a charge gyrates a magnetic field line. Note that $\tau_{\rm gyro}$ is independent of the velocity of the charged particle. Hence, protons have a gyration period that is $1,836 = m_{\rm p}/m_{\rm e}$ times longer than that of electrons. In addition positive and negative charges gyrate in opposite directions.

Using that a virialized system (i.e., a galaxy or dark matter halo) obeys $\langle v \rangle \sim \sqrt{GM/R}$, we have that $\tau_{\rm c} = R/\langle v \rangle \propto (G\rho)^{-1/2}$. All **dynamical times** of a virialized, gravitational system have this same $(G\rho)^{-1/2}$-scaling. This includes the 'free-fall time', $\tau_{\rm ff}$, which is the characteristic time scale on which a pressure-less system collapes under its own weight, and the 'orbital time', $\tau_{\rm orb}$, which is the orbital period of a particle, typically on a closed circular orbit. All these time scales are approximately the same (they only differ by $\mathcal{O}(1)$ pre-factors).

Note that

$$\frac{R}{\lambda_{\rm J}} = \frac{\tau_{\rm s}}{\tau_{\rm ff}}$$

Hence, if $R < \lambda_{\rm J}$ then $\tau_{\rm s} < \tau_{\rm ff}$ and the system can respond hydrodynamically to changes in the gravity, i.e., pressure can adjust to balance gravity and the system is stable against gravitational collapse. On the other hand, if $R > \lambda_{\rm J}$ then gravity wins. This is the famous **Jeans stability criterion**, which we will discuss in more detail in Part III.

The plasma oscillation time and gyration period are more commonly presented in terms of their corresponding frequencies; the **plasma frequency** $\omega_{\rm p} = 2\pi/\tau_{\rm p}$, which is the natural frequency with which a plasma oscillated in response to a charge separation, and the **gyration frequency** $\omega_{\rm c} = 2\pi/\tau_{\rm gyro} = |q|B/mc$, also known as the **cyclotron frequency**, which is the frequency with which electric charges gyrate magnetic field lines. As we will see in Part V, plasmas are opaque to radiation (=EM

oscillations) with a frequencies $\omega < \omega_{\rm p}$, which is why the Earth's ionosphere is opaque to radio emission with frequencies below ~ 10MHz. In addition, since in most cases we have that $\omega_{\rm c} < \omega_{\rm p}$, unless the magnetic field strength is enormous, the cyclotron radiation produced by the electrons as they gyrate the magnetic field lines typically remains trapped inside the plasma (see Part V for details).

Relaxation time: An additional time scale that is of great importance is the relaxation time, which is roughly defined as the time scale on which a system that is perturbed returns to an equilibrium. As discussed in Chapter 1, there are different ways in which a many-body system can achieve this, and which of these mechanisms dominates mainly depends on the type of fluid. For example, galaxies can undergo **violent relaxation**, which is an example of a **collective effect**. Violent relaxation is fast and operates on of order the crossing time. Another example of collective effects that cause relaxation is **Landau damping**, which causes the damping of Langmuir waves in a plasma (and which is very similar in nature to violent relaxation). Another relaxation mechanism that we will encounter in this course is **phase mixing**, which destroys the phase coherence of the response to a perturbations.

Probably the most well-known relaxation mechanism, though, is **two-body relaxation**, which is relaxation due to two-body collisions among the constituent particles. Two-body relaxation always drives the system towards **thermal equilibrium** in which the velocity distribution becomes a **Maxwell-Boltzmann** (MB) distribution. Two-body relaxation is the main mechanism by which collisional fluids such as neutral gases and liquids relax. The time scale on which such a fluid relaxes is of the order of the collision time, $\tau_{\rm coll}$, which is typically extremely fast. Hence, we can assume that the (local) velocities of the particles in a collisional fluid follows a MB distribution. This in turn gives rise to the concept of an **equation of state** which we can use to compute the pressure in the fluid, and which, as we will see, causes **closure** in the set of hydrodynamical equations.

In the case of a collisionless fluid, the two-body relaxation time is much longer. Formally, if a system is truly collisionless, then the two-body relaxation time should be infinite. However, as already mentioned in the previous chapter, collisionless systems are only *approximately* collisionless. On sufficiently long time-scales collisions will still drive the system towards a MB distribution. In a galaxy or dark matter halo, the two-body relaxation time is typically much longer than the Hubble time and such systems are, for all practical purposes, truly collisionless. Globular clusters, on other hand, can have two-body relaxation times of order a Gyr, and we may thus expect that they have been sculpted by two-body collisions. Finally, as discussed

in Chapter 1, plasmas have two-body relaxation times that are only a few hours. Hence, on astrophysical times scales they will have relaxed and established a MB distribution.

It is important to distinguish the *local* relaxation time, which is the time on which the *local* velocity distribution becomes MB, versus the (significantly longer) *global* relaxation time. The latter requires the entire system to adopt a single MB distribution, and thus to be characterized by a single (effective) temperature. In others words, a system that is globally relaxed has to be isothermal. Although local relaxation is common, global relaxation is rare. For example, stars are locally relaxed (because $\tau_{\rm relax} \sim \tau_{\rm coll}$), but have strong temperature gradients. This is mainly a consequence of the continuous supply of (nuclear) energy from within, which 'percolates' through the system, until it reaches the photosphere from where it is released into space. The time scale on which a collisional system tries to establish *global* equilibrium is governed by what are known as the **transport coefficients**. Examples are the diffusion coefficient (transport of number density), the viscosity (transport of momentum), the conduction coefficient (transport of energy or heat), and the electric conductivity (transport of electic charge). Effectively these are all diffusive, due to the random walks of particles undergoing many collisions.

Galaxies and dark matter halos have relaxation times that exceed the Hubble time. Hence, their local velocity distributions are NOT expected to be MB. Yet, they can appear 'relaxed' and thus to be in an equilibrium configuration (they reached that state through collective effects such as violent relaxtion). As we will see, though, this is only a **quasi-equilibrium**: *given enough time, any many-body system will evolve toward thermal equilibrium!* An interesting case are globular clusters, which have relaxation times that can be substantially shorter than the Hubble time. Hence, one may thus expect their local velocity distributions to be MB distributions. There is one caveat though; stars in the tail of MB distribution exceed the escape velocity and escape the system, a process called **evaporation**. Typically the velocity dispersion (a measure for the effective temperature) of a globular cluster is higher in the center than in its outskirts. Two-body collisions among the stars will cause a conductive heat flux that tries to establish thermal equilibrium across the globular (the same happens in the central regions of a halo of self-interacting dark matter). However, as we will see, gravitational systems have a negative heat capacity, which means that if heat escapes the core, the core gets hotter!! Consequently, the conductive flux increases, and the system enters what is called the **gravothermal catastrophe**, which leads to **core collapse**. As the central region gets hotter and hotter, evaporation becomes more pronounced, and ultimately the system will evaporate entirely. Hence, gravitational

many-body systems can never achieve *global* relaxation. The only way for them to be in thermal equilibrium, is to be in thermal equilibrium with the rest of the Universe, which is called the **heat death** of the Universe.

As you may already know, perhaps from taking a course in Galactic Dynamics, the **two-body relaxation time** for a gravitational N-body system is often given as

$$\tau_{\rm relax} = \frac{N}{8\,\ln N}\,\tau_{\rm c}$$

Since N and the crossing time $\tau_{\rm c}$ describe *global* properties of the entire system, you might be inclined to consider this the global relaxation time of the system. However, that would be incorrect; after all, a gravitational N-body system never achieves global relaxation. Rather, the above expression should be considered a very crude approximation for the time-scale on which the system relaxes locally (yes, this is weird, as there are no local quantities appearing in the expression). As we will see in Chapter 9, a more accurate estimate for the local two-body relaxation time can be obtained from the **Fokker-Planck equation**, and is given by

$$\tau_{\rm relax}(\vec{r}) = 0.34\frac{\sigma_{\rm v}^3(\vec{r})}{G^2\,m\,\rho(\vec{r})\,\ln\Lambda}$$

Here $\sigma_{\rm v}(\vec{r})$ and $\rho(\vec{r})$ are the local velocity dispersion and density, m is the mass of the constituent particles, and $\ln\Lambda$ is called the Coulomb logarithm. It is straightforward to show that if the system has a uniform density, and is virialized, then the two expressions for $\tau_{\rm relax}$ are virtually identicial.

CHAPTER 3

Classical Dynamics: a primer

Mechanics of a single particle:

Let $\vec{r}$ be the radius vector of a particle of mass m from some given origin and $\vec{v}$ its velocity vector (i.e., $\vec{v} = \mathrm{d}\vec{r}/\mathrm{d}t$).

Newton's second law of motion: there exist frames of reference (called inertial frames) in which the motion of the particle is described by

$$\boxed{\vec{F} = m\vec{a} = m\frac{\mathrm{d}^2\vec{r}}{\mathrm{d}t^2} = \frac{\mathrm{d}\vec{p}}{\mathrm{d}t} \equiv \dot{\vec{p}}}$$

where $\vec{a}$ is the acceleration and $\vec{p} = m\vec{v}$ is the **linear momentum**. Note that, in general, $\vec{F} = \vec{F}(\vec{r}, \vec{v})$; an example of a force that depends on velocity is friction. Throughout these lectures we limit ourselves to systems in which m is constant. Note, though, that if $m = m(t)$ then $\vec{F} = m\vec{a}$ is no longer valid. Rather, $\vec{F} = \dot{\vec{p}} = \dot{m}\vec{v} + m\vec{a}$.

The above **equation of motions** (one for each degree of freedom) are *second-order differential equations*. General theorems governing differential calculus guarantee that if two boundary conditions per degree of freedom are specified, i.e., $\vec{r}$ and $\vec{v}$ at some initial time t_0, then the equations of motion can be integrated to determine $\vec{r}(t)$ for all t. This is the main goal of classical dynamics.

The **angular momentum** of the particle about a point $\mathcal{O}$, is given by

$$\vec{L} = \vec{r} \times \vec{p}$$

where $\vec{r}$ is now the radius vector from $\mathcal{O}$. The **torque** about $\mathcal{O}$ is defined by

$$\vec{\mathcal{T}} = \vec{r} \times \vec{F} = \vec{r} \times \dot{\vec{p}}$$

Using that

$$\frac{\mathrm{d}}{\mathrm{d}t}[\vec{r} \times \vec{p}] = \vec{v} \times \vec{p} + \vec{r} \times \dot{\vec{p}} = \vec{r} \times \dot{\vec{p}}$$

where we have used that the cross-product between $\vec{v}$ and $\vec{p}$ vanishes, we see that

$$\vec{\mathcal{T}} = \frac{\mathrm{d}}{\mathrm{d}t}[\vec{r} \times \vec{p}] = \frac{\mathrm{d}\vec{L}}{\mathrm{d}t} \equiv \dot{\vec{L}}$$

From this equation, and the equation of motion, we infer two **conservation laws**:

If the force acting on a particle is zero, its linear momentum is conserved
If the torque acting on a particle is zero, its angular momentum is conserved

The work done by an external force $\vec{F}$ on a particle going from point $\vec{x}_1$ to point $\vec{x}_2$ is given by

$$\mathcal{W}_{12} = \int_{\vec{x}_1}^{\vec{x}_2} \vec{F} \cdot \mathrm{d}\vec{s}$$

Using that $\vec{F} = m\,\mathrm{d}\vec{v}/\mathrm{d}t$ and that the path length $\mathrm{d}\vec{s} = \vec{v}\,\mathrm{d}t$, it is easy to see that $W_{12} = T_2 - T_1$ where $T_i = \frac{1}{2}mv_i^2$ is the **kinetic energy** of the particle at position $\vec{x}_i$.

Throughout these lecture we will mainly focus on a very special kind of force, known as a **conservative force**. A conservative force depends only on position, not on velocity, and is such that the work done in going from $\vec{x}_1$ to $\vec{x}_2$ is independent of the path taken, i.e.,

$$\oint \vec{F} \cdot \mathrm{d}\vec{s} = 0$$

Note that if friction or other dissipative forces are present, the system cannot be conservative (after all, for friction $\vec{F} \cdot \mathrm{d}\vec{s}$ is always positive).

For a force that *is* conservative, the force is *curl-free*, i.e., $\nabla \times \vec{F} = 0$, and can always be written as a gradient of a scalar function of position, i.e.,

$$\vec{F} = -\nabla V(\vec{r})$$

where V is called the **potential**, or **potential energy**. Substituting this expression in that for $\mathcal{W}_{12}$ we see that $\mathcal{W}_{12} = V_1 - V_2$. Combined with the fact that $\mathcal{W}_{12} = T_2 - T_1$ we see that $T_1 + V_1 = T_2 + V_2$.

Hence, using that the total energy $E = T + V$, we have a third conservation law:

If the force acting on a particle is conservative, its total energy is conserved

If the potential depends explicitely on time, i.e., $V = V(t)$, then energy is no longer conserved. Instead we have that

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\partial E}{\partial \vec{v}} \cdot \frac{\mathrm{d}\vec{v}}{\mathrm{d}t} + \frac{\partial E}{\partial V}\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\partial V}{\partial t}$$

(see Chapter 24 for details). As we will see later in these lecture notes, this is the principle that underlies **violent relaxation**.

Mechanics of many-particle systems:

Consider a system of N particles of mass m_i, and with position vectors $\vec{r}_i$ and momenta $\vec{p}_i = m_i\,\vec{v}_i$. The force acting on particle i can be split in an **external** force due to sources from outside the system, and **internal** forces due to the other $N-1$ particles of the system, i.e.,

$$\boxed{m_i\,\ddot{\vec{r}}_i = \dot{\vec{p}}_i = \vec{F}_{\mathrm{ext},i} + \sum_j \vec{F}_{ji}}$$

where $\vec{F}_{ji}$ is the force of particle j acting on particle i.

Throughout we assume that the internal forces obey Newton's third law of motion (action=reaction) such that $\vec{F}_{ji} = -\vec{F}_{ij}$, and that all internal forces are **central**, which means that the direction of $\vec{F}_{ji}$ is in the direction of $\vec{r}_{ji} = \vec{r}_j - \vec{r}_i$. Since all central forces are **conservative**, we can write $\vec{F}_{ji}$ as the gradient of a scalar function, i.e.,

$$\vec{F}_{ji} = -\nabla U(\vec{r}_j - \vec{r}_i)$$

where $U(\vec{r})$ is the potential corresponding to the internal force. If the external forces are also conservative than

$$\vec{F}_{\mathrm{ext},i} = -\nabla V(\vec{r}_i)$$

where V is the potential corresponding to the external force.

For a many-body system in which all forces are conservative we define

$$\begin{aligned}
\textbf{Total Mass} \quad & M = \sum_i m_i \\
\textbf{Center of Mass} \quad & \vec{R} = \frac{1}{M}\sum_i m_i \vec{r}_i \\
\textbf{Total (linear) Momentum} \quad & \vec{P} = \sum_i m_i \mathbf{v}_i = M\,\frac{\mathrm{d}\vec{R}}{\mathrm{d}t} \\
\textbf{Total Angular Momentum} \quad & \vec{L} = \sum_i r_i \times \vec{p}_i \\
\textbf{Total Kinetic Energy} \quad & K = \sum_i \frac{1}{2} m_i\, v_i^2 \\
\textbf{Total Potential Energy} \quad & W = \sum_i V_{\mathrm{ext},i} + \frac{1}{2}\sum_{i\neq j} U_{ij}
\end{aligned}$$

Note that the potential energy has separate terms related to the external and internal forces. The latter involves a summation over all particle pairs (the factor 1/2 is to avoid double-counting). As long as the internal forces are central, then

$$\frac{\mathrm{d}\vec{L}}{\mathrm{d}t} = \mathcal{T}_{\mathrm{ext}} = \sum_i \vec{r}_i \times \vec{F}_{\mathrm{ext},i}$$

The total energy of a many-body system is simply $E = K + W$. Note that some textbooks use different notations (i.e., T and V for total kinetic and potential energy).

The three conservation laws mentioned above for a single particle translate to many-body systems as follows:

If the total external force is zero, total linear momentum is conserved
If the applied (=external) torque is zero, total angular momentum is conserved
If all forces (internal & external) are conservative, total energy is conserved

Generalized Coordinates:

In a many body system consisting of N particles, we have $3N$ coordinates that specify the positions of the particles. If there are no constraints, such that all $3N$ coordinates are *independent*, then we say that the system has $n = 3N$ **degrees of freedom**. If, on the other hand, the system has k **holonomic** constraints, which are constraints of the form

$$f(\vec{r}_1, \vec{r}_2, ..., \vec{r}_N, t) = 0$$

then the k expressions can be used to eliminate k of the $3N$ coordinates and the system only has $n = 3N - k$ degrees of freedom. An example of a many-body system with (many) holonomic constraints is a rigid body, where each pair of particles satisfies $(\vec{r}_i - \vec{r}_j)^2 - c_{ij} = 0$, where c_{ij} is a constant.

A system with n degrees of freedom can be characterized by a set of n **generalized coordinates** q_i $(i = 1, ..., 3N - k)$, which are related to the 'old' coordinates by equations of the form $\vec{r}_j = \vec{r}_j(q_1, q_2, ..., q_n, t)$ with $j = 1, ..., N$, which implicitely contain the $k = 3N - n$ constraints.

Unlike Cartesian coordinates, or other orthogonal coordinate systems (i.e., cylindrical or spherical), the generalized coordinates, q_i, in general can **NOT** be divided into conventional groups of three (spatial vectors). In fact, all sorts of quantities can be invoked to serve as generalized coordinates, even quantities with dimensions other

than length! If there are zero constraints, such that $n = 3N$, as is often the case for the many-body systems encountered in these lecture notes, then it is often convenient (but not necessary) to use an appropriate coordinate system as generalized coordinates. For instance, in the case of an external central force field, $V = V(r)$, it is convenient for q_i to reflect the spherical coordinates of the N particles.

The Least-Action Principle:

In general, solving the Newtonian equations of motion involves having to solve a set of $3N$ second-order differential equations plus k holonomic equations that specify the constraints. This can be extremely challenging, even for small N, especially when using a non-Cartesian coordinate system, or when having to deal with non-inertial frames. Fortunately, two far more convenient formalisms have been developed, the Lagrangian and Hamiltonian frameworks. These yield equations of motion that are valid for any set of generalized coordinates; an invariance to the choice of coordinates is automatically baked in, which implies no issue with fictitious forces or with complicated vector arithmetic that comes with having to deal with curvi-linear coordinate systems. In addition, the Lagrangian and Hamiltonian formalisms connect directly to other branches of physics, including electromagnetism, quantum theory and particle physics. Most importantly, both the Lagrangian and Hamiltonian formalisms are based on one of the most important principles in all of physics: the least-action principle.

Let $\vec{q} = (q_1, q_2, ..., q_n)$ be a n-dimensional vector, where $n = 3N - k$ is the number of degrees of freedom. Hence $\vec{q}$ uniquely specifies the position of all N particles in the system. The vector $\vec{q}$ lives in a n-dimensional space known as **configuration space**, C. Each point in C specifies a configuration of the system, the evolution of which gives rise to a curve (or 'path') in C.

Consider two configurations, an initial one $\vec{q}_1 = \vec{q}(t_1)$ and a final one $\vec{q}_2 = \vec{q}(t_2)$. There are infinitely many paths in C that connect $\vec{q}_1$ and $\vec{q}_2$. According to the **least action principle** the actual path taken by the system is an extremum of the **action**

$$\boxed{S = \int_{t_1}^{t_2} \mathcal{L}(\vec{q}, \dot{\vec{q}}, t)\, \mathrm{d}t}$$

Here $\mathcal{L} = K - W$ is the difference of the kinetic energy and potential energy of the system and is called the **Lagrangian**.

Note that the action is a *functional*, i.e., it is a function of the path which is itself a function. The principle of least action, also known as **Hamilton's principle**, states

that the dynamics are such that the *variation* of the line integral S for fixed t_1 and t_2 is zero, i.e., $\delta S = 0$. Hence, the name 'least' action is somewhat of a misnomer; the path doesn't necessarily minimize S, as it can also be a maximum or a saddle point. A more appropriate name would therefore be the 'principle of stationary action'.

Lagrangian Dynamics:
Using the **calculus of variations** (see any textbook on advanced mechanics), the condition $\delta S = 0$ translates to

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0\,, \qquad (i = 1, 2, ..., n)}$$

These equations are known as the **Euler-Lagrange equations** (or Lagrange equations, for short). These n *second-order differential equations* serve as the equations of motion within the Lagrangian formalism of classical dynamics.

To see the connection to Newtonian mechanics, consider a system of N particles of mass m, with no constraints on their coordinates, such that $n = 3N$. Assume that the particles are subject to an external, conservative force field characterized by a potential, V, and let q_i be the $3N$ Cartesian coordinates of the N particles. Then Newton's equations of motion read

$$\dot{p}_i = -\frac{\partial V}{\partial q_i}$$

where $p_i = m\,\dot{q}_i$ is the momentum, and V is the potential. The total kinetic and potential energy of the system are $K = \frac{1}{2}\sum_i m\dot{q}_i^2$ and $W = \sum_i V(q_i)$. Hence, $\partial\mathcal{L}/\partial\dot{q}_i = m\,\dot{q}_i = p_i$ and $\partial\mathcal{L}/\partial q_i = -\partial V/\partial q_i$. We thus see that the Euler-Lagrange equations reduce to Newton's equations of motion.

However, there are two very important reasons for working with Lagrange's equations rather than Newton's. The first is that Lagrange's equations hold in any coordinate system, while Newton's are restricted to an inertial frame. The second is the ease with which we can deal with constraints in the Lagrangian system.

In the above example, in which all q_i are Cartesian coordinates, we have that $\partial\mathcal{L}/\partial\dot{q}_i = m\dot{q}_i = p_i$, where p_i is the linear momentum. This notion suggests an obvious *extension* to the concept of momentum: the **generalized momentum**, also

known as the **conjugate momentum**, associated with the generalized coordinate q_i is defined as

$$\boxed{p_i \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}_i}}$$

Note that if q_i is not a Cartesian coordinate then p_i does not necessarily have the dimensions of a linear momentum!! Unfortunately, we will use the same symbol p to refer to a generalized momentum and a linear momentum. It (hopefully) should be clear from the context which meaning is intended.

Note that if the Lagrangian of a system does not contain a given coordinate q_j (although it may contain the corresponding velocity $\dot{q}_j$), then the coordinate is said to be **cyclic** or **ignorable**. The Euler-Lagrange equation for a cyclic coordinate reduces to

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right) = \frac{\mathrm{d}p_j}{\mathrm{d}t} = 0$$

Hence, we see that

The generalized momentum conjugate to a cyclic coordinate is conserved (i.e., is a constant of motion)

Noether's Theorem:

Conservation laws in Lagrangian dynamics are intimately related to **Noether's theorem** which states that

With each continuous symmetry of the Lagrangian corresponds a conserved quanity

Let's first examine what we mean by a *continuous symmetry*. Consider a one-parameter family of maps

$$q_i(t) \to Q_i(\lambda, t)\,, \qquad \lambda \in \mathbb{R}$$

such that $Q_i(0,t) = q_i(t)$. Hence, λ is a continuous variable that characterizes a coordinate transformation $q_i \to Q_i$. This transformation is said to be a continuous symmetry of the Lagrangian if

$$\frac{\partial}{\partial\lambda}\mathcal{L}\left(Q_i(\lambda,t), \dot{Q}(\lambda,t), t\right)_{\lambda=0} = 0$$

To prove Noether's theorem, we use that

$$\frac{\partial\mathcal{L}}{\partial\lambda} = \frac{\partial\mathcal{L}}{\partial Q_i}\frac{\partial Q_i}{\partial\lambda} + \frac{\partial\mathcal{L}}{\partial\dot{Q}_i}\frac{\partial\dot{Q}_i}{\partial\lambda}$$

where we have used **Einstein's summation convention**[1], to write

$$\left(\frac{\partial\mathcal{L}}{\partial\lambda}\right)_{\lambda=0} = \frac{\partial\mathcal{L}}{\partial q_i}\frac{\partial Q_i}{\partial\lambda} + \frac{\partial\mathcal{L}}{\partial\dot{q}_i}\frac{\partial\dot{Q}_i}{\partial\lambda}$$

Using the Euler-Lagrange equations we can rewrite this as

$$\left(\frac{\partial\mathcal{L}}{\partial\lambda}\right)_{\lambda=0} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}_i}\right)\frac{\partial Q_i}{\partial\lambda} + \frac{\partial\mathcal{L}}{\partial\dot{q}_i}\frac{\partial\dot{Q}_i}{\partial\lambda} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}_i}\frac{\partial Q_i}{\partial\lambda}\right)$$

We thus see that if $(\partial\mathcal{L}/\partial\lambda)_{\lambda=0} = 0$, which implies that λ corresponds to a continuous symmetry of the Lagrangian, then the quantity

$$\frac{\partial\mathcal{L}}{\partial\dot{q}_i}\frac{\partial Q_i}{\partial\lambda} = \sum_i \frac{\partial\mathcal{L}}{\partial\dot{q}_i}\frac{\partial Q_i}{\partial\lambda}$$

is conserved.

[1] a repeated index is summed over, i.e., $a_i\,b_i = \sum_i a_i\,b_i$

Example: Consider a closed system of N particles with Lagrangian

$$\mathcal{L} = \frac{1}{2}\sum_i m_i \dot{\vec{r}}_i^2 - \sum_{i \neq j} V(|\vec{r}_i - \vec{r}_j|)$$

This Lagrangian is symmetric under the continuous transformation $\vec{r}_i(t) \to \vec{r}_i(t) + \lambda\vec{n}$ for any $\lambda \in \mathbb{R}$ and any vector $\vec{n}$. This is simply a statement that space is homogeneous and that a translation of the system by $\lambda\vec{n}$ does not modify the equations of motion. From Noether's theorem we infer that

$$\sum_i \frac{\partial \mathcal{L}}{\partial \dot{\vec{r}}_i} \cdot \vec{n} = \sum_i \vec{p}_i \cdot \vec{n}$$

is conserved. Note that $\vec{p}_i \cdot \vec{n}$ is simply the component of the linear momentum of particle i in the direction of $\vec{n}$. Hence, Noether's theorem states that the *total* linear momentum of the system in the direction of $\vec{n}$ is conserved. And since this is valid for any $\vec{n}$, we have that the homogeneity of space implies a translational invariance of the Lagrangian which in turn, via Noether's theorem, implies a conservation of total linear momentum!

Similarly, the isotropy of space implies a rotational invariance of $\mathcal{L}$, which relates to the conservation of total angular momentum, while the homogeneity of time implies that $\mathcal{L}$ is invariant under translations $t \to t + \lambda$, which in turn implies the conservation of total energy.

Implications of Noether's theorem:

Invariance of $\mathcal{L}$ under	**time**	translation $\Leftrightarrow$ conservation of **energy**
Invariance of $\mathcal{L}$ under	**spatial**	translation $\Leftrightarrow$ conservation of **linear momentum**
Invariance of $\mathcal{L}$ under	**rotational**	translation $\Leftrightarrow$ conservation of **angular momentum**

Hamiltonian Dynamics:

In the Lagrangian formalism, the central role is played by the Lagrangian $\mathcal{L}(q_i, \dot{q}_i, t)$, where q_i $(i = 1, 2, ..., n)$ are generalized coordinates for the n degrees of freedom. The equations of motion are given by the n Euler-Lagrange equations, which are second-order differential equations that require 2n initial conditions to solve (typically the q_i and $\dot{q}_i$ at some initial time t_0).

In 1830 Hamilton introduced a new formalism, motivated by an attempt to put q_i and $\dot{q}_i$ on a more symmetric footing. This is done by replacing the $\dot{q}_i$ by the generalized momenta $p_i = \partial\mathcal{L}/\partial\dot{q}_i$. Note that $p_i = p_i(q_i, \dot{q}_i, t)$. In terms of these generalized momenta the Euler-Lagrange equations take on the form

$$\dot{p}_i = \frac{\partial \mathcal{L}}{\partial q_i}\,, \qquad (i = 1, 2, ..., n)$$

The Hamiltonian framework, outlined below, does not really add anything new, nor is solving the equations of motion in the Hamiltonian framework superior to that in the Lagrangian framework. Rather, the Hamiltonian framework merely gives us an alternative (and more powerful) view of dynamics that provides valuable insights into many areas of physics. Within classical mechanics it forms the basis for further developments such as Hamilton-Jacobi theory (covered in the next chapter) and chaos (not covered in these lecture notes). Outside of classical mechanics the Hamiltonian framework provides much of the language used to construct statistical mechanics (see Chapters 6 and 7) and quantum mechanics.

In the Lagrangian framework $\vec{q} = (q_1, q_2, ..., q_n)$ defines a point in n-dimensional **configuration space**, C. Although $\vec{q}$ completely specifies the **configuration** of the system, completely describing its **state** requires both $\vec{q}$ and $\vec{p} = (p_1, p_2, ..., p_n)$. In classical dynamics, once the state is completely described at some time t_0, we have all the information needed to completely and uniquely describe the state at any time in the future or the past. The state of a system $(\vec{q}, \vec{p})$ lives in $2n$-dimensional **phase-space** Γ. The state-variables $(\vec{q}, \vec{p})$ are known as the **canonical coordinates** of the system. Note that, in principle, one could also define a state-space using $(\vec{q}, \dot{\vec{q}})$. However, since this set of variables is not canonical (see next chapter for a definition), working in phase-space turns out to be far more powerful.

As the system evolves, it describes a path in phase-space. But unlike the paths in configuration space, paths in phase-space can never cross (self-intersect), as this would violate the **deterministic** nature of classical dynamics (see also Chapter 6).

Hamilton's goal was to formulate mechanics in terms of $2n$ *first-order* differential equations for the $2n$ state-variables, rather than in terms of n *second-order* differential

equations for the n generalized coordinates, as in the Lagrangian framework. This implies that we need to replace our Lagrangian, $\mathcal{L}(q_i, \dot{q}_i, t)$, with a new function, $\mathcal{H}(q_i, p_i, t)$, called the **Hamiltonian**, which contains the same information as the Lagrangian. The procedure for switching variables in this manner is provided by the **Legendre transformation** (see Appendix E).

In particular, the Hamiltonian $\mathcal{H}(q_i, p_i, t)$ is generated by the Legendre transform

$$\boxed{\mathcal{H}(q_i, p_i, t) = \dot{q}_i p_i - \mathcal{L}(q_i, \dot{q}_i, t)}$$

which should be regarded as the equation that defines the Hamiltonian. Note that the first term uses the Einstein summation convention and should thus be read as a sum over n terms. Alternatively, we can also write the above expression in vector form as

$$\boxed{\mathcal{H}(\vec{q}, \vec{p}, t) = \dot{\vec{q}} \cdot \vec{p} - \mathcal{L}(\vec{q}, \dot{\vec{q}}, t)}$$

The total derivative of the Hamiltonian is

$$\mathrm{d}\mathcal{H} = \frac{\partial \mathcal{H}}{\partial q_i}\,\mathrm{d}q_i + \frac{\partial \mathcal{H}}{\partial p_i}\,\mathrm{d}p_i + \frac{\partial \mathcal{H}}{\partial t}\,\mathrm{d}t$$

An alternative expression for $\mathrm{d}\mathcal{H}$ can be obtained from the Legendre transformation:

$$\begin{aligned}
\mathrm{d}\mathcal{H} &= \dot{q}_i\,\mathrm{d}p_i + p_i\,\mathrm{d}\dot{q}_i - \mathrm{d}\mathcal{L} \\
&= \dot{q}_i\,\mathrm{d}p_i + p_i\,\mathrm{d}\dot{q}_i - \left(\frac{\partial \mathcal{L}}{\partial q_i}\,\mathrm{d}q_i + \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\,\mathrm{d}\dot{q}_i + \frac{\partial \mathcal{L}}{\partial t}\,\mathrm{d}t\right) \\
&= \dot{q}_i\,\mathrm{d}p_i - \frac{\partial \mathcal{L}}{\partial q_i}\,\mathrm{d}q_i - \frac{\partial \mathcal{L}}{\partial t}\,\mathrm{d}t \\
&= \dot{q}_i\,\mathrm{d}p_i - \dot{p}_i\,\mathrm{d}q_i - \frac{\partial \mathcal{L}}{\partial t}\,\mathrm{d}t
\end{aligned}$$

Here we have used the definition of the generalized momentum in the third step, and the Euler-Lagrange equation in the fourth step.

Since the two expressions for $\mathrm{d}\mathcal{H}$ have to be identical, we obtain the following $2n+1$ equation

$$\boxed{\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i}\,, \qquad \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i}\,, \qquad \frac{\partial \mathcal{L}}{\partial t} = -\frac{\partial \mathcal{H}}{\partial t}}$$

These are **Hamilton's equations**. Note that the first $2n$, which in vector form read

$$\boxed{\dot{\vec{q}} = \frac{\partial \mathcal{H}}{\partial \vec{p}}\,, \qquad\qquad \dot{\vec{p}} = -\frac{\partial \mathcal{H}}{\partial \vec{q}}}$$

are the $2n$ first-order differential equations that replace the n second-order Euler-Lagrange equations as the equations of motion . The latter equation implies that

$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = \frac{\partial \mathcal{H}}{\partial q_i}\,\dot{q}_i + \frac{\partial \mathcal{H}}{\partial p_i}\,\dot{p}_i + \frac{\partial \mathcal{H}}{\partial t} = \frac{\partial \mathcal{H}}{\partial t} = -\frac{\partial \mathcal{L}}{\partial t}$$

where the second step follows from Hamilton's equations of motion. Hence, if the Lagrangian does not explicitly depend on time, then the Hamiltonian is a constant of motion (i.e., as the system evolves in phase-space, its value remains fixed)[2]

It can be proven that[3] **if** the equations of transformation that define the generalized coordinates, $\vec{r}_m = \vec{r}_m(q_1, q_2, ..., q_n, t)$, do **not** depend explicitly on time, **and** the potential V is independent of velocity ($\partial V/\partial \dot{\vec{r}}_m = 0$, i.e., no friction), **then** the Hamiltonian is equal to the total energy, i.e.,

$$\mathcal{H} = K + W = E$$

Note, though, that this is not to be considered the definition of the Hamiltonian, which instead is given by the Legendre transformation. Nevertheless, since we are mainly concerned with conservative forces, for which $\partial V/\partial \dot{\vec{r}}_m = 0$, and with systems without any holonomic constraints (such that $n = 3N$), in almost all cases considered in these lecture notes the Hamiltonian will indeed be equal to the total energy

If the potential is also time-independent, then time does not explicitly appear in the Lagrangian and we thus have that

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = \frac{\partial \mathcal{H}}{\partial t} = -\frac{\partial \mathcal{L}}{\partial t} = 0$$

i.e., the total energy of the system is conserved.

[2]For a more detailed discussion regarding the physical interpretation of the total and partial derivatives, see Part III.

[3]See for example the book "Classical Mechanics" by John R. Taylor

CHAPTER 4

Hamilton-Jacobi Theory

As we have seen in the previous chapter, in Hamiltonian dynamics we describe a system's state using the phase-space coordinates q_i and p_i, which are the generalized coordinates and the generalized momenta, respectively. The latter are conjugate to the generalized velocities $\dot{q}_i$, and are therefore also known as the conjugate momenta. The set $(\vec{q}, \vec{p})$ are called **canonical variables** of the system. They play a special role in dynamics, and it is because of this that the Hamiltonian formalism of dynamics is so powerful.

We start this chapter on Hamilton-Jacobi theory by introducing a few key concepts in dynamics, namely Poisson brackets and canonical transformations.

Poisson brackets:

Given two functions $A(q_i, p_i)$ and $B(q_i, p_i)$ of the **canonical** phase-space coordinates q_i and p_i, the **Poisson bracket** of A and B is defined as

$$\{A, B\} = \frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i}$$

where the Einstein summation convention has been implied. In vector form, we have

$$\{A, B\} = \frac{\partial A}{\partial \vec{q}} \cdot \frac{\partial B}{\partial \vec{p}} - \frac{\partial A}{\partial \vec{p}} \cdot \frac{\partial B}{\partial \vec{q}}$$

Note that some textbooks use square-brackets to indicate Poisson brackets, i.e., $[A, B]$. Throughout these lecture notes we adopt the curly-bracket notation.

It is straightforward to verify the following properties of Poisson brackets:

$$\begin{aligned}
\{A, B\} &= -\{B, A\} \\
\{\alpha A + \beta B, C\} &= \alpha\{A, C\} + \beta\{B, C\} \quad \forall \alpha, \beta \in \mathbb{R} \\
\{AB, C\} &= A\{B, C\} + B\{A, C\} \\
\{\{A, B\}, C\} + \{\{B, C\}, A\} + \{\{C, A\}, B\} &= 0
\end{aligned}$$

The first of these rules implies that $\{A, A\} = 0$. The second expresses the **linearity** of the Poisson bracket, and the fourth rule is known as the **Jacobi identity**.

Suppose we choose functions A and B to be the canonical variables $\vec{q}$ and $\vec{p}$ themselves. Then, one infers that they obey

$$\boxed{\{q_i, q_j\} = 0, \qquad \{p_i, p_j\} = 0, \qquad \{q_i, p_j\} = \delta_{ij}}$$

These relations are known as the **canonical commutation relations**. Any set $(\vec{Q}, \vec{P})$ of canonical variables has to obey these commutation relations, otherwise they are not a canonical set.

Poisson brackets are an extremely powerful construct in classical mechanics and beyond. First of all, they played an important role in carrying out the original transition from classical to quantum mechanics in that there is a simple *correspondence principle* according to which the classical Poisson brackets is to be replaced by a suitably defined **commutator** of the corresponding quantum operators. Furthermore, we have that, for any function $f(\vec{q}, \vec{p}, t)$,

$$\begin{aligned} \frac{\mathrm{d}f}{\mathrm{d}t} &= \frac{\partial f}{\partial q_i}\,\dot{q}_i + \frac{\partial f}{\partial p_i}\dot{p}_i + \frac{\partial f}{\partial t} \\ &= \frac{\partial f}{\partial q_i}\frac{\partial \mathcal{H}}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial \mathcal{H}}{\partial q_i} + \frac{\partial f}{\partial t} \\ &= \{f, \mathcal{H}\} + \frac{\partial f}{\partial t} \end{aligned}$$

This is sometimes called **Poisson's equation of motion**. It shows that the time-evolution of any dynamical variable is governed by the Hamiltonian through the Poisson bracket of the variable with the Hamiltonian. It contains Hamilton's equations of motion as a special case; if we substitute for f one of the canonical variables then we obtain

$$\boxed{\dot{q}_i = \{q_i, \mathcal{H}\}\,, \qquad \dot{p}_i = \{p_i, \mathcal{H}\}}$$

By introducing the $2n$-dimensional vector $\vec{w} = (\vec{q}, \vec{p})$, the **Hamiltonian equations of motion** can be written as

$$\boxed{\dot{\vec{w}} = \{\vec{w}, \mathcal{H}\}}$$

This elegant, super-compact form for the Hamiltonian equations of motion make it clear that the generalized coordinates and generalized momenta can be treated on equal footing.

Another example is to consider as f the Hamiltonian itself. Using that the Poisson bracket of a function with itself vanishes (we say that any function Poisson *commutes*

with itself), we obtain that

$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = \frac{\partial \mathcal{H}}{\partial t}$$

Hence, as we have already seen in Chapter 3, the Hamilonian will be conserved unless it has an explicit time dependence.

More generally, if we can find a function $f(\vec{q}, \vec{p})$ without any explicit time dependence for which the Poisson bracket with the Hamiltonian vanishes, i.e..

$$\{f, \mathcal{H}\} = 0\,,$$

then f is a **constant of motion** (i.e., $\mathrm{d}f/\mathrm{d}t = 0$). We say that f **Poisson commutes** with the Hamiltonian. If f *does* have an explicit time dependence, i.e., $f = f(\vec{q}, \vec{p}, t)$, then the condition for f to be a constant of motion becomes

$$\{\mathcal{H}, f\} = \frac{\partial f}{\partial t}$$

If f and g are constants of motion, then from the Jacobi identity we see that

$$\{\{f, g\}, \mathcal{H}\} = \{f, \{g, \mathcal{H}\}\} + \{\{f, \mathcal{H}\}, g\} = 0$$

which means that $\{f, g\}$ is also a constant of motion (**Poisson theorem**). We say that the constants of motion form a closed algebra under the Poisson bracket.

Finally, a constant of motion that does not have an explicit time dependence, and which thus Poisson commutes with the Hamiltonian, is called an **integral of motion**. Two integrals of motions that Poisson commute with each other are said to be in **involution**. As we will see later, such integrals of motion are extremely important for understanding the orbital structure of galaxies.

Canonical Transformations:

Canonical transformations are transformations of the form $(\vec{q}, \vec{p}) \rightarrow (\vec{Q}, \vec{P})$ between two canonical coordinate systems that leaves the equations of motion invariant.

In order to reveal the form of these transformations, we first demonstrate the **non-uniqueness of the Lagrangian**.

Consider a transformation

$$\mathcal{L} \rightarrow \mathcal{L}' = \mathcal{L} + \frac{\mathrm{d}F}{\mathrm{d}t}$$

where $F = F(\vec{q}, t)$. Under this transformation the **action integral** becomes

$$S' = \int_{t_1}^{t_2} \mathcal{L}' \, \mathrm{d}t = \int_{t_1}^{t_2} \mathcal{L} \mathrm{d}t + \int_{t_1}^{t_2} \frac{\mathrm{d}F}{\mathrm{d}t} \, \mathrm{d}t = S + F(t_2) - F(t_1)$$

The equations of motion derive from $\delta S = 0$ (i.e., **the least action principle**). Here the variations δS are such that the path between $q_1 = q(t_1)$ and $q_2(t_2)$ is varied, while the beginning and end points are kept fixed (i.e., $\delta F(t_2) = \delta F(t_1) = 0$). Hence, we have that $\delta S' = \delta S$, which also implies that the equations of motion are kept invariant.

Now consider our transformation $(\vec{q}, \vec{p}) \rightarrow (\vec{Q}, \vec{P})$ with corresponding Lagrangians $\mathcal{L}(\vec{q}, \dot{\vec{q}}, t)$ and $\mathcal{L}'(\vec{Q}, \dot{\vec{Q}}, t)$. We start by writing the Lagrangians in terms of the corresponding Hamiltonians (see Chapter 3):

$$\begin{aligned} \mathcal{L}(\vec{q}, \vec{p}, t) &= \vec{p} \cdot \dot{\vec{q}} - \mathcal{H}(\vec{q}, \vec{p}, t) \\ \mathcal{L}'(\vec{Q}, \vec{P}, t) &= \vec{P} \cdot \dot{\vec{Q}} - \mathcal{H}'(\vec{Q}, \vec{P}, t) \end{aligned}$$

In order for the equations of motion to be invariant, we have the requirement that

$$\begin{aligned} \mathcal{L}(\vec{q}, \vec{p}, t) &= \mathcal{L}'(\vec{Q}, \vec{P}, t) + \frac{\mathrm{d}F}{\mathrm{d}t} \\ \Leftrightarrow \frac{\mathrm{d}F}{\mathrm{d}t} &= \vec{p} \cdot \dot{\vec{q}} - \mathcal{H}(\vec{q}, \vec{p}, t) - \left[\vec{P} \cdot \dot{\vec{Q}} - \mathcal{H}'(\vec{Q}, \vec{P}, t) \right] \\ \Leftrightarrow \mathrm{d}F &= p_i \mathrm{d}q_i - P_i \mathrm{d}Q_i + (\mathcal{H}' - \mathcal{H}) \mathrm{d}t \end{aligned}$$

If we take $F = F(\vec{q}, \vec{Q}, t)$ then we also have that

$$\mathrm{d}F = \frac{\partial F}{\partial q_i} \mathrm{d}q_i + \frac{\partial F}{\partial Q_i} \mathrm{d}Q_i + \frac{\partial F}{\partial t} \mathrm{d}t$$

Equating the two expressions for the differential $\mathrm{d}F$ yields the transformation rules:

$$\boxed{p_i = \frac{\partial F}{\partial q_i}, \qquad P_i = -\frac{\partial F}{\partial Q_i}, \qquad \mathcal{H}' = \mathcal{H} + \frac{\partial F}{\partial t}}$$

The function $F(\vec{q}, \vec{Q}, t)$ is called the **generating function** of the **canonical transformation** $(\vec{q}, \vec{p}) \rightarrow (\vec{Q}, \vec{P})$. For simplicity, in what follows we only consider generating functions that do not explicitly depend on time, such that $\mathcal{H}' = \mathcal{H}$.

In order to transform $(\vec{q}, \vec{p}) \rightarrow (\vec{Q}, \vec{P})$ one proceeds as follows:

- Find a function $F(\vec{q}, \vec{Q})$ so that $p_i = \partial F/\partial q_i$. Inversion of this equation yields $Q_i(q_j, p_j)$.
- Substitute $Q_i(q_j, p_j)$ in the equation $P_i = \partial F/\partial Q_i$ to obtain $P_i(q_j, p_j)$.

Example: consider the generating function $F(\vec{q}, \vec{Q}) = q_i Q_i$. According to the transformation rules we have that

$$p_i = \frac{\partial F}{\partial q_i} = Q_i \,, \qquad P_i = -\frac{\partial F}{\partial Q_i} = -q_i$$

We thus have that $Q_i = p_i$ and $P_i = -q_i$: the canonical transformation has changed the roles of the generalized coordinates and the generalized momenta, eventhough the equations of motion have remained invariant! This demonstrates that Hamilton's original goal, to devise a dynamics in which all state variables are on a more symmetric footing, has indeed been achieved: there is no special status to either generalized coordinates or their conjugate momenta!

The example discussed above considers a generating function of the form $F = F(\vec{q}, \vec{Q})$, i.e., one that depends on both the old and new generalized coordinates. But this is not the only choice for a generating function for a canonical transformation. Any function of mixed (old and new) variables, whether generalized coordinates or generalized momenta will work, although each has its own transformation rules, as summarized below:

$F_1(\vec{q}, \vec{Q})$:	$p_i = +\partial F_1/\partial q_i$	$P_i = -\partial F_1/\partial Q_i$
$F_2(\vec{q}, \vec{P})$:	$p_i = +\partial F_2/\partial q_i$	$Q_i = +\partial F_2/\partial P_i$
$F_3(\vec{p}, \vec{Q})$:	$q_i = -\partial F_3/\partial p_i$	$P_i = -\partial F_3/\partial Q_i$
$F_4(\vec{p}, \vec{P})$:	$q_i = -\partial F_4/\partial p_i$	$Q_i = +\partial F_4/\partial P_i$

Properties of the four basic canonical transformations

An important property of **canonical transformations** is that they **leave Poisson brackets invariant**. In other words,

$$\{A, B\}_{\vec{q},\vec{p}} = \{A, B\}_{\vec{Q},\vec{P}}$$

where the index indicates the canonical coordinate system used. And since Hamilton's equations of motion can be expressed in terms of Poisson brackets, we thus see that

Hamilton's equations of motion are **invariant** under **canonical transformations**

Summary: In Newtonian mechanics, the $3N$ equations of motion (which are second-order differential equations) plus k holonomic equations depend on which coordinate system one adopts. In Lagrangian dynamics, the $n = 3N - k$ Euler-Lagrange equations (which are also second-order differential equations) are invariant to changes in the generalized coordinate system (i.e., are invariant to transformations $\vec{q} \to \vec{Q}$). In Hamiltonian dynamics, the $2n$ equations of motion (which are first-order differential equations) are invariant to *canonical* transformations $(\vec{q}, \vec{p}) \to (\vec{Q}, \vec{P})$.

The Hamilton-Jacobi equation:

We have all experienced the difficulty of solving problems in physics using the 'wrong' (as in 'not-optimal') coordinate system. For example, working in Cartesian coordinates if the problem really requires, say, spherical coordinates. Although it is always possible to find the correct answer, with enough will-power, you end up making your life unnecessarily difficult. The ability to change coordinate systems can drastically simplify a problem. An obvious question that comes to mind is whether there is a particular canonical transformation $(\vec{q}, \vec{p}) \to (\vec{Q}, \vec{P})$ for which the solution to the equations of motion become extremely simple. As we will see, under the right conditions the answer is a solid 'Yes'.

Suppose we can find a canonical transformation $(\vec{q}, \vec{p}) \to (\vec{Q}, \vec{P})$ for which the new Hamiltonian $\mathcal{H}'(\vec{Q}, \vec{P}) = \mathcal{H}'(\vec{P})$. In other words, for which all the new generalized coordinates are **cyclic** (or **ignorable**). The equations of motion now become

$$\dot{P}_i = -\frac{\partial \mathcal{H}'}{\partial Q_i} = 0\,, \qquad \dot{Q}_i = \frac{\partial \mathcal{H}'}{\partial P_i} = \text{constant}$$

Hence, the new generalized momenta P_i are **integrals of motion**, and since $\partial \mathcal{H}'/\partial P_i$ only depends on $\vec{P}$, it has no time dependence either. Hence, the solution for the dynamics is trivially given by

$$Q_i(t) = \omega_i\, t + Q_i(0)\,, \qquad P_i(t) = P_i(0)$$

While the generalized momenta are conserved, the generalized coordinates increase linearly with time at a rate $\omega_i \equiv \partial \mathcal{H}'/\partial P_i$. Clearly a beautifully simple and elegant solution (the topology of which we will discuss a bit later).

The key question now becomes how to find the generator of the canonical transformation $(\vec{q}, \vec{p}) \to (\vec{Q}, \vec{P})$ which leads to only cyclic Q_i. In what follows we focus exclusively on conservative systems for which $\partial \mathcal{H}/\partial t = 0$ (i.e., no explicit time dependence for the Hamiltonian), and we assume that $\mathcal{H} = E$. We will consider a generator of the second kind $F_2(\vec{q}, \vec{P}, t)$ for which the transformation rules are

$$p_i = \frac{\partial F_2}{\partial q_i}\,, \qquad Q_i = \frac{\partial F_2}{\partial P_i}\,, \qquad \mathcal{H}' = \mathcal{H} + \frac{\partial F_2}{\partial t}$$

Let's consider a generator that does not explicitely depend on time, i.e., $\partial F_2/\partial t = 0$. Then $\mathcal{H}(\vec{q}, \vec{p}) = \mathcal{H}'(\vec{P}) = E$. If we now substitute $\partial F_2/\partial q_i$ for p_i in the original Hamiltian we obtain

$$\boxed{\mathcal{H}\left(q_i, \frac{\partial F_2}{\partial q_i}\right) = E}$$

This is the (restricted) **Hamilton-Jacobi equation**. It is a partial differential equation (PDE) of n variables, q_i. The solution for the generating function is called **Hamilton's characteristic function** and is typically indicated by the symbol $W(\vec{q}, \vec{P})$, rather than $F_2(\vec{q}, \vec{P})$.

Hence, all we got to do is solve the Hamilton-Jacobi equation for Hamilton's characteristic function, use that to derive the new canonical coordinates $\vec{Q}$ and $\vec{P}$, and our dynamical system is solved trivially. Sounds easy... However, solving a PDE is extremely difficult, and the Hamilton-Jacobi equation is no exception. But the real strength of the Hamilton-Jacobi equation is not in providing yet another method of solving the equations of motion; rather, its main utility lies in the structure it reveals about classical dynamics. Before we adress this, though, it is important to point out one condition under which solving the Hamilton-Jacobi equation is drastically simplified, namely if the generator, Hamilton's characteristic function, is **separable**, i.e.,

$$W(\vec{q}, \vec{P}) = \sum_{i=1}^{n} W_i(q_i, \vec{P})$$

Under this condition the n-dimensional PDE reduces to a set of n first-order ODEs, which are easily solved by quadrature.

If the Hamilton-Jacobi equation is **separable**,
we say that the Hamiltonian is **integrable**

Note than an **integrable Hamiltonian** with n degrees of freedom thus has n independent **integrals of motion** (i.e., in our example these are the n generalized momenta P_i). Integrals of motion are independent if they are in **involution**, i.e., $\{P_i, P_j\} = 0$. Note that this condition is automatically satisfied because $\vec{P}$ is canonical and therefore satisfies the **canonical commutation relations**. This indicates an important aspect of integrable systems:

> **Liouville' Theorem of Integrable Systems:** If a system of n degrees of freedom has n mutually Poisson commuting integrals of motion $I_1, I_2, ..., I_n$ (i.e., n integrals of motion in involution), then the system is integrable.

The requirement that the integrals of motion Poisson commmute, i.e., $\{I_i, I_j\} = 0$, is simply another way of saying that we can view I_i as canonical momentum variables.

As you might expect, integrable Hamiltonians are rare (and precious). Most systems, especially those encountered in astrophysics, are not-integrable (chaotic systems notably among them). In a non-integrable system, there are no n integrals of motion in involution, and their is no solution for the Hamilton-Jacobi equation. Note, though, that this refers to a *global* solution. In almost all cases in which the Hamiltonian is not integrable there are large parts of phase-space in which you can locally find a transformation for which the equations of motion take on the simple form depicted above.

Recommended exercise: Students are strongly encouraged, at this point, to study **Worksheet 2**, which examines the integrability of 10 Hamiltonian systems.

Integrability of galaxies:

We end this chapter by discussing the integrability of galaxies, which are N-body systems in which all forces (gravity) are central. As discussed in Worksheet 2, an N-body system in which all forces are central (i.e., gravity) is **not integrable** for $N > 2$. This begs the question: does that mean that every galaxy is not integrable?

At first sight, the answer would have to be a strong yes, since integrability requires that the galaxy must have $3N$ integrals of motion in involution. However, in general an N-body system only has 10 integrals of motion (the so-called Galilean invariants; see Worksheet 2), and those are not even in involution. However, if N is large the system becomes collisionless. This means that the potential $V(\vec{q}_1, \vec{q}_2, ..., \vec{q}_N)$ can be written as

$$V(\vec{q}_1, \vec{q}_2, ..., \vec{q}_N) = \sum_i V_{\rm ext}(\vec{q}_i)$$

i.e., we consider the force on particle i due to the other $N-1$ particles as if it arises from a (time-independent), smooth, external potential. Hence, the Hamiltonian is

$$\mathcal{H}(\vec{q}_1, \vec{q}_2, ..., \vec{q}_N, \vec{p}_1, \vec{p}_2, ..., \vec{p}_N) = \sum_{i=1}^{N} \left[\frac{\vec{p}_i^2}{2m} + V_{\rm ext}(\vec{q}_i) \right] = \sum_{i=1}^{N} \mathcal{H}_i(\vec{q}_i, \vec{p}_i)$$

If the system as a whole has spherical symmetry, such that $V_{\rm ext}(\vec{q}) = V_{\rm ext}(r)$, then this Hamiltonian is simply the sum of n independent central force problems, each of which **is** integrable (see example 7 in Worksheet 2).

Hence, a gravitational, spherical N-body system is integrable as long as N is sufficiently large such that the system is, to good approximation, collisionless. So what are the $3N$ integrals of motion in involution for such a system? Well, each star conserves energy, and angular momentum and thus the 3N integrals of motion in involution are E_i, $L_{z,i}$ and $\vec{L}_i^2$ for $i = 1, ..., N$.

If $V_{\rm ext}(\vec{q}) = V_{\rm ext}(R, z)$, such that we have axisymmetry, rather than spherical symmetry, the situation changes. Since θ is a cyclic variable, we have a conserved quantity (in addition to energy), which is L_z. So each star has two integrals of motion in involution; E and L_z. This is not sufficient to guarantee integrability, and in general an axisymmetric galaxy is indeed **not** integrable. However, under certain conditions there can be a third integral of motion I_3, which is in involution with E and L_z, and the system **can** thus be integrable. More general, though, only certain parts of phase-space admit a third integral, and the system consists of both regular orbits (those that admit three integrals of motion in involution) and chaotic, or irregular orbits (those that only have E and L_z as integrals of motion).

CHAPTER 5

Action-Angle Variables

Consider an integrable Hamiltonian with n degrees of freedom and with $(I_1, I_2, ..., I_n)$ a set of n integrals of motion in involution. Now define $I_a = \frac{1}{2}(I_1 + I_2)$ and $I_b = \frac{1}{2}(I_1 - I_2)$. By construction, I_a and I_b are also integrals of motion, and it is easy to see that $\{I_a, I_b\} = 0$. Hence, the set $(I_a, I_b, I_3, ..., I_n)$ is also a set of n integrals of motion in involution. In other words, there is no unique set of canonical variables that solve the Hamilton-Jacobi equations. This begs the question, which one is the most optimal set? The answer to that question is the set of **action-angle variables**.

Before we go and discuss this special, powerful set of canonical variables, let's take a closer look at the trivial solution of an integrable Hamiltonian system:

$$Q_i(t) = \omega_i t + Q_i(0)\,, \qquad P_i(t) = P_i(0)$$

Hence, while the generalized momentum is an integral of motion, the generalized coordinates grow linearly in time with a rate given by $\omega_i = \partial\mathcal{H}'/\partial P_i$. At first sight this seems weird, as it seems to imply that the system must somehow expand indefinitely, while yet conserving the total energy. The solution is that, in almost all integrable Hamiltonians of interest, the motions $Q_i(t)$ are all periodic (i.e., 'cyclic'), and that ω_i indicates the corresponding **frequency** (which is why we picked this symbol). We distinguish two different kinds of 'periodic motion', which is best illustrated using a simple pendulum (see Fig. 1).

- **Libration:** at low energy, the pendulum swings back and forth between states in which the generalized momentum p corresponding to the generalized coordinate q (i.e., the angle θ) vanishes. The corresponding trajectory in (q, p)-phase-space is *closed* (i.e., in the case of the pendulum in question, it is an ellipse), such that both q and p evolve periodically and with the same frequency, i.e., the state returns to the same position in phase-space at the end of each period. In the case depicted in Fig. 1, q is an angle that is restricted to a range smaller than $[0, 2\pi]$.

- **Rotation:** at high enough energy, the pendulum continuously rotates, and the momentum never vanishes. The momentum now is some periodic function of q, which itself evolves without bounds. Although the trajectory in phase-space

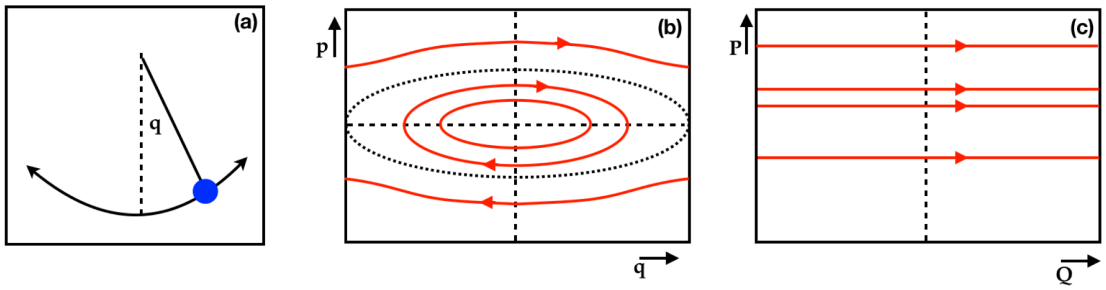


Figure 1: *Left-hand panel (a) shows a pendulum; an integrable Hamiltonian system of one degree of freedom, the angle q. The middle planel (b) shows possible trajectories in (q, p) phase-space. For sufficiently low energy the solutions are librations, indicated by elliptical trajectories. After energy increases, and one crosses the* **separatrix**, *indicated by the black dotted ellipse, one enters the part of phase-space in which the pendulum undergoes rotations (note how these trajectoriess never cross the $p = 0$ line). Finally, the right-hand panel shows trajectories in the phase-space (Q, P), which are the canonical variables obtained from (q, p) using a transformation with Hamilton's characteristic function. Note how in this new coordinate system the generalized momenta P are all integrals of motion. Both q and Q range from 0 to 2π, and one should think of phase-space as a cylinder that is cut open along its axis and rolled out.*

is not closed, each period the evolution of the system is the same, leading to a trajectory that repeats itself with a translation. In the case depicted in Fig. 1, q is an angle that continues to grow, but modulo 2π.

The areas in phase-space where the phase-space trajectories correspond to **librations** and **rotations** are separated by a **separatrix** (dotted ellipse in the middle panel of Fig. 1). The trajectory that traces out the separatrix corresponds to the case where the pendulum has just the right amount of energy to make it to the top, where it ends up with zero momentum.

We can treat integrable Hamiltonian systems with the **action-angle formalism** if *each* pair (q_i, p_i) has either a librating or a rotating motion. The **actions** I_i take the role of the generalized momenta and are defined by

$$\boxed{I_i = \oint p_i \, \mathrm{d}q_i}$$

NOTE: no Einstein summation convention here; $p_i \mathrm{d}q_i$ is not a sum over all i, but rather the product for i only.

The 'angles' θ_i are the corresponding generalized coordinates, and are proper angles. We'll now demonstrate what is so special about $(\vec{\theta}, \vec{I})$ as a set of canonical variables.

If the Hamiltonian is **integrable** then Hamilton's characteristic function, which is a generating function for a type-2 canonical transformation, meaning that W is of type $F_2(\vec{q}, \vec{P})$, is **separable**, such that

$$W(\vec{q}, \vec{P}) = \sum_i W_i(q_i, P_1, P_2, ..., P_n) = \sum_i W_i(q_i, \vec{P})$$

According to the transformation rules for a type-2 canonical transformation we have that

$$p_i = \frac{\partial W}{\partial q_i} = \frac{\partial W_i}{\partial q_i} = p_i(q_i, \vec{P})$$

Hence, the action

$$I_i = \oint p_i \, \mathrm{d}q_i = \oint p_i(q_i, \vec{P}) \, \mathrm{d}q_i = I_i(\vec{P})$$

where the latter follows from the fact that we are integrating over the periodic motion in q_i. Thus, we see that the actions are functions of the generalized momenta, P_i, only, for which the corresponding generalized coordinates are cyclic. Inversly, we have that $P_i = P_i(\vec{I})$. And since P_i are integrals of motion, so are the I_i; after all

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \sum_i \frac{\partial I}{\partial P_i} \dot{P}_i = 0$$

since $\dot{P}_i = 0$. And since $\mathcal{H} = \mathcal{H}(\vec{P})$, we thus have that $\mathcal{H} = \mathcal{H}(\vec{I})$, which indicates that the angles, θ_i (the generalized coordinates corresponding to the actions) are all cyclic!

Using the transformation rules for type-2 canonical transformations we have that

$$\theta_i = \frac{\partial W}{\partial I_i} = \sum_i \frac{\partial W_i}{\partial I_i} = \theta_i(q_1, ..., q_n, I_1, ..., I_n)$$

Similarly, from Hamilton's equations of motion we infer that

$$\omega_i \equiv \dot{\theta}_i = \frac{\partial \mathcal{H}}{\partial I_i} = \omega_i(I_1, ..., I_n)$$

And since the actions are integral of motion, we thus see that the ω_i are constants as well (independent of time)! Hence, we have the standard solution for an integrable Hamiltonian;

$$\theta_i(t) = \omega_i t + \theta_i(0) \,, \qquad I_i(t) = I_i(0)$$

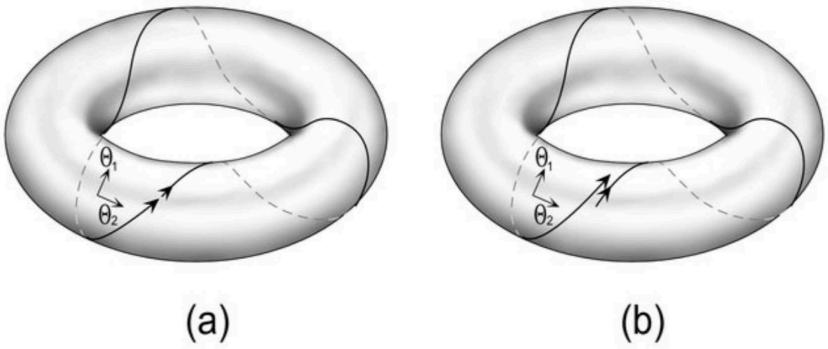


Figure 2: *Comparison of phase-space trajectories on 2-tori. The trajectory traced out on the 2-torus in (a) corresponds to a resonance, with* $\omega_1/\omega_2 = 3$*, causing the trajectory to close itself after three cycles in* θ_1 *(corresponding to one cycle in* θ_2*). The trajectory on the 2-torus in (b) is non-resonant (*ω_1 *and* ω_2 *are incommensurable) and will eventually cover the entire surface of the 2-torus. [from Masoliver & Ros, arXiv: 1012.4384]*

So far, nothing special really, because the same holds for $(\vec{Q}, \vec{P})$. To see what makes the action-angle variables so special, we now compute by how much the angle θ_i changes during one period of its periodic motion:

$$\Delta\theta_i = \oint \frac{\partial \theta_i}{\partial q_i} \mathrm{d}q_i = \oint \frac{\partial^2 W}{\partial I_i \, \partial q_i} \mathrm{d}q_i = \frac{\partial}{\partial I_i} \oint \frac{\partial W}{\partial q_i} \mathrm{d}q_i = \frac{\partial}{\partial I_i} \oint p_i \, \mathrm{d}q_i = \frac{\partial I_i}{\partial I_i} = 1$$

Here we have used that $\theta_i = \partial W / \partial I_i$ and that I_i is a constant and can thus be taken out of the integral. Since $\theta_i(t) = \omega_i t + \theta_i(0)$ we also have that $\Delta\theta_i = \omega_i \, T$, where T is the period of the periodic motion in θ_i. Combining the two expressions for $\Delta\theta$ we see that

$$\omega_i = \dot{\theta}_i = \frac{1}{T}$$

Hence, the time-derivative of the angle-variable is the actual **frequency** of the motion in the 'direction' associated with the i^{th} degree of freedom. This characteristic is *unique* to the action-angle variables, and is what makes them so powerful: *The action-angle formalism allows one to determine the frequencies of periodic motion without having to calculate the exact trajectories for the motion.* The 'recipe' for doing so is as follows:

- Calculate $I_i = \oint p_i \mathrm{d}q_i$
- Express the original Hamiltonian as a function of the actions; $\mathcal{H} = \mathcal{H}(I_1, ..., I_n)$
- Compute the frequencies using $\omega_i = \partial \mathcal{H} / \partial I_i$

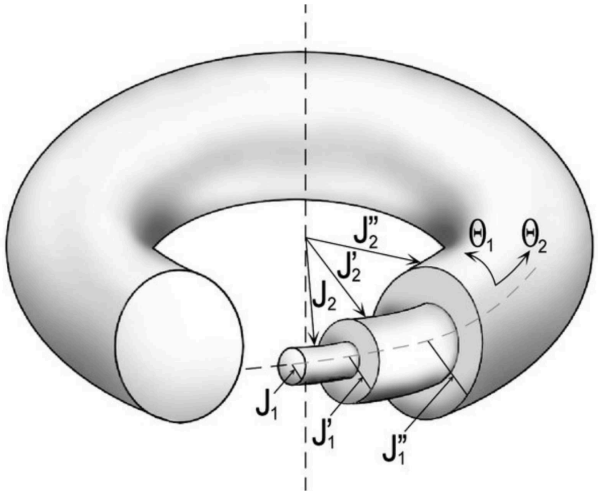


Figure 3: *The nesting of 2-tori for several values of the actions* (J_1, J_2). *This is only an illustration of how 2-tori foliate phase-space; in reality the foliation is in 4-dimensional phase-space. [from Masoliver & Ros, arXiv: 1012.4384]*

The action-angle variables also give a powerful insight into the structure of integrable Hamiltonians. If a Hamiltonian is integrable, and all its motion is periodic, then there exists a set of action-angle variables. Holding the actions $\vec{I} = (I_1, ..., I_n)$ fixed, the corresponding angles trace out a n**-torus** in phase-space. To develop a feeling for what an n-torus is, start by considering a system with $n = 1$ degree of freedom (i.e., the pendulum of Fig. 1). It's motion, in action-angle coordinates (not the angle q depicted in the left-hand panel of Fig. 1) is such that θ varies periodically from 0 to 2π (see right-hand panel of Fig. 1). Hence, in action-angle phase-space it traces out a circle, i.e., the particle traces out a 1D topology in 2D phase-space.

If $n = 2$, then there are two angles that periodically trace out the range from 0 to 2π. The topology that this corresponds to is a 2-torus, which is equivalent to a donut (see Fig. 2). The actions I_1 and I_2, in this case, quantify the surface of this donut, and the system moves over the surface of this donut; one motion is the periodic motion with which it circulates the donut-hole (motion along angle θ_2 in Fig. 2), the other is the periodic motion with which it circulates the hull of the donut (motion along angle θ_1 in Fig. 2). If the two corresponding frequencies are incommensurable, then over time the phase-space trajectory of the particle will densely cover the entire surface of the donut. Hence, the particle is restricted to move on a 2D surface in 4D phase-space. If the frequencies are commensurable (i.e., ω_1/ω_2 is rational), then the trajectory is closed, and the phase-space trajectory is a 1D manifold (a line) traced

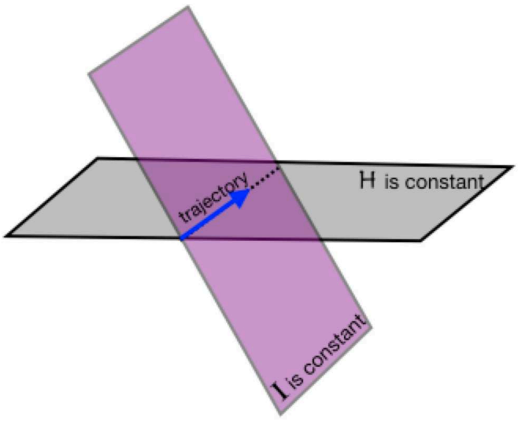


Figure 4: *Trajectory of a particle is restricted to intersection of surfaces of constant $\mathcal{H}$ and constant I. These planes represent 5D-manifolds in 6D phase-space, with the intersection being a 4D-manifold. With 3 integrals of motion (in involution), the particle trajectory is restricted to a 3D-manifold in 6D phase-space.*

out on the surface of the 2-torus.

In general, if there are n degrees of freedom, then there are n angles that trace out the range from 0 to 2π, and the topology is that of an n-torus. The surface of the n-dimensional torus is characterized by the n corresponding actions. Unless 2 or more of the frequencies ω_i are commensurable, the system evolves along an n-dimensional surface in $2n$-dimensional phase-space.

The phase-space trajectories in an **integrable** Hamiltonian system of n-degrees of freedom are restricted to an n-dimensional manifold (an n-torus) in $2n$ dimensional phase-space. All of accessible phase-space is covered, we say **foliated**, with a nesting of n-tori (see Fig. 3).

Fig. 4 shows a simple, but instructive, geometric interpretation of this result. Let $I = I(\vec{q}, \vec{p})$ be an integral of motion. Then, the vector $(\dot{\vec{q}}, \dot{\vec{p}}) = (\partial\mathcal{H}/\partial\vec{p}, -\partial\mathcal{H}/\partial\vec{q})$ is tangent to the surface $I(\vec{q}, \vec{p})$ =constant. This is easy to see from the fact that

$$\begin{aligned} \nabla I \cdot (\dot{\vec{q}}, \dot{\vec{p}}) &= \left(\frac{\partial I}{\partial \vec{q}}, \frac{\partial I}{\partial \vec{p}}\right) \cdot (\dot{\vec{q}}, \dot{\vec{p}}) \\ &= \frac{\partial I}{\partial q_i}\frac{\partial \mathcal{H}}{\partial p_i} - \frac{\partial I}{\partial p_i}\frac{\partial \mathcal{H}}{\partial q_i} = \{I, \mathcal{H}\} = 0 \end{aligned}$$

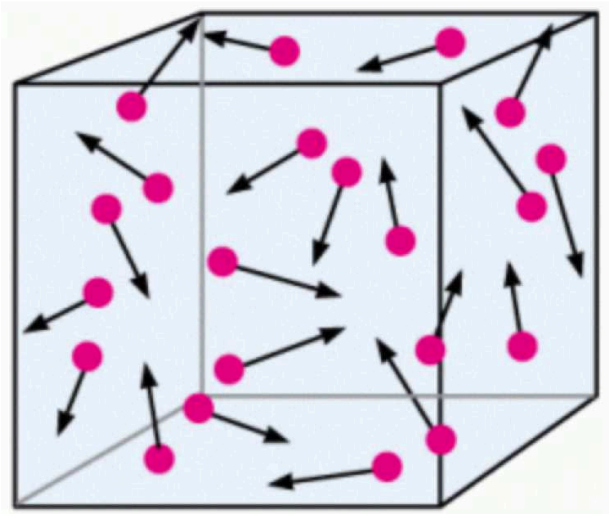
We also have that $(\dot{\vec{q}}, \dot{\vec{p}})$ is tangent to the surface $\mathcal{H}(\vec{q}, \vec{p})$ =constant, because

$$\nabla\mathcal{H} \cdot (\dot{\vec{q}}, \dot{\vec{p}}) = \left(\frac{\partial \mathcal{H}}{\partial \vec{q}}, \frac{\partial \mathcal{H}}{\partial \vec{p}}\right) \cdot (\dot{\vec{q}}, \dot{\vec{p}}) = -\dot{\vec{p}} \cdot \dot{\vec{q}} + \dot{\vec{q}} \cdot \dot{\vec{p}} = 0$$

Hence, the particle's trajectory has to move on the intersection of the two surfaces, as depicted in Fig. 4:

> For each integral of motion (in involution), the dimensionality of the manifold traced out by the particle is reduced by one.

Part II: Kinetic Theory



Kinetic theory is the body of theory which explains the physical properties of matter in terms of the motions of its constituent particles. The various 'fluids' encountered in this course only differ in how their constituent particles interact, and these differences give rise to a rich variety of physics. Hence, understanding kinetic theory is essential for developing an understanding of how a galaxy differs from the air that you breathe, or from the plasma that makes up the Sun. The following chapters develop the kinetic theory used to describe all these various fluids from the ground up, starting from Hamiltonian dynamics.

The material covered in this part is described in more detail in the following excellent textbooks:

- *The Physics of Fluids and Plasmas* by A. Choudhuri
- *Introduction to Plasma Theory* by D.R. Nicholson
- *Statistical Mechanics* by K. Huang
- *Modern Classical Physics* by K.Thorne & R. Blandford
- *Galactic Dynamics* by J. Binney & S. Tremaine
- *Dynamics and Evolution of Galactic Nuclei* by D. Merritt

CHAPTER 6

From Liouville to Boltzmann

We start this chapter by reminding ourselves of a few fundamental concepts in dynamics.

Degrees of freedom: The minimum number of independent coordinates needed to specify the position and configuration of a system in configuration space. In what follows we use $n_{\rm dof}$ to indicate the number of degrees of freedom. Throughout we will consider 3D systems of N particles without any **holonomic constraints** such that $n_{\rm dof} = 3N$.

State of a system: the information needed to uniquely quantify the system at a moment in time. For an N-body system this is typically the positions and velocities (or momenta) of all the particles, in which case the state-vector has dimensionality $2n_{\rm dof} = 6N$. Throughout we shall assume that all particles are identical without any internal properies other than mass. Such particles are sometimes called **monoatoms**, and can be treated as point particles.

Phase-Space: The phase-space of a dynamical system is a space in which all possible states of a system are represented, with each possible state corresponding to one unique point in that phase-space. For our monoatoms, the dimensionality of phase-space is $2n_{\rm dof} = 6N$.

`Caution:` I will use 'phase-space' to refer to **both** this $2n_{\rm dof}$-dimensional space, in which each state is associated with a point in that space, as well as to the 6-dimensional space $(\vec{x}, \vec{v})$ in which each individual particle is associated with a point in that space. In order to avoid confusion, in this chapter I will refer to the former as Γ-space, and the latter as μ-space.

Canonical Coordinates: in classical mechanics, canonical coordinates are coordinates q_i and p_i in phase-space that are used in the Hamiltonian formalism and that satisfy the **canonical commutation relations**:

$$\{q_i, q_j\} = 0, \qquad \{p_i, p_j\} = 0, \qquad \{q_i, p_j\} = \delta_{ij}$$

Let N be the number of constituent particles in our fluid. Typically, N will be a large number. How do you (classically) describe such a system? The **microstate** of a system composed of N monoatoms is completely described by

$$\vec{\Gamma} = (\vec{q}_1, \vec{q}_2, ..., \vec{q}_N, \vec{p}_1, \vec{p}_2, ..., \vec{p}_N)$$

which corresponds to a single point in our $6N$-dimensional phase-space (Γ-space). As already discussed in Part I, the dynamics of our fluid of N monoatoms is described by its **Hamiltonian**

$$\mathcal{H}(\vec{q}_i, \vec{p}_i, t) \equiv \mathcal{H}(\vec{q}_1, \vec{q}_2, ..., \vec{q}_N, \vec{p}_1, \vec{p}_2, ..., \vec{p}_N, t)$$

and the corresponding **equations of motion** are:

$$\boxed{\dot{\vec{q}}_i = \frac{\partial \mathcal{H}}{\partial \vec{p}_i}\,; \qquad \dot{\vec{p}}_i = -\frac{\partial \mathcal{H}}{\partial \vec{q}_i}}$$

In what follows we will often adopt a shorthand notation, which also is more 'symmetric'. We introduce the 6D vector $\vec{w} \equiv (\vec{q}, \vec{p})$, i.e., the 6D array one obtains when combining the 3 components of $\vec{q}$ with the 3 components of $\vec{p}$. Using **Poisson brackets**, we can then write the **Hamiltonian equations of motion** as

$$\boxed{\dot{\vec{w}}_i = \{\vec{w}_i, \mathcal{H}\}}$$

Thus, given $\vec{w}_i$ for all $i = 1, 2, ..., N$, at any given time t_0, one can compute the Hamiltonian and solve for the equations of motion to obtain $\vec{w}_i(t)$. This specifies a unique **trajectory** $\vec{\Gamma}(t)$ in this phase-space (see panel [a] of Fig. 5). Note that no two trajectories $\vec{\Gamma}_1(t)$ and $\vec{\Gamma}_2(t)$ are allowed to cross each other. If that were the case, it would mean that the same initial state can evolve differently, which would be a violation of the **deterministic character** of classical physics. The Hamiltonian formalism described above basically is a **complete treatment** of fluid dynamics. In practice, though, it is utterly useless, simply because N is HUGE, making it impossible to specify the complete set of initial conditions. We neither have (nor want) the detailed information that is required to specify a **microstate**. We are only interested in the average behavior of the **macroscopic** properties of the system, such as density, temperature, pressure, etc. With each such **macrostate** corresponds a huge number of microstates, called a **statistical ensemble**.

The ensemble is described statistically by the **N-body distribution function**

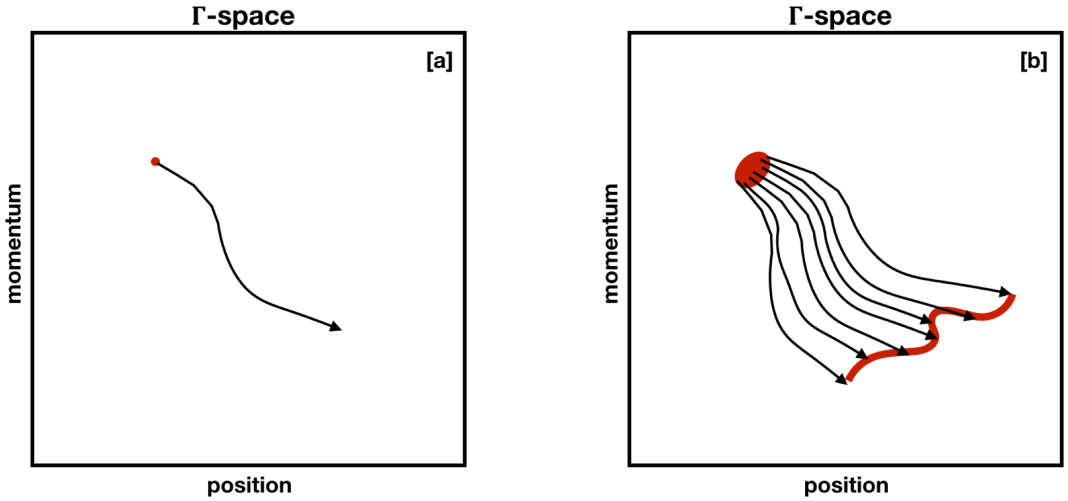


Figure 5: *Illustration of evolution in* Γ*-space. The* x*- and* y*-axes represent the* $3N$*-dimensional position-vector and momentum-vector, respectively. Panel (a) shows the evolution of a state (indicated by the red dot). As time goes on, the potitions and momentum of all the particles change (according to the Hamiltonian equations of motion), and the state moves around in* Γ*-space. Panel (b) shows the evolution of an ensemble of microstates (called a macrostate). As neighboring states evolve slightly differently, the volume in* Γ*-space occupied by the original microstates (the red, oval region) is stretched and sheared into a 'spagetti-like' feature. According to Liouville's theorem, the volume of this spagetti-like feature is identical to that of the original macrostate (i.e., the flow in* Γ*-space is incompressible). Note also, that two trajectories in* Γ*-space can NEVER cross each other.*

$$f^{(N)}(\vec{w}_i) \equiv f^{(N)}(\vec{w}_1, \vec{w}_2, ..., \vec{w}_N) = f^{(N)}(\vec{q}_1, \vec{q}_2, ..., \vec{q}_N, \vec{p}_1, \vec{p}_2, ..., \vec{p}_N)$$

which expresses the ensemble's probability distribution, i.e., $f^{(N)}(\vec{w}_i)\,\mathrm{d}V$ is the probability that the actual microstate is given by $\vec{\Gamma}(\vec{q}_i, \vec{p}_i)$, where $\mathrm{d}V = \prod_{i=1}^{N} \mathrm{d}^6\vec{w}_i = \prod_{i=1}^{N} \mathrm{d}^3\vec{q}_i\,\mathrm{d}^3\vec{p}_i$. This implies the following **normalization condition**

$$\int \mathrm{d}V\, f^{(N)}(\vec{w}_i) = 1$$

In our statistical approach, we seek to describe the evolution of the N-body distribution function, $f^{(N)}(\vec{w}_i, t)$, rather than that of a particular microstate, which instead is given by $\vec{\Gamma}(\vec{w}_i, t)$. Since probability is locally conserved, it must obey a **continuity equation**; any change of probability in one part of phase-space must be compensated by a flow of probability into or out of neighboring regions. The continuity equation of a (continuum) density field, $\rho(\vec{x})$, is given by

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\, \vec{v}) = 0$$

which expresses that the local change in the mass enclosed in some volume is balanced by the divergence of the flow out of that volume. In the case of our probability distribution $f^{(N)}$ we have that ∇ is in $6N$-dimensional phase-space, and includes $\partial/\partial\vec{q}_i$ and $\partial/\partial\vec{p}_i$, i.e.,

$$\nabla = \frac{\partial}{\partial \vec{w}_i} = \left(\frac{\partial}{\partial \vec{x}_i}, \frac{\partial}{\partial \vec{p}_i}\right) = \left(\frac{\partial}{\partial \vec{x}_1}, \frac{\partial}{\partial \vec{x}_2}, ..., \frac{\partial}{\partial \vec{x}_N}, \frac{\partial}{\partial \vec{p}_1}, \frac{\partial}{\partial \vec{p}_2}, ..., \frac{\partial}{\partial \vec{p}_N}\right)$$

Similarly, the 'velocity vector' in our $6N$-dimensional Γ-space is given by

$$\dot{\vec{w}} \equiv (\dot{\vec{q}}_i, \dot{\vec{p}}_i) = (\dot{\vec{q}}_1, \dot{\vec{q}}_2, ..., \dot{\vec{q}}_N, \dot{\vec{p}}_1, \dot{\vec{p}}_2, ..., \dot{\vec{p}}_N)$$

Hence, the continuity equation for $f^{(N)}$, which is known as the **Liouville equation**, can be written as

$$\frac{\partial f^{(N)}}{\partial t} + \nabla \cdot (f^{(N)}\, \dot{\vec{w}}) = 0$$

Using the fact that the gradient of the product of a vector and a scalar can be written as the sum of the scalar times the divergence of the vector, plus the dot-product of the vector and the gradient of the scalar (see Appendix A), we have that

$$\nabla \cdot (f^{(N)}\, \dot{\vec{w}}) = f^{(N)}\, \nabla \cdot \dot{\vec{w}} + \dot{\vec{w}} \cdot \nabla f^{(N)}$$

If we write out the divergence of $\dot{\vec{w}}$ as

$$\nabla \cdot \dot{\vec{w}} = \sum_{i=1}^{N} \left[\frac{\partial \dot{\vec{q}}_i}{\partial \vec{q}_i} + \frac{\partial \dot{\vec{p}}_i}{\partial \vec{p}_i}\right]$$

and use the Hamiltonian equations of motion to write $\dot{\vec{q}}_i$ and $\dot{\vec{p}}_i$ as gradients of the Hamiltonian, we find that

$$\nabla \cdot \dot{\vec{w}} = \sum_{i=1}^{N} \left[\frac{\partial}{\partial \vec{q}_i}\left(\frac{\partial \mathcal{H}}{\partial \vec{p}_i}\right) - \frac{\partial}{\partial \vec{p}_i}\left(\frac{\partial \mathcal{H}}{\partial \vec{q}_i}\right)\right] = \sum_{i=1}^{N} \left[\frac{\partial^2 \mathcal{H}}{\partial \vec{q}_i\, \partial \vec{p}_i} - \frac{\partial^2 \mathcal{H}}{\partial \vec{p}_i\, \partial \vec{q}_i}\right] = 0$$

Thus, we obtain the important result that

In a Hamiltonian system the flow in Γ-space is **incompressible**

This is generally known as the **Liouville Theorem**. It implies that the volume in Γ-space occupied by a macrostate does NOT change under Hamiltonian evolution. Although the microstates that make up the macrostate can disperse, the volume they occupy stays connected and constant; it typically will change shape, but its total volume remains fixed (see panel [b] of Fig. 5).

Using this result, we can write the **Liouville equation** in any of the following forms:

$$\frac{\partial f^{(N)}}{\partial t} + \dot{\vec{w}} \cdot \nabla f^{(N)} = 0$$

$$\frac{\partial f^{(N)}}{\partial t} + \sum_{i=1}^{N} \left(\dot{\vec{q}}_i \cdot \frac{\partial f^{(N)}}{\partial \vec{q}_i} + \dot{\vec{p}}_i \cdot \frac{\partial f^{(N)}}{\partial \vec{p}_i} \right) = 0$$

$$\frac{\mathrm{d}f^{(N)}}{\mathrm{d}t} = 0$$

$$\frac{\partial f^{(N)}}{\partial t} + \{f^N, \mathcal{H}\} = 0$$

The second expression follows from the first by simply writing out the terms of the divergence. The third expression follows from the second one upon realizing that $f^{(N)} = f^{(N)}(t, \vec{q}_1, \vec{q}_2, ..., \vec{q}_3, \vec{p}_1, \vec{p}_2, ..., \vec{p}_N)$ and using the fact that for a function $f(x, y)$ the infinitessimal $\mathrm{d}f = (\partial f/\partial x)\,\mathrm{d}x + (\partial f/\partial y)\,\mathrm{d}y$. Finally, the fourth expression follows from the second upon using the Hamiltonian equations of motion and the expression for the Poisson brackets, and will be used abundantly below.

Rather than describing the evolution of a single microstate, $\vec{\Gamma}(t)$, the Liouville equation describes the evolution of an ensemble of microstates (a macrostate). If anything, this makes computations even harder; for starters, the N-point distribution function $f^{(N)}$ is a function of $6N$ variables, which is utterly unmanageable. However, the Liouville equation is an important, powerful starting point for the development of a macroscopic description of the dynamics.

In particular, we seek to describe the evolution of the phase-space distribution func-

tion (DF)

$$f(\vec{q},\vec{p}) = \frac{\mathrm{d}^6 N}{\mathrm{d}^3\vec{q}\,\mathrm{d}^3\vec{p}}$$

which describes the density of particles in 6D phase-space $(\vec{q},\vec{p})$. In what follows, we shall refer to this 6-dimensional phase-space as μ-space, to distinguish it from the 6N-dimensional Γ-space. And we shall refer to the above DF as the 1-point DF, $f^{(1)}$, in order to distinguish it from the N-point DF, $f^{(N)}$, which appears in the Liouville equation. Whereas the latter describes the ensemble density of micro-states in Γ-space, the former describes the density of particles in μ-space.

Kinetic Theory: One can derive an equation for the time-evolution of the 1-point DF, starting from the Liouville equation. First we make the assumption that all particles are (statistically) identical. This implies that $f^{(N)}$ is a **symmetric function** of $\vec{w}_i$, such that

$$f^{(N)}(...,\vec{w}_i,...,\vec{w}_j,...) = f^{(N)}(...,\vec{w}_j,...,\vec{w}_i,...) \qquad \forall(i,j)$$

In words; if you flip the indices of any two particles, nothing changes. This allows us to derive an equation describing the evolution of the 1-point distribution function $f^{(1)}(\vec{w})$, as follows.

We first define the **reduced** or **k-particle DF**, which is obtained by integrating the N-body DF, $f^{(N)}$, over $N-k$ six-vectors $\vec{w}_i$. Since $f^{(N)}$ is symmetric in $\vec{w}_i$, without loss of generality we may choose the integration variables to be $\vec{w}_{k+1},\vec{w}_{k+2},...,\vec{w}_N$:

$$\boxed{f^{(k)}(\vec{w}_1,\vec{w}_2,...,\vec{w}_k,t) \equiv \frac{N!}{(N-k)!}\int \prod_{i=k+1}^{N} \mathrm{d}^6\vec{w}_i\, f^{(N)}(\vec{w}_1,\vec{w}_2,...,\vec{w}_N,t)}$$

where the choice of the prefactor will become clear in what follows.

In particular, the **1-particle distribution function** is

$$f^{(1)}(\vec{w}_1,t) \equiv N\int \prod_{i=2}^{N} \mathrm{d}^6\vec{w}_i\, f^{(N)}(\vec{w}_1,\vec{w}_2,...,\vec{w}_N,t)$$

Because of the prefactor, we now have that

$$\int \mathrm{d}^6\vec{w}_1\, f^{(1)}(\vec{w}_1,t) = N \int \prod_{i=1}^{N} \mathrm{d}^6\vec{w}_i\, f^{(N)}(\vec{w}_1,\vec{w}_2,...,\vec{w}_N,t) = N$$

where we have used the normalization condition of $f^{(N)}$. Hence, $f^{(1)}(\vec{q},\vec{p},t) = \mathrm{d}N/\mathrm{d}^3\vec{q}\,\mathrm{d}^3\vec{p}$ is the number of particles in the phase-space volume $\mathrm{d}^3\vec{q}\,\mathrm{d}^3\vec{p}$ centered on $(\vec{q},\vec{p})$.

That $f^{(1)}(\vec{w},t)$ is an important, relevant DF is evident from the following. Consider an **observable** $Q(\vec{w})$ that involves only quantities that depend additively on the phase-space coordinates of **single, individual particles** [i.e., $Q_{\text{ensemble}} = Q(\vec{w}_1) + Q(\vec{w}_2) + ... + Q(\vec{w}_N)$]. Examples are velocity, kinetic energy, or any other velocity moment v^k. The expectation value, $\langle Q\rangle$, can be written as

$$\langle Q\rangle = \int \mathrm{d}^6\vec{w}_1 ... \mathrm{d}^6\vec{w}_N f^{(N)}(\vec{w}_1,\vec{w}_2,...,\vec{w}_N) \sum_{i=1}^{N} Q_i$$

Since all particles are statistically identical, we can rewrite this as

$$\boxed{\langle Q\rangle = \int \mathrm{d}^6\vec{w}_1\, Q(\vec{w}_1)\, f^{(1)}(\vec{w}_1)}$$

Hence, computing the expectation value for any observable $Q(\vec{w})$ only requires knowledge of the 1-particle DF. And since all our macroscopic continuum properties of the fluid (i.e., density, bulk velocity, and internal energy) depend additively on the phase-space coordinates, the **1-particle DF** suffices for a macroscopic description of the fluid. Hence, our goal is to derive an evolution equation for $f^{(1)}(\vec{q},\vec{p},t)$. We do so as follows.

For the time evolution of each reduced DF we can write

$$\begin{aligned}
\frac{\partial f^{(k)}}{\partial t} &= \frac{N!}{(N-k)!} \int \prod_{i=k+1}^{N} \mathrm{d}^6\vec{w}_i\, \frac{\partial f^{(N)}}{\partial t}(\vec{w}_1,\vec{w}_2,...,\vec{w}_N) \\
&= \frac{N!}{(N-k)!} \int \prod_{i=k+1}^{N} \mathrm{d}^6\vec{w}_i\, \{\mathcal{H}, f^{(N)}\}
\end{aligned}$$

where the first step simply follows from operating the time derivative on the definition of the reduced k-particle DF, and the second step follows from the **Liouville equation**.

Next we substitute the Hamiltonian, which in general can be written as

$$\boxed{\mathcal{H}(\vec{q}_i, \vec{p}_i) = \sum_{i=1}^{N} \frac{\vec{p}_i^{\,2}}{2m} + \sum_{i=1}^{N} V(\vec{q}_i) + \frac{1}{2} \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} U(|\vec{q}_i - \vec{q}_j|)}$$

Note that the Hamiltonian contains three terms; a **kinetic** energy term, a term describing the potential energy due to an **external force** $\vec{F}_i = -\nabla V(\vec{q}_i)$ that only depends on the position of particle i (i.e., an example would be the gravitational field of Earth when describing it's atmosphere), and the potential energy $U(|\vec{q}_i - \vec{q}_j|)$ related to **two-body interactions** between particles i and j. The force on particle i due to the latter depends on the positions of all the other $N-1$ particles. Note that the factor of $1/2$ is to avoid double-counting of the particle pairs. Examples of the two-body interactions can be the VanderWaals force in the case of a liquid, the Coulomb force in the case of a plasma, or the gravitational force in the case of dark matter halo.

Substituting this expression for the Hamiltonian in the equation for the time-evolution of the reduced DF yields, after some tedious algebra (see Appendix H), an expression for the evolution of the **k-particle DF**

$$\boxed{\frac{\partial f^{(k)}}{\partial t} = \{\mathcal{H}^{(k)}, f^{(k)}\} + \sum_{i=1}^{k} \int \mathrm{d}^3\vec{q}_{k+1}\, \mathrm{d}^3\vec{p}_{k+1}\, \frac{\partial U(|\vec{q}_i - \vec{q}_{k+1}|)}{\partial \vec{q}_i} \cdot \frac{\partial f^{(k+1)}}{\partial \vec{p}_i}}$$

Here $\mathcal{H}^{(k)}$ is the Hamiltonian for the k-particles, which is simply given by

$$\mathcal{H}^{(k)}(\vec{w}_1, \vec{q}_2, ..., \vec{q}_k) = \sum_{i=1}^{k} \frac{\vec{p}_i^{\,2}}{2m} + \sum_{i=1}^{k} V(\vec{q}_i) + \frac{1}{2} \sum_{i=1}^{k} \sum_{\substack{j=1 \\ j \neq i}}^{k} U(|\vec{q}_i - \vec{q}_j|)$$

Note that the above expression for the evolution of the k-particle DF is not a closed function; it depends on $f^{(k+1)}$. Hence, if you want to solve for $f^{(k)}$ you first need to solve for $f^{(k+1)}$, which requires that you solve for $f^{(k+2)}$, etc. Thus, we have a

hierarcical set of N coupled differential equations, which is called the **BBGKY hierarchy** (after **B**ogoliubov, **B**orn, **G**reen, **K**irkwood and **Y**von, who independently developed this approach between 1935 and 1946).

Of particular interest to us is the expression for the **1-particle DF:**

$$\boxed{\frac{\partial f^{(1)}}{\partial t} = \{\mathcal{H}^{(1)}, f^{(1)}\} + \int \mathrm{d}^3\vec{q}_2\, \mathrm{d}^3\vec{p}_2\, \frac{\partial U(|\vec{q}_1 - \vec{q}_2|)}{\partial \vec{q}_1} \cdot \frac{\partial f^{(2)}}{\partial \vec{p}_1}}$$

Note that $\mathcal{H}^{(1)}$ is the 1-particle Hamiltonian which is simply

$$\mathcal{H}^{(1)} = \mathcal{H}^{(1)}(\vec{q}, \vec{p}) = \frac{p^2}{2m} + V(\vec{q})$$

where we emphasize once more that $V(\vec{x})$ is the external potential. The first term in the evolution equation for the 1-particle DF (the Poisson brackets) is called the **streaming term**; it describes how particles move in the absence of collisions. The second term is called the **collision integral**, and describes how the distribution of particles in phase-space is impacted by two-body collisions. Note that it depends on the **2-particle DF** $f^{(2)}(\vec{q}_1, \vec{q}_2, \vec{p}_1, \vec{p}_2)$, which shouldn't come as a surprise given that accounting for two-body collisions requires knowledge of the phase-space coordinates of the two particles in question.

Thus, we started with the **Liouville equation**, governing a complicated function of N variable, and it looks like all we have achieved is to replace it with a set of N coupled equations. However, the **BBKGY hierarchy** is useful since it allows us to make some simplifying assumptions (which will be sufficiently accurate under certain conditions), that truncates the series. Also, it is important to point out that the BBGKY hierarchy is completely general; *the only real assumption we have made thus far is that the system obeys Hamiltonian dynamics!*

From here on out, though, we can target specific fluids (i.e., collisionless fluids, neutral fluids, plasmas) by specifying details about the two-body interaction potential $U(|\vec{q}_i - \vec{q}_j|)$ and/or the external potential $V(\vec{q})$. Let us start by considering the easiest example, namely the **collisionless fluid**. Here we have two methods of proceeding. First of all, we can simply set $U(|\vec{q}_i - \vec{q}_j|) = 0$ (i.e., ignore two-body interactions) and realize that we can compute $V(\vec{q})$ from the density distribution

$$\rho(\vec{q},t) = m \int \mathrm{d}^3\vec{p}\, f^{(1)}(\vec{q},\vec{p},t)$$

where m is the particle mass, using the Poisson equation

$$\nabla^2\Phi = 4\pi G\rho$$

where we have used $\Phi(\vec{q},t) = V(\vec{q},t)/m$ to coincide with the standard notation for the gravitational potential used throughout these lecture notes. This implies that the collision integral vanishes, and we are left with a **closed equation** for the 1-particle DF, given by

$$\boxed{\frac{\mathrm{d}f^{(1)}}{\mathrm{d}t} = \frac{\partial f^{(1)}}{\partial t} + \{f^{(1)}, \mathcal{H}^{(1)}\} = \frac{\partial f^{(1)}}{\partial t} + \vec{v}\cdot\frac{\partial f^{(1)}}{\partial \vec{x}} - \nabla\Phi\cdot\frac{\partial f^{(1)}}{\partial \vec{v}} = 0}$$

Here we have used the more common $(\vec{x},\vec{v})$ coordinates in place of the canonical $(\vec{q},\vec{p})$, and the fact that $\dot{\vec{p}} = -m\nabla\Phi$. This equation is the **Collisionless Boltzmann Equation** (CBE) which is the fundamental equation describing a collisionless system (i.e., a galaxy or dark matter halo). It expresses that *the flow of particles in μ-space is incompressible, and that the local phase-space density around any particle is fixed.* The evolution of a collisionless system of particles under this CBE is depicted in the left-hand panel of Fig. 6. Although the CBE is a simple looking equation, recall that $f^{(1)}$ is still a 6D function. Solving the CBE is tedious and not something that is typically done. As we will see in the next Chapter, instead what we do is to (try to) solve **moment equations** of the CBE.

For completeness, let us now derive the CBE using a somewhat different approach. This time we treat the gravity among the individual particles, and we do NOT assume upfront that we can account for gravity in terms of a 'smooth' potential $V(\vec{q})$. Hence, we set $V = 0$, and

$$U(|\vec{q}_1 - \vec{q}_2|) = -\frac{Gm^2}{|\vec{q}_1 - \vec{q}_2|}$$

is now the gravitational potential energy due to particles 1 and 2. Starting from our BBGKY expression for the 1-particle DF, we need to come up with a description for the 2-particle DF. In general, we can always write

$$\boxed{f^{(2)}(\vec{q}_1,\vec{q}_2,\vec{p}_1,\vec{p}_2) = f^{(1)}(\vec{q}_1,\vec{p}_1)\, f^{(1)}(\vec{q}_2,\vec{p}_2) + g(\vec{q}_1,\vec{q}_2,\vec{p}_1,\vec{p}_2)}$$

This the first step in what is called the **Mayer cluster expansion**. We can write this in (a self-explanatory) shorthand notation as

$$f^{(2)}(1,2) = f^{(1)}(1)\, f^{(1)}(2) + g(1,2)$$

The next step in the expansion involves the 3-particle DF:

$$f^{(3)}(1,2,3) = f(1)\, f(2)\, f(3) + f(1)\, g(2,3) + f(2)\, g(1,3) + f(3)\, g(1,2) + h(1,2,3)$$

and so onward for k-particle DFs with $k > 3$. The function $g(1,2)$ is called the **two-point correlation function**. It describes how the phase-space coordinates of two particles are correlated. Note that if they are NOT correlated then $g(1,2) = 0$. This is reminiscent of probability statistics: if x and y are two independent random variables then $P(x,y) = P(x)\, P(y)$. Similarly, $h(1,2,3)$ describes the **three-point correlation function**; the correlation among particles 1, 2 and 3 that is not already captured by their mutual two-point correlations described by $g(1,2)$, $g(2,3)$ and $g(1,3)$.

Now, let's assume that the phase-space coordinates of two particles are uncorrelated; i.e., we set $g(1,2) = 0$. This implies that the 2-particles DF is simply the products of two 1-particle DFs, and thus that the evolution equation for $f^{(1)}$ is closed! In fact, using that $\partial\mathcal{H}^{(1)}/\partial\vec{q} = 0$ and $\partial\mathcal{H}^{(1)}/\partial\vec{p} = \vec{p}/m = \vec{v}$ we obtain that

$$\frac{\partial f^{(1)}}{\partial t} + \vec{v}\cdot\frac{\partial f^{(1)}}{\partial \vec{x}} = \int \mathrm{d}^3\vec{q}_2\, \mathrm{d}^3\vec{p}_2\, f^{(1)}(\vec{q}_2,\vec{p}_2)\, \frac{\partial U(|\vec{q}_1 - \vec{q}_2|)}{\partial \vec{q}_1}\cdot\frac{\partial f^{(1)}}{\partial \vec{p}_1}$$

Taking the operator outside of the **collision integral** (note that $f^{(1)}$ in the operator has $\vec{q}_1$ and $\vec{p}_1$ as arguments), and performing the integral over $\vec{p}_2$ yields

$$\frac{\partial f^{(1)}}{\partial t} + \vec{v}\cdot\frac{\partial f^{(1)}}{\partial \vec{x}} - \frac{\partial f^{(1)}}{\partial \vec{p}_1}\cdot\frac{\partial}{\partial \vec{q}_1}\left[\frac{1}{m}\int \mathrm{d}^3\vec{q}_2\, \rho(\vec{q}_2)\, U(|\vec{q}_1 - \vec{q}_2|)\right] = 0$$

Using that

$$\Phi(\vec{x}) = -G\int \mathrm{d}^3\vec{x}'\, \frac{\rho(\vec{x}')}{|\vec{x} - \vec{x}'|}$$

this finally can be written as

$$\frac{\partial f^{(1)}}{\partial t} + \vec{v}\cdot\frac{\partial f^{(1)}}{\partial \vec{x}} - \nabla\Phi\cdot\frac{\partial f^{(1)}}{\partial \vec{v}} = 0$$

which we recognize as the CBE! Thus, *we infer that in a collisionless system the phase-space coordinates of the particles are uncorrelated!* This is an important point. It implies that collisions among particles introduce correlations, and thus that the collisional aspect of a fluid is ultimately captured by the correlation functions $g(1,2)$, $h(1,2,3)$, etc.

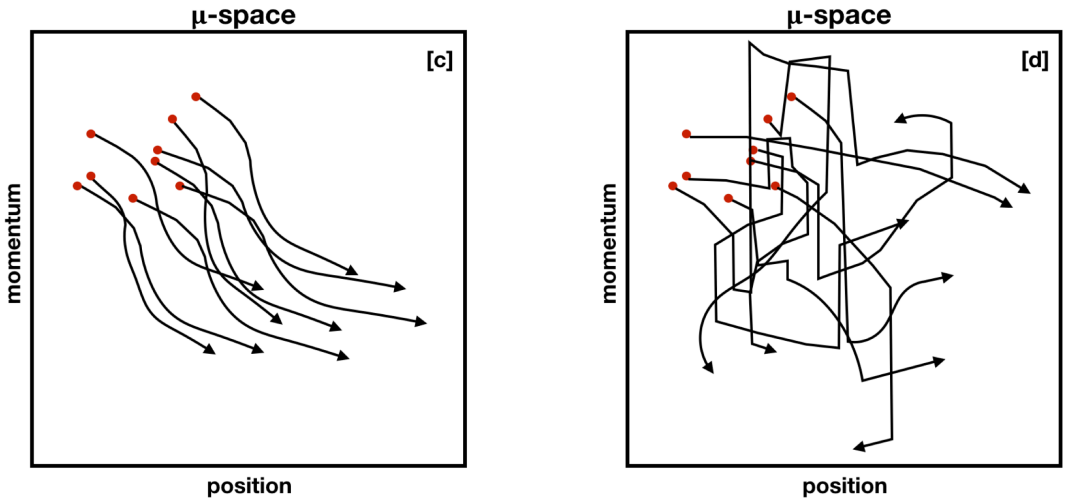


Figure 6: *Illustration of evolution in μ-space. The x- and y-axes represent the* 3*-dimensional position-vector and momentum-vector, respectively. Panel (a) shows the evolution of a collection of particles (indicated by the red dots) in a collisionless system governed by the CBE. As time goes on, the potitions and momentum of all the particles change (according to the Hamiltonian equations of motion), and the particles move around in μ-space smoothly (no abrubt changes). Note that, unlike in Γ-space, trajectories of individual particles are allowed to cross each other. Panel (b) shows the evolution of particles in a collisional system. The collisions are highly localized and cause abrupt changes in momentum. The dynamics of this system is described by the Boltzmann equation.*

If we want to describe say a **collisional, neutral fluid**, we need to decide on how to treat these correlation functions. If we do this for a neutral fluid, which means a fluid in which the interaction potentials are only effective over very small distances (i.e., $U(r) = 0$ for r larger than some small, characteristic collision scale, $r_{\rm coll}$), then one derives what is called the **Boltzmann equation**, which is given by

$$\boxed{\frac{\partial f^{(1)}}{\partial t} + \vec{v} \cdot \frac{\partial f^{(1)}}{\partial \vec{x}} - \nabla\Phi \cdot \frac{\partial f^{(1)}}{\partial \vec{v}} = I[f^{(1)}]}$$

Here $I[f^{(1)}]$ is the collision integral which now is only a function of the 1-particle DF, making the Boltzmann equation a closed equation. It describes how, due to collisions, particles are 'kicked' in and out of certain parts of phase-space. The right-hand panel of Fig. **??** shows an illustration of evolution under the Boltzmann equation.
Rigorously deriving an expression for $I[f^{(1)}]$ from the BBGKY hierarchy is fiddly and outside the scope of this course (see for example the textbook *"Statistical Mechanics"* by Kerson Huang). Instead, in the next Chapter we will use a more heuristic approach, which relies on making the assumptions

- **dilute gas**; density is sufficiently low so that only binary collisions need to be considered
- collisions can be thought of as **instantaneous** and **perfectly localized**.
- **Molecular chaos**: velocities of colliding particles are uncorrelated

The first two assumptions are reasonable, but the **molecular chaos** assumption (introduced by Boltzmann, who referred to it as the **Stosszahlansatz**, which translates to 'assumption regarding number of collisions') has a long and interesting history. Mathematically, the assumption implies that

$$f^{(2)}(\vec{q},\vec{q},\vec{p}_1,\vec{p}_2) = f^{(1)}(\vec{q},\vec{p}_1)\, f^{(1)}(\vec{q},\vec{p}_2)$$

which thus assumes that $g(\vec{q},\vec{q},\vec{p}_1,\vec{p}_2) = 0$. Note that this is different from the assumption we made above when describing collisionless fluids, as here it is only assumed that *at a given location* the *momenta* are uncorrelated. This is a weaker assumption than setting $g(\vec{q}_1,\vec{q}_2,\vec{p}_1,\vec{p}_2) = 0$. At first sight this seems a reasonable assumption; after all, in a dilute gas particles move (relatively) long distances between collisions (i.e., $\lambda_{\rm mfp} \gg r_{\rm coll}$). Although collisions introduce correlations among the particles, each particle is expected to have many collisions with other particles before colliding with a particular particle again. It seems reasonable to postulate that these intermittent collisions erase correlations again. However, this apparently unremarkable assumption effectively introduces an **arrow of time**, as briefly discussed in the colored text-box at the end of this chapter.

Finally, we point out that if treating a **collisional plasma** in which the interactions are due to long-range Coulomb forces, then the standard approach is to assume that $h(1,2,3) = 0$ (i.e., assume that the three-body correlation function is zero). Making several other assumptions (i.e., plasma is spatially homogeneous and the 2-particle

correlation function $g(1,2)$ relaxes much faster than the 1-particle DF $f^{(1)}$) this allows one to derive an expression for $g(1,2)$ from the evolution equation of the 2-particle DF, which can then be substituted in the evolution equation for the 1-particles DF. The result is called the **Lenard-Balescu** equation. It is an example of a **Fokker-Planck** equation, which is a generic equation used to describe the time evolution of the probability density function of the velocity of a particle under the influence of stochastic forces (here the Coulomb collisions) that mainly cause small deflections. The Fokker-Planck equation is also used to describe gravitational N-body systems in which the impact of collisions is not negligble (i.e., describing two-body relaxation in a globular cluster), and will be discusssed in detail in Chapter 9.

As a final remark for this Chapter, we have thus far only considered the case of a single species of mono-atoms. If we consider different types of particles, then we have to introduce a separate distribution function for each type. If the different types of particles can interact with each other, this then has to be accounted for in the collision terms.

Molecular Chaos and the Arrow of Time

The assumption of "molecular chaos" (also known as "Stosszahlansatz") which allows one to write down a closed equation for the time-evolution of the 1-point DF, was used by L. Boltzmann to proof his famous **H-Theorem**, which basically states that entropy should always increase (i.e., it is supposed to be a proof of the **second law of thermodynamics**). This in turn implies **time-asymmetry**, giving rise to the thermodynamic **arrow of time**. However, as first brought to bear by J. Loschmidt, it should not be possible to deduce an irreversible process from time-symmetric dynamics and a time-symmetric formalism (i.e., the dynamics that result from the Liouville equation, which has no underlying assumptions, is perfectly time-reversible!). The origin of this **"Loschmidt paradox"**, as it is known, is the questionable assumption of "molecular chaos". After all, once the particles are allowed to collide, their velocity directions and positions in fact do become correlated. Molecular chaos basically assumes that the subsequent collisions with all the other particles somehow erases this correlation again. To what extent this is true, and whether the H-theorem really proofs the second law of thermodynamics is a topic of ongoing debate, especially among philosophers of science.

The material in this text-box is not part of the curriculum for this course

CHAPTER 7

From Boltzmann to Navier-Stokes

In the previous chapter we derived the **BBGKY hierarchy** of equations:

$$\boxed{\begin{aligned}
&\frac{\partial f^{(1)}}{\partial t} = \{\mathcal{H}^{(1)}, f^{(1)}\} + \int \mathrm{d}^3\vec{q}_2\,\mathrm{d}^3\vec{p}_2\,\frac{\partial U(|\vec{q}_1-\vec{q}_2|)}{\partial \vec{q}_1}\cdot\frac{\partial f^{(2)}}{\partial \vec{p}_1}\\
&\quad\cdot\\
&\quad\cdot\\
&\quad\cdot\\
&\frac{\partial f^{(k)}}{\partial t} = \{\mathcal{H}^{(k)}, f^{(k)}\} + \sum_{i=1}^{k}\int \mathrm{d}^3\vec{q}_{k+1}\,\mathrm{d}^3\vec{p}_{k+1}\,\frac{\partial U(|\vec{q}_i-\vec{q}_{k+1}|)}{\partial \vec{q}_i}\cdot\frac{\partial f^{(k+1)}}{\partial \vec{p}_i}
\end{aligned}}$$

Here $k = 1, 2, ..., N$, $f^{(k)}$ is the k-particle DF, which relates to the N-particle DF ($N > k$) according to

$$f^{(k)}(\vec{w}_1, \vec{w}_2, ..., \vec{w}_k, t) \equiv \frac{N!}{(N-k)!}\int \prod_{i=k+1}^{N} \mathrm{d}^6\vec{w}_i\, f^{(N)}(\vec{w}_1, \vec{w}_2, ..., \vec{w}_N, t)\,,$$

and $\mathcal{H}^{(k)}$ is the k-particle Hamiltonian given by

$$\mathcal{H}^{(k)} = \sum_{i=1}^{k}\frac{\vec{p}_i^{\,2}}{2m} + \sum_{i=1}^{k} V(\vec{q}_i) + \frac{1}{2}\sum_{i=1}^{k}\sum_{\substack{j=1\\ j\neq i}}^{k} U(|\vec{q}_i - \vec{q}_j|)$$

with $V(\vec{q})$ the potential associated with an external force, and $U(r)$ the two-body interaction potential between two (assumed equal) particles separated by a distance $r = |\vec{q}_i - \vec{q}_j|$.

In order to close this set of N equations, one needs to make certain assumptions that truncate the series. One such assumption is that all particles are uncorrrelated (both spatially and in terms of their momenta), such that

$$f^{(2)}(\vec{q}_1, \vec{q}_2, \vec{p}_1, \vec{p}_2) = f^{(1)}(\vec{q}_1, \vec{p}_1)\, f^{(1)}(\vec{q}_2, \vec{p}_2)$$

which is equivalent to setting the **correlation function** $g(1,2) = 0$. As we have shown in the previous Chapter, the first equation in the BBGKY hierarchy is now closed, and yields the **Collisionless Boltzmann Equation** (CBE), which can be written as

$$\boxed{\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \dot{\vec{x}} \cdot \frac{\partial f}{\partial \vec{x}} + \dot{\vec{v}} \cdot \frac{\partial f}{\partial \vec{v}} = 0}$$

which is the fundamental evolution equation for collisionless systems. If the forces between particles are gravitational in nature, then $\dot{\vec{v}} = \nabla\Phi$, with $\Phi(\vec{x})$ the **gravitational potential** which related to the density distribution via the **Poisson equation**. Note that in this case the gravitational potential experienced by each particle is treated as due to the external potential $V(\vec{q})$, and not as a two-body interaction $U(r)$. As we have seen in the previous chapter, one can also obtain the CBE by setting $V(\vec{q}) = 0$, and incorporating gravity as a two-body interaction potential, as long as one then assumes that the interacting particles are uncorrelated.

NOTE: we have used the shorthand notation f for the 1-particle DF $f^{(1)}$. In what follows *we will adopt that notation throughout, and only use the superscript-notation whenever confusion might arise.*

If, on the other hand, we want to describe a dilute, neutral fluid in which the particles only have short-range interactions (such that $U(r) \simeq 0$ outside of some small distance $r_{\rm coll}$), then we can make the assumption of **molecular chaos** which also allows us to close the BBGKY hierarchy, yielding the **Boltzmann Equation**:

$$\boxed{\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \dot{\vec{x}} \cdot \frac{\partial f}{\partial \vec{x}} + \dot{\vec{v}} \cdot \frac{\partial f}{\partial \vec{v}} = I[f]}$$

where $I[f]$ is the **collision integral**, which describes how the phase-space density around a particle (or fluid element) changes with time due to collisions. Note that the third term, including the particle acceleration $\dot{\vec{v}}$, describes the accelerations due to an external potential (e.g., a gravitational field in which the fluid is located). It does NOT represent the acceleration due to the two-body interactions, which is instead captured by the collision integral.

Let us now take a closer look at this collision integral $I[f]$. It basically expresses the Eulerian time-derivative of the DF due to collisions, i.e.,$I[f] = (\partial f/\partial t)_{\rm coll}$. Recall

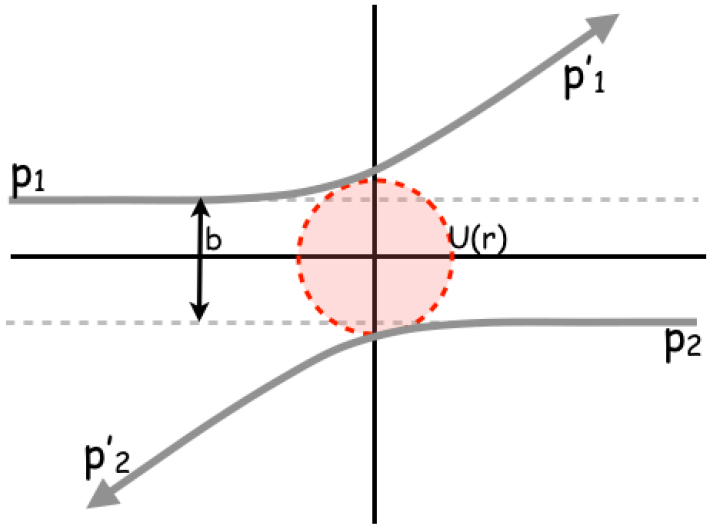


Figure 7: *Illustration of 'collision' between two particles with momenta p_1 and p_2 due to interaction potential $U(r)$. The impact parameter of the collision is b.*

that we have made the assumption of a dilute gas, so that we only need to consider two-body interactions. In what follows, we make the additional assumption that all collisions are **elastic** [actually, this is sort of implied by the fact that we assume that the dynamics are Hamiltonian]. An example is shown in Fig. 7, where $\vec{p}_1 + \vec{p}_2 \to \vec{p}_1\,' + \vec{p}_2\,'$. Since we assumed a short-range, instantaneous and localized interaction, so that the external potential doesn't significantly vary over the interaction volume (the dashed circle in Fig. 7), we have

$$\begin{aligned} \textbf{momentum conservation:} \quad & \vec{p}_1 + \vec{p}_2 = \vec{p}_1\,' + \vec{p}_2\,' \\ \textbf{energy conservation:} \quad & |\vec{p}_1|^2 + |\vec{p}_2|^2 = |\vec{p}_1\,'|^2 + |\vec{p}_2\,'|^2 \end{aligned}$$

where as throughout we have assumed equal mass particles.

We can write the rate at which particles of momentum $\vec{p}_1$ at location $\vec{x}$ experience collisions $\vec{p}_1 + \vec{p}_2 \to \vec{p}_1\,' + \vec{p}_2\,'$ as

$$\mathcal{R} = \omega(\vec{p}_1, \vec{p}_2 | \vec{p}_1\,', \vec{p}_2\,')\, f^{(2)}(\vec{x}, \vec{x}, \vec{p}_1, \vec{p}_2)\, \mathrm{d}^3\vec{p}_2\, \mathrm{d}^3\vec{p}_1\,'\, \mathrm{d}^3\vec{p}_2\,'$$

Here $f^{(2)}(\vec{x}, \vec{x}, \vec{p}_1, \vec{p}_2)$ is the 2-particle DF, expressing the probability that at location $\vec{x}$, you encounter two particles with momenta $\vec{p}_1$ and $\vec{p}_2$, respectively. The function $\omega(\vec{p}_1, \vec{p}_2 | \vec{p}_1\,', \vec{p}_2\,')$ depends on the interaction potential $U(\vec{r})$ and can be calculated

(using kinetic theory) via differential cross sections. Note that momentum and energy conservation is encoded in the fact that $\omega(\vec{p}_1, \vec{p}_2|\vec{p}_1\,', \vec{p}_2\,') \propto \delta^3(\vec{P} - \vec{P}')\,\delta(E - E')$ with $\delta(x)$ the Dirac delta function, $\vec{P} = \vec{p}_1 + \vec{p}_2$ and $\vec{P}' = \vec{p}_1\,' + \vec{p}_2\,'$.

In addition, we have **time-reversibility**, so that it is equally likely that the inverse process $(-\vec{p}_1\,' + -\vec{p}_2\,' \to -\vec{p}_1 + -\vec{p}_2)$ happens. This implies that

$$\omega(\vec{p}_1, \vec{p}_2|\vec{p}_1\,', \vec{p}_2\,') = \omega(\vec{p}_1\,'\,\vec{p}_2\,'|\vec{p}_1, \vec{p}_2)$$

Using our assumption of **molecular chaos**, which states that the momenta of the interacting particles are independent, we have that

$$f^{(2)}(\vec{x}, \vec{x}, \vec{p}_1, \vec{p}_2) = f^{(1)}(\vec{x}, \vec{p}_1)\, f^{(1)}(\vec{x}, \vec{p}_2)$$

so that the **collision integral** can be written as

$$\boxed{I[f] = \int \mathrm{d}^3\vec{p}_2\, \mathrm{d}^3\vec{p}_1\,'\, \mathrm{d}^3\vec{p}_2\,'\, \omega(\vec{p}_1\,', \vec{p}_2\,'|\vec{p}_1, \vec{p}_2)\, [f(\vec{p}_1\,')\, f(\vec{p}_2\,') - f(\vec{p}_1)\, f(\vec{p}_2)]}$$

where we have suppressed the x arguments of f in order to avoid unnecessary clutter. The first term within the square brackets describes the **replenishing collisions**, in which particles at $(\vec{x}, \vec{p}_1\,')$ are scattered into $(\vec{x}, \vec{p}_1)$. The second term with the square brackets describes the **depleting collisions**, in which particles at $(\vec{x}, \vec{p}_1)$ are kicked out of their phase-space volume into $(\vec{x}, \vec{p}_1\,')$.

We can use the above expression to derive that the equilibrium solution for the velocity distribution in a homogeneous fluid is given by the Maxwell-Boltzmann distribution. The expression for an equilibrium distribution function, f_{eq} is that $\partial f_{\mathrm{eq}}/\partial t = 0$ (i.e., the DF at any given location doesn't evolve with time). If we ignore a potential external potential (i.e., $V = 0$), and we take into consideration that an equilibrium solution must indeed be spatially homogeneous, such that $\partial f_{\mathrm{eq}}/\partial \vec{q} = 0$, then we have that the streaming term $\{\mathcal{H}, f_{\mathrm{eq}}\} = 0$. Hence, having an equilibrium requires that the collision integral vanishes as well. As is apparent from the above expression, this will be the case if

$$f(\vec{x}, \vec{p}_1\,')\, f(\vec{x}, \vec{p}_2\,') - f(\vec{x}, \vec{p}_1)\, f(\vec{x}, \vec{p}_2) = 0$$

This condition is known as **detailed balance**, and can be written as

$$\log[f(\vec{p}_1)] + \log[f(\vec{p}_2)] = \log[f(\vec{p}_1\,')] + \log[f(\vec{p}_2\,')]$$

This has the form of a conservation law, and implies that $\log[f_{\rm eq}]$ must be equal to a sum of conserved quantities, $A(\vec{p})$, that obey

$$A(\vec{p}_1) + A(\vec{p}_2) = A(\vec{p}_1\,') + A(\vec{p}_2\,')$$

Quantities $A(\vec{p})$ for which this is the case are called **collisional invariants**. There are three such quantities of interest to us

$A = 1$	**particle number conservations**
$A = \vec{p}$	**momentum conservation**
$A = \vec{p}^{\,2}/(2m)$	**energy conservation**

and we thus expect that

$$\log[f_{\rm eq}(\vec{p})] \propto a_1 \, + a_2 \, \vec{p} + a_3 \, |\vec{p}|^2$$

with a_1, a_2 and a_3 some constants. This notion can be used to demonstrate that the equilibrium distribution must be of the form of a **Maxwell-Boltzmann distribution**

$$f_{\rm eq}(p) = \frac{n}{(2\pi m k_{\rm B} T)^{3/2}} \, \exp\left[-\frac{p^2}{2mk_{\rm B}T}\right]$$

(see "*Statistical Mechanics*" by Kerson Huang for a detailed derivation), and also Chapter 9.

As a small aside, we mention that for a system in thermal equilibrium we have that $I[f_{\rm eq}] = 0$ (i.e., the collisions no longer cause a net change of the distribution function. Hence, for a system that is not too far from equilibrium, a reasonable approximation of the collision integral is given by

$$I[f] \simeq \frac{-(f - f_{\rm eq})}{\tau_{\rm coll}}$$

where $\tau_{\rm coll} = \lambda_{\rm mfp}/\langle v \rangle$ is the collision time (see Chapter 2).

We have seen that if the logarithm of the DF is a sum of collisional invariants (which it is if the system is in equilibrium), then the collision integral vanishes. In addition, as we will now demonstrate, for a collisional invariant $A(\vec{p})$ we also have that

$$\boxed{\int \mathrm{d}^3\vec{p}\, A(\vec{p}) \left(\frac{\partial f}{\partial t}\right)_{\mathrm{coll}} = 0}$$

which will be useful for what follows. To see that this equality holds, we first introduce

$$\mathcal{I}_1 = \int \mathrm{d}^3\vec{p}_1\, \mathrm{d}^3\vec{p}_2\, \mathrm{d}^3\vec{p}_1\,'\, \mathrm{d}^3\vec{p}_2\,'\, \omega(\vec{p}_1\,', \vec{p}_2\,'|\vec{p}_1, \vec{p}_2)\, A(\vec{p}_1)\, [f(\vec{p}_1\,')\, f(\vec{p}_2\,') - f(\vec{p}_1)\, f(\vec{p}_2)]$$

which is the collision integral multiplied by $A(\vec{p}_1)$ and integrated over $\vec{p}_1$. Note that now *all* momenta are integrated over, such that they are basically nothing but dummy variables. Re-labelling $1 \leftrightarrow 2$, and reordering yields

$$\mathcal{I}_2 = \int \mathrm{d}^3\vec{p}_1\, \mathrm{d}^3\vec{p}_2\, \mathrm{d}^3\vec{p}_1\,'\, \mathrm{d}^3\vec{p}_2\,'\, \omega(\vec{p}_1\,', \vec{p}_2\,'|\vec{p}_1, \vec{p}_2)\, A(\vec{p}_2)\, [f(\vec{p}_1\,')\, f(\vec{p}_2\,') - f(\vec{p}_1)\, f(\vec{p}_2)]$$

i.e., everything is unchanged except for the argument of our collisional invariant. And since the momenta are dummy variables, we have that $\mathcal{I}_2 = \mathcal{I}_1$. Rather than swapping indices 1 and 2, we can also swap $\vec{p} \leftrightarrow \vec{p}\,'$. This gives us two additional integrals:

$$\mathcal{I}_3 = -\int \mathrm{d}^3\vec{p}_1\, \mathrm{d}^3\vec{p}_2\, \mathrm{d}^3\vec{p}_1\,'\, \mathrm{d}^3\vec{p}_2\,'\, \omega(\vec{p}_1, \vec{p}_2|\vec{p}_1\,', \vec{p}_2\,')\, A(\vec{p}_1\,')\, [f(\vec{p}_1\,')\, f(\vec{p}_2\,') - f(\vec{p}_1)\, f(\vec{p}_2)]$$

and

$$\mathcal{I}_4 = -\int \mathrm{d}^3\vec{p}_1\, \mathrm{d}^3\vec{p}_2\, \mathrm{d}^3\vec{p}_1\,'\, \mathrm{d}^3\vec{p}_2\,'\, \omega(\vec{p}_1, \vec{p}_2|\vec{p}_1\,', \vec{p}_2\,')\, A(\vec{p}_2\,')\, [f(\vec{p}_1\,')\, f(\vec{p}_2\,') - f(\vec{p}_1)\, f(\vec{p}_2)]$$

where the minus sign comes from the fact that we have reversed $f(\vec{p}_1)\, f(\vec{p}_2) - f(\vec{p}_1\,')\, f(\vec{p}_2\,')$. Because of time-reversibility $\omega(\vec{p}_1\,', \vec{p}_2\,'|\vec{p}_1, \vec{p}_2) = \omega(\vec{p}_1, \vec{p}_2|\vec{p}_1\,', \vec{p}_2\,')$, and we thus have that $\mathcal{I}_4 = \mathcal{I}_3 = \mathcal{I}_2 = \mathcal{I}_1$. Hence $\mathcal{I}_1 = [\mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4]/4$, which can be written as

$$\mathcal{I}_1 = \frac{1}{4}\int \mathrm{d}^3\vec{p}_1\, \mathrm{d}^3\vec{p}_2\, \mathrm{d}^3\vec{p}_1\,'\, \mathrm{d}^3\vec{p}_2\,'\, \omega(\vec{p}_1\,', \vec{p}_2\,'|\vec{p}_1, \vec{p}_2) \times \\ \{A(\vec{p}_1) + A(\vec{p}_2) - A(\vec{p}_1\,') - A(\vec{p}_2\,')\}\, [f(\vec{p}_1\,')\, f(\vec{p}_2\,') - f(\vec{p}_1)\, f(\vec{p}_2)]$$

Since $A(\vec{p})$ is a collisional invariant, the factor in curly brackets vanishes, which in turn assures that $\mathcal{I}_1 = 0$, which completes our proof.

Thus far, we have derived the Boltzmann equation, and we have been able to write down an expression for the collision integral under the assumptions of (i) short-range, elastic collisions and (ii) molecular chaos. How do we proceed from here? The Boltzmann equation with the above expression for the collision integral is a non-linear integro-differential equation, and solving such an equation is extremely difficult. Fortunately, in the fluid limit we don't really need to. Rather, we are interested what happens to our macroscopic quantities that describe the fluid (ρ, $\vec{u}$, P, ε, etc). We can use the Boltzmann equation to describe the time-evolution of these macroscopic quantities by considering **moment equations** of the Boltzmann equation.

In mathematics, the n^{th}-moment of a real-valued, continuous function $f(x)$ is

$$\mu_n = \int x^n f(x)\,\mathrm{d}x$$

If $f(x)$ is normalized, so that it can be interpreted as a probability function, then $\mu_n = \langle x^n \rangle$.

In our case, consider the scalar function $Q(\vec{v})$. The expectation value for Q at location $\vec{x}$ at time t is given by

$$\langle Q \rangle = \langle Q \rangle(\vec{x}, t) = \frac{\int Q(\vec{v})\, f(\vec{x}, \vec{v}, t)\,\mathrm{d}^3\vec{v}}{\int f(\vec{x}, \vec{v}, t)\,\mathrm{d}^3\vec{v}}$$

Using that

$$n = n(\vec{x}, t) = \int f(\vec{x}, \vec{v}, t)\,\mathrm{d}^3\vec{v}$$

we thus have that

$$\boxed{\int Q(\vec{v})\, f(\vec{x}, \vec{v}, t)\,\mathrm{d}^3\vec{v} = n\,\langle Q \rangle}$$

We will use this abundantly in what follows. In particular, define

$$g(\vec{x},t) = \int Q(\vec{v})\, f(\vec{x},\vec{v},t)\, \mathrm{d}^3\vec{v}$$

Then, in relation to fluid dynamics, there are a few functions $Q(\vec{v})$ that are of particular interest:

$Q(\vec{v}) = 1$	$\Rightarrow$	$g(\vec{x},t) = n(\vec{x},t)$	number density
$Q(\vec{v}) = m$	$\Rightarrow$	$g(\vec{x},t) = \rho(\vec{x},t)$	mass density
$Q(\vec{v}) = m\,\vec{v}$	$\Rightarrow$	$g(\vec{x},t) = \rho(\vec{x},t)\,\vec{u}(\vec{x},t)$	momentum flux density
$Q(\vec{v}) = \frac{1}{2}m(\vec{v}-\vec{u})^2$	$\Rightarrow$	$g(\vec{x},t) = \rho(\vec{x},t)\,\varepsilon(\vec{x},t)$	specific energy density

where we have defined the **macroscopic** velocity (also known as the bulk velocity of streaming velocity) $\vec{u}(\vec{x}) \equiv \langle\vec{v}\rangle$, which is the mean velocity of all particles at the volume element centered on $\vec{x}$, and the **specific internal energy** $\varepsilon \equiv \langle(\vec{v}-\vec{u})^2/2\rangle$ (here 'specific' means 'per unit mass').

This indicates that we can obtain dynamical equations for the macroscopic fluid quantities by multiplying the Boltzmann equation with appropriate functions, $Q(\vec{v})$, and integrating over all of velocity space.

Hence, we seek to solve equations of the form

$$\int Q(\vec{v}) \left[\frac{\partial f}{\partial t} + \vec{v}\cdot\nabla f - \nabla\Phi\cdot\frac{\partial f}{\partial \vec{v}}\right] \mathrm{d}^3\vec{v} = \int Q(\vec{v}) \left(\frac{\partial f}{\partial t}\right)_{\mathrm{coll}} \mathrm{d}^3\vec{v}$$

In what follows, we restrict ourselves to $Q(\vec{v})$ that are **collisional invariants** so that the integral on the right-hand side vanishes, and we are left with

$$\int Q(\vec{v})\,\frac{\partial f}{\partial t}\,\mathrm{d}^3\vec{v} + \int Q(\vec{v})\,\vec{v}\cdot\nabla f\,\mathrm{d}^3\vec{v} - \int Q(\vec{v})\,\nabla\Phi\cdot\frac{\partial f}{\partial \vec{v}}\,\mathrm{d}^3\vec{v} = 0$$

Since mass, momentum and energy are all conserved in elastic, short-range collisions we have that the momentum integral over the collision integral will be zero for the zeroth, first and second order moment equations! In other words, *although collisional and collisionless systems solve different Boltzmann equations, their zeroth, first and second moment equations are identical!*

We now split the above equation in three terms:

$$\begin{aligned}
\mathbf{I} \quad & \int Q(\vec{v})\,\frac{\partial f}{\partial t}\,\mathrm{d}^3\vec{v} \\
\mathbf{II} \quad & \int Q(\vec{v})\,v_i\frac{\partial f}{\partial x_i}\,\mathrm{d}^3\vec{v} \\
\mathbf{III} \quad & \int Q(\vec{v})\,\frac{\partial \Phi}{\partial x_i}\,\frac{\partial f}{\partial v_i}\,\mathrm{d}^3\vec{v}
\end{aligned}$$

where we have that $\mathbf{I} + \mathbf{II} - \mathbf{III} = 0$, as long as Q is a collisional invariant.

We now proceed to rewrite each of these three integrals in turn.

Integral I

The first integral can be written as

$$\int Q(\vec{v})\,\frac{\partial f}{\partial t}\,\mathrm{d}^3\vec{v} = \int \frac{\partial Q f}{\partial t}\,\mathrm{d}^3\vec{v} = \frac{\partial}{\partial t}\int Q f\,\mathrm{d}^3\vec{v} = \frac{\partial}{\partial t} n\langle Q\rangle$$

where we have used that both $Q(\vec{v})$ and the integration volume are independent of time.

Integral II

Using similar logic, the second integral can be written as

$$\int Q(\vec{v})\, v_i \frac{\partial f}{\partial x_i}\, \mathrm{d}^3\vec{v} = \int \frac{\partial Q\, v_i\, f}{\partial x_i}\, \mathrm{d}^3\vec{v} = \frac{\partial}{\partial x_i} \int Q\, v_i\, f\, \mathrm{d}^3\vec{v} = \frac{\partial}{\partial x_i}\Big[n\, \langle Q\, v_i \rangle\Big]$$

Here we have used that

$$Q\, v_i\, \frac{\partial f}{\partial x_i} = \frac{\partial (Q\, v_i\, f)}{\partial x_i} - f\, \frac{\partial Q\, v_i}{\partial x_i} = \frac{\partial (Q\, v_i\, f)}{\partial x_i}$$

where the last step follows from the fact that neither v_i nor Q depend on x_i.

Integral III

For the third, and last integral, we are going to define $\vec{F} = \nabla\Phi$ and $\nabla_v \equiv (\partial/\partial v_x, \partial/\partial v_y, \partial/\partial v_z)$, i.e., ∇_v is the equivalent of ∇ but in velocity space. This allows us to write

$$\begin{aligned}
\int Q\, \vec{F} \cdot \nabla_v f\, \mathrm{d}^3\vec{v} &= \int \nabla_v \cdot (Qf\vec{F}) \mathrm{d}^3\vec{v} - \int f\, \nabla_v \cdot (Q\vec{F})\, \mathrm{d}^3\vec{v} \\
&= \int Qf\vec{F} \mathrm{d}^2 S_v - \int f\, \frac{\partial Q F_i}{\partial v_i}\, \mathrm{d}^3\vec{v} \\
&= -\int fQ \frac{\partial F_i}{\partial v_i}\, \mathrm{d}^3\vec{v} - \int f F_i\, \frac{\partial Q}{\partial v_i}\, \mathrm{d}^3\vec{v} \\
&= -\int f \frac{\partial \Phi}{\partial x_i}\, \frac{\partial Q}{\partial v_i}\, \mathrm{d}^3\vec{v} = -\frac{\partial \Phi}{\partial x_i} n \left\langle \frac{\partial Q}{\partial v_i} \right\rangle
\end{aligned}$$

Here we have used Gauss' divergence theorem (see Appendix C), and the fact that the integral of $Qf\vec{F}$ over the surface S_v (which is a sphere with radius $|\vec{v}| = \infty$) is equal to zero. This follows from the 'normalization' requirement that $\int f\, \mathrm{d}^3\vec{v} = n$. We have also used that $F_i = \partial\Phi/\partial x_i$ is independent of v_i.

Combining the above expressions for **I**, **II**, and **III**, we obtain that

$$\boxed{\frac{\partial}{\partial t} n\langle Q \rangle + \frac{\partial}{\partial x_i}\Big[n\langle Q v_i \rangle\Big] + \frac{\partial \Phi}{\partial x_i} n \left\langle \frac{\partial Q}{\partial v_i} \right\rangle = 0}$$

In what follows we refer to this as the **master-moment-equation** (in index-form).

Now let us consider $Q = m$, which is indeed a collisional invariant, as required. Substitution in the master-moment equation, and using that $\langle m \rangle = m$, that $mn = \rho$ and that $\langle m v_i \rangle = m \langle v_i \rangle = m u_i$, we obtain

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_i}{\partial x_i} = 0}$$

which is known as the **continuity equation**. It states that in order for the density at some position $\vec{x}$ to change, there needs to be a divergence of mass flux in or out of the volume element centered on $\vec{x}$.

Next we consider $Q = m v_j$, which is also a collisional invariant. Using that $n \langle m v_j v_i \rangle = \rho \langle v_i v_j \rangle$ and that

$$\frac{\partial \Phi}{\partial x_i} n \left\langle \frac{\partial m v_j}{\partial v_i} \right\rangle = \frac{\partial \Phi}{\partial x_i} \rho \left\langle \frac{\partial v_j}{\partial v_i} \right\rangle = \frac{\partial \Phi}{\partial x_i} \rho \delta_{ij} = \rho \frac{\partial \Phi}{\partial x_j}$$

substitution of $Q = m v_j$ in the master-moment equation yields

$$\frac{\partial \rho u_j}{\partial t} + \frac{\partial \rho \langle v_i v_j \rangle}{\partial x_i} + \rho \frac{\partial \Phi}{\partial x_j} = 0$$

Next we use that

$$\frac{\partial \rho u_j}{\partial t} = \rho \frac{\partial u_j}{\partial t} + u_j \frac{\partial \rho}{\partial t} = \rho \frac{\partial u_j}{\partial t} - u_j \frac{\partial \rho u_k}{\partial x_k}$$

where, in the last step, we have used the continuity equation. Substitution in the above equation, and using that k is a mere dummy variable (which can therefore be replaced by i), we obtain that

$$\begin{aligned}
& \rho \frac{\partial u_j}{\partial t} - u_j \frac{\partial \rho u_i}{\partial x_i} + \frac{\partial \rho \langle v_i v_j \rangle}{\partial x_i} + \rho \frac{\partial \Phi}{\partial x_j} = 0 \\
\Leftrightarrow \quad & \rho \frac{\partial u_j}{\partial t} - \left[\frac{\partial \rho u_i u_j}{\partial x_i} - \rho u_i \frac{\partial u_j}{\partial x_i} \right] + \frac{\partial \rho \langle v_i v_j \rangle}{\partial x_i} + \rho \frac{\partial \Phi}{\partial x_j} = 0 \\
\Leftrightarrow \quad & \rho \frac{\partial u_j}{\partial t} + \rho u_i \frac{\partial u_j}{\partial x_i} + \frac{\partial \left[\rho \langle v_i v_j \rangle - \rho u_i u_j \right]}{\partial x_i} + \rho \frac{\partial \Phi}{\partial x_j} = 0
\end{aligned}$$

In order to make sense of the $\rho \langle v_i v_j \rangle - \rho u_i u_j$ term, it is important to understand the difference between the microscopic velocity v_i, which is the actual velocity of a

particle, and the macroscopic (or 'streaming') velocity u_i, which is the average of v_i, averaged over all particles in some arbitrary small volume. This volume, which is often called a **fluid element** has to be large enough to have many particles to average over, but small enough such that we can ignore any gradients in density or potential across it. In general, we can split the microscopic velocity of a particle, $\vec{v}$, in a streaming velocity, $\vec{u}$ and a 'random' velocity $\vec{w}$:

$$\vec{v} = \vec{u} + \vec{w}$$

where $\langle \vec{v} \rangle = \vec{u}$ and $\langle \vec{w} \rangle = 0$, and $\langle \cdot \rangle$ indicates an average over the fluid element. It is convenient to introduce the **stress tensor**

$$\sigma_{ij} \equiv -\rho \langle w_i w_j \rangle = -\rho \langle v_i v_j \rangle + \rho u_i u_j$$

where the second step follows from the fact that $\vec{w} = \vec{v} - \vec{u}$. Substituting the expression for the stress tensor, we finally obtain the **momentum equations**:

$$\boxed{\frac{\partial u_j}{\partial t} + u_i \frac{\partial u_j}{\partial x_i} = \frac{1}{\rho} \frac{\partial \sigma_{ij}}{\partial x_i} - \frac{\partial \Phi}{\partial x_j}}$$

When applied to a collisionless fluid, these momentum equations are known as the **Jeans equations** (see Part IV for details). For a collisional fluid, we will see that

$$\sigma_{ij} = -P\delta_{ij} + \tau_{ij}$$

with P the hydrodynamic pressure and

$$\tau_{ij} = \mu \left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \delta_{ij} \frac{\partial u_k}{\partial x_k} \right] + \eta \, \delta_{ij} \frac{\partial u_k}{\partial x_k}$$

is called the **deviatoric stress tensor**, with μ and η the coefficients of **shear viscosity** and **bulk viscosity**, respectively (see Part III) for details. If the above expressions for the stress tensor is substituted into the moment equations one obtains the **Navier-Stokes equations**. In the limit of zero viscocity ($\mu = \eta = 0$) these reduce to what are called the **Euler equations** used to describe **inviscid fluids**.

At this point, it is instructive to realize that when we integrated the **Boltzmann equation** over velocity space to obtain the **master moment equation** we 'lost' the **collision integral** (any velocity moment of the collision integral vanishes, as long as it involves a collisional invariant). Hence. *the master moment equation for a collisional fluid is identical to that for a collisionless fluid*! As we will see in Parts

III and IV, *the information regarding the collisionality of the fluid is hidden inside the stress tensor*, σ_{ij}: for a collisionless fluid σ_{ij}, which is manifest symmetric, has 6 unknowns (three diagonal elements and three off-diagonal elements), while for a collisional fluid there are only three unknowns: P, μ and η. Most importantly, these three quantities are actually related to other macroscopic quantities of the fluid, such as density and/or temperature via a number of **constitutive equations**. As we will see, this subtle difference has far-reaching implications!

Finally, it is left as an exercise for the reader (or look at Appendix J) to show that substitution of $Q = mv^2/2$ in the master moment equation yields the **energy equation** (in Lagrangian index form):

$$\boxed{\frac{\partial}{\partial t}\left[\rho\left(\frac{u^2}{2}+\varepsilon\right)\right] = -\frac{\partial}{\partial x_k}\left[\rho\left(\frac{u^2}{2}+\varepsilon\right)\,u_k - \sigma_{jk}u_j + \rho\langle w_k \frac{1}{2}w^2\rangle\right] - \rho u_k \frac{\partial \Phi}{\partial x_k}}$$

where $\varepsilon = \frac{1}{2}\langle w^2\rangle$ is the **specific internal energy**.

This energy equation is rarely ever used for collisionless fluids. For collisional fluids, though, it plays an important role. In particular, for a collisional fluid it can be recast in the more useful form:

$$\rho\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} = -P\frac{\partial u_k}{\partial x_k} + \mathcal{V} - \frac{\partial F_{\mathrm{cond},k}}{\partial x_k}$$

where $\mathcal{V}$ is the **rate of viscous dissipation** and $\vec{F}_{\mathrm{cond}}$ is the **conductive heat flux**. For a more detailed discussion of the energy equation, see Part III.

CHAPTER 8

Stochasticity & the Langevin Equation

We start this chapter on stochastic forces with a few definitions:

Random Variable: a (1-dimensional) random variable is a scalar function $x(t)$, where t is usually time, for which the future evolution is not determined by any set of initial data knowable to us.

Examples of random variables are the sequence of outcomes of a repeatedly flipped coin (here both x and t are discrete), the value of my stock portfolio, the x-coordinate of a particle undergoing Brownian motion, the force experienced by a globular cluster as it orbits a dark matter halo full of substructure, a star moving inside a galaxy or globular cluster. In the latter case, the force is stochastic due to finite-N effects.

Random Process: (aka stochastic process) is an ensemble of realizations of random variables $x(t)$, that all represent the same physical entity. Any particular $x(t)$ is called a realization of the random process.

In general, a random process is completely specified by the *set* of probability distibutions

$$\begin{aligned} P_1 &= P_1(x_1,t_1)\mathrm{d}x_1 \\ P_2 &= P_2(x_2,t_2;x_1,t_1)\mathrm{d}x_2\,\mathrm{d}x_1 \\ &\cdot \\ &\cdot \\ P_n &= P_n(x_n,t_n,....,x_2,t_2,x_1,t_1)\mathrm{d}x_n\,...\,\mathrm{d}x_2\,\mathrm{d}x_1 \end{aligned}$$

Here $t_n > t_{n-1} > ... > t_2 > t_1$ and P_n is the probability that a realization $x(t)$ drawn at random from the random process takes on values between x_1 and $x_1 + \mathrm{d}x_1$ at t_1, between x_2 and $x_2 + \mathrm{d}x_2$ at t_2, etc.

Ensemble Averages: the ensemble average of a function $f(x(t))$ of a random variable is defined as

$$\langle f(t)\rangle = \int f(x)\,P_1(x,t)\,\mathrm{d}x$$

Similarly, we can compute ensemble averages that evolve multiple epochs, such as

$$\langle x(t_1)x(t_2)\rangle = \int x_1\, x_2\, P_2(x_2, t_2; x_1, t_1)\, \mathrm{d}x_2 \mathrm{d}x_1$$

Stationarity: a random process is said to be **stationary** iff its probability distributions P_n (for all n) depend only on time differences and not on absolute time, i.e.,

$$P_n(x_n, t_n + \tau; ...; x_2, t_2 + \tau; x_1, t_1 + \tau) = P_n(x_n, t_n, ..., x_2, t_2, x_1, t_1) \qquad \forall \tau$$

Note: stationarity does <u>not</u> mean that there is no time evolution in probability distributions; for instance, if at time t_1 a particle is placed in a heat bath of temperature T with zero velocity, i.e., $P_1(v_1, t_1) = \delta(v_1)$, then $P_2(v_2, t_2; v_1, t_1)$ *does* depend on time (the particle experiences random collisions with the other particles in the heat bath). The system is stationary, though, as long as $T \neq T(t)$, since in that case $P_2(v_2, t_2; v_1, t_1)$ only depends on $t_2 - t_1$.

Ergodicity: Let $f(x)$ be any function of a random variable $x(t)$. A stationary random process is said to be ergodic iff the time average of a realization,

$$\overline{f} = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} f[x(t')]\, \mathrm{d}t'$$

is equal to the ensemble average $\langle f(x)\rangle$. Note that because of the assumed stationarity $P_1(x, t) = P_1(x)$. For $f(x) = x$ ergodicity means that the time average of a give realization $x(t)$ (when averaged over a sufficiently long time) is equal to an average over many realizations at any given time.

Markov Process: a random process is said to be Markov (or 'Markovian') iff all future probabilities are determined completely by its most recently known values, i.e.,

$$P_n(x_n, t_n | x_{n-1}, t_{n-1}; ...; x_2, t_2; x_1, t_1) = P_2(x_n, t_n | x_{n-1}, t_{n-1})$$

Note that here we have made use of **conditional probabilities** which are related to the unconditional probabilities according to **Bayes theorem**:

$$P_n(x_n, t_n | x_{n-1}, t_{n-1}; ...; x_2, t_2; x_1, t_1) = \frac{P_n(x_n, t_n; x_{n-1}, t_{n-1}; ...; x_2, t_2; x_1, t_1)}{P_{n-1}(x_{n-1}, t_{n-1}; ...; x_2, t_2; x_1, t_1)}$$

Hence, a stationary Markov process is completely specified by

$$P_1(x) \qquad \text{and} \qquad P_2(x_2, t|x_1) = \frac{P_2(x_2, t; x_1, 0)}{P_1(x_1)}$$

Throughout what follows we shall assume our random processes to be stationary, ergodic and Markovian, unless stated otherwise. Note that a true, continuous Markovian process in physics does not exist (physical trajectories $x(t)$ are differentiable, and therefore cannot be perfectly Markovian). However, if $x(t)$ is examined with a sufficiently coarse time-resolution it can become discretely Markovian, which is the situation we are considering here.

Stochasticity in Many-Body Systems: Now that we have these preliminaries out of the way, let's examine the impact of stochasticity in astrophysical many-body systems. Consider an N-body system (i.e., a gas, a galaxy of stars, a plasma of charged particles, etc) and let $(\vec{q}_0, \vec{p}_0)$ be the initial phase-space coordinates of some particle (hereafter the **subject mass**). As the particle evolves, it describes a path $\mu_1(t) = (\vec{q}(t), \vec{p}(t))$ in phase-space. If the system is in (quasi)-equilibrium, and I launch a second particle from $(\vec{q}_0, \vec{p}_0)$ at some later time, then its phase-space trajectory $\mu_2(t) = (\vec{q}(t), \vec{p}(t))$ will be different from $\mu_1(t)$. This is due to the fact that the forces that the subject mass experiences are **stochastic**, i.e., $F(t)$ is a **random variable**, and as a consequence, so is $\mu(t)$. This stochasticity is a manifestation of the collisions between the subject mass and the other $N-1$ particles in the system. To clarify, in principle if I knew the exact phase-space coordinates of all N particles in my system then the $u(t)$ trajectories are entirely deterministic, and there is no stochasticity in the sense of 'unpredictability'. In practice, though, we never know the exact $(\vec{q}, \vec{p})$ for all degrees of freedom. Rather, we have some macroscopic information about the system (i.e., its density distribution, and/or its velocity distribution). Each different microstate corresponding to that macrostate will yield a different $\mu(t)$, and this gives rise to a **random process**. In the case of a 'collisionless' fluid (i.e., a galaxy, or a collisionless plasma), recall that collisionless does <u>not</u> mean 'no collisions'. It merely means that the subject mass undergoes simultaneous interactions (=collisions) with many other particles, which manifests *approximately* as if the system is collisionless. In reality, the system is always somewhat collisional (i.e., the relaxation time is large, but not infinite!)

We now proceed to develop a mathematical treatment of this collisionality-induced stochasticity. Let $\langle \mu(t) \rangle$ be the ensemble averaged trajectory of the random process

$\mu(t)$ described above. Then we write that

$$\vec{F}(\vec{x},t) = \langle\vec{F}(\vec{x})\rangle + \delta\vec{F}(\vec{x},t)$$

where $\langle\vec{F}(\vec{x})\rangle$ is the force field that gives rise to $\langle\mu(t)\rangle$ starting from the initial conditions $(\vec{q}_0,\vec{p}_0)$. The residual $\delta\vec{F}$ is the stochastic force, which has that $\langle\delta\vec{F}\rangle = 0$. We can write $\langle\vec{F}(\vec{x})\rangle$ as a sum of a velocity-independent, conservative force $\vec{F}_{\rm c}(\vec{x}) = -\nabla V(\vec{x})$, and a velocity dependent force $\vec{F}_{\rm nc}(\vec{x},\vec{v})$. The idea that there might be a velocity-dependent force component comes from the notion that the outcome of collisions depends on velocity, simply because a fast moving subject mass sees a different velocity distribution for the particles it interacts with, than a slow moving subject mass. Hence, $\langle\mu(t)\rangle$ is <u>not</u> the same as the trajectory in the absence of a stochastic force. Only in the limit $N\to\infty$, we have that both $\delta\vec{F}\to 0$ and $\vec{F}_{\rm nc}(\vec{x},\vec{v})\to 0$, which implies that the trajectory becomes equal to its average, and identical to that in the absence of any stochastic forces.

Here is another thought-experiment that should convince you of the presence of a drag-force. Consider a box filled with a gas in thermal equilibrium (a *thermal bath*). The equilibrium velocity distribution, $P_{\rm eq}(\vec{v})$, is a **Maxwell-Boltzmann distribution** with a mean velocity $\langle\vec{v}\rangle = 0$ and a velocity dispersion $\sigma^2 = k_{\rm B}T/m$. Consider a random particle at some time t_0 with velocity $\vec{v}_0$. The particle starts out with a velocity PDF $P(\vec{v}) = \delta^{(3)}(\vec{v}-\vec{v}_0)$. If we wait long enough (i.e., in the limit $t\to\infty$), we know that, due to the collisions (stochastic forces), the particle will take on a velocity PDF $P(\vec{v}) = P_{\rm eq}(\vec{v})$. Hence, over time the expectation value for the particle's speed evolves from $v_0 = |\vec{v}_0|$ to 0 (i.e., on average the particle must experience some drag force). At the same time, the PDF broadens from a delta-function into a Maxwell-Boltzmann distribution; in addition to friction, the random collisions also cause a **diffusion** (as we will see in more detail below).

In what follows we simplify our picture by ignoring the spatial dependence; basically what follows is valid for particles ('subjects') moving in an infinite, homogeneous (and isotropic) sea of background particles ('field particles') of density ρ. The hope is that we can use what follows in a realistic, non-homogeneous system with density distribution $\rho(\vec{x})$, by adopting what is known as the **local approximation**; the impact of stochasticity at a position $\vec{x}$ in a non-homogeneous system can be approximated by that of an infinite, homogeneous system of density $\rho = \rho(\vec{x})$. This is a fair assumption as long as the 'collisionality' is dominated by collisions with relatively nearby particles. We will address the validity of this local approximation in the next chapter.

Using Taylor series expansion, and the local approximation, we write

$$F_{\rm nc}(\vec{x},\vec{v}) = F_{\rm nc}(v) = c_0 + c_1\, v + c_2\, v^2 + \mathcal{O}(v^3)$$

where $v = |\vec{v}|$. Note that $c_0 = 0$, as the velocity independent component of the force is already accounted for through $\vec{F}_{\rm c}$. We also assumed here that the velocity dependent force always acts in the direction of the velocity vector of the subject (which is the only logical possibility if the background is homogeneous and isotropic). Let's truncate this series at first-order, such that the equation of motion for our subject mass becomes

$$\boxed{\boxed{m\,\frac{{\rm d}\vec{v}}{{\rm d}t} = \vec{F}(\vec{x},t) = -\nabla V(\vec{x}) - \gamma\vec{v}(t) + \delta\vec{F}(t)}}$$

where we have replaced c_1 by γ. Note that the $\gamma\vec{v}(t)$ term is a **friction** term, with γ the friction coefficient ($[\gamma] = {\rm g}s^{-1}$). The quantity $1/\gamma$ is sometimes referred to as the **mobility**.

The above equation of motion is known as the **Langevin equation**, which is an example of a *stochastic differential equation*. Although it looks like an ordinary differential equation, it is a different beast because $\delta F(t)$ is a random variable. Clearly, there is a different solution for each realization $\delta F(t)$. So how are we supposed to solve this? The goal is to solve for ensemble averages such as $\langle\mu(t)\rangle$ or $\langle v(t_1)v(t_2)\rangle$ using statistical properties of the random process, i.e., using information about the probability distribution of $\delta F(t)$. Note that such a probability distribution has to be a **functional**: you give it a function $\delta F(t)$, and it returns a scalar, which in this case has to be between zero and one.

Note that, because of our assumption of homogeneity of the field particles we have that $\nabla V = 0$, which simplifies our Langevin equation (solving the Langevin equation for an arbitrary potential $V(\vec{x})$, as in the local approximation discussed above, involves solving an unpleasant non-linear stochastic differential equation, which is outside the scope of these lecture notes). If we further simplify matters by considering a 1D system, then the **Langevin equation** reduces to

$$\boxed{\boxed{m\frac{{\rm d}v}{{\rm d}t} = -\gamma v + \delta F(t)}}$$

Multiplying by an integration factor ${\rm e}^{\gamma t/m}$ and rearranging yields

$$\frac{{\rm d}v}{{\rm d}t}{\rm e}^{\gamma t/m} + \frac{\gamma}{m}v{\rm e}^{\gamma t/m} = \frac{\rm d}{{\rm d}t}\left[v{\rm e}^{\gamma t/m}\right] = \frac{1}{m}\delta F(t)\,{\rm e}^{\gamma t/m}$$

which is easily solved for the time dependence of the velocity

$$v(t) = v_0\,\mathrm{e}^{-\gamma t/m} + \frac{1}{m}\int_0^t \delta F(t')\,\mathrm{e}^{-\gamma(t'-t)/m}\,\mathrm{d}t'$$

Taking the ensemble average, using that $\langle \delta F\rangle = 0$, we obtain

$$\langle v(t)\rangle = v_0\,\mathrm{e}^{-\gamma t/m}$$

Thus, the speed of the subject mass will slow down over time on a 'dissipation time' $\tau_\mathrm{d} \sim m/\gamma$. Since slowing down means a decrease in kinetic energy (and since $V = 0$, the potential energy is zero), γ represents **dissipation**. A word of caution is required here. Everything we have discussed so far is valid for a Hamiltonian system, which is non-dissipative. What we mean by 'dissipation' here is that the subject mass experiences, on average, dissipation. The energy lost by the subject is transferred to the field particles, through the collisions that are the source for the stochastic nature of δF. Hence, the system as a whole does not dissipate energy, consistent with the system as a whole being Hamiltonian. Put differently, collisions/interactions cause an internal exchange of energies and momenta, but leave the total energy and momentum conserved.

Let's return to our solution for $v(t)$ and use it to determine the average orbit of the subject:

$$\langle x(t)\rangle = x_0 + \int_0^t \mathrm{d}t'\,\langle v(t')\rangle = x_0 + \frac{m}{\gamma}v_0\,(1-\mathrm{e}^{-\gamma t/m})$$

As expected, this represents a particle that moves in a straight line with a velocity that is decaying with time.

It get's more interesting if we try to compute the expectation values of quadratic quantities. In particular, we find that

$$\langle v(t_1)v(t_2)\rangle = \langle v(t_1)\rangle\,\langle v(t_2)\rangle + \frac{1}{m^2}\int_0^{t_1}\mathrm{d}t_1'\int_0^{t_2}\mathrm{d}t_2'\,\langle \delta F(t_1')\,\delta F(t_2')\rangle\,\mathrm{e}^{\gamma(t_1'+t_2'-t_1-t_2)/m}$$

where we have made use of the fact that $\langle \delta F\rangle = 0$ to drop the two cross terms.

Hence, evaluation of $\langle v(t_1)v(t_2)\rangle$, which describes how the velocity of the subject at t_1 is correlated with that at t_2, requires knowledge of the **correlation function** $\langle \delta F(t_1)\,\delta F(t_2)\rangle$. Typically, if $t_2 - t_1 > \tau_\mathrm{coll}$ (see Chapter 2 for definition of the collision time) the forces will be uncorrelated, i.e., $\langle \delta F(t_1)\,\delta F(t_2)\rangle = 0$ for $t_2 - t_1 > \tau_\mathrm{coll}$.

Typically $\tau_{\rm coll}$ is much shorter than any other timescale of interest, and we can effectively take the limit $\tau_{\rm coll} \to 0$. In that case, we can write

$$\langle \delta F(t_1)\, \delta F(t_2) \rangle = 2D\, \gamma^2\, \delta(t_2 - t_1)$$

Here the factor γ^2 has been put in for convenience, and D is a parameter that characterizes the strength of the correlation (with $[D] = \ {\rm cm}^2\ {\rm s}^{-1}$). It is called a **diffusion coefficient** for reasons that will become clear shortly. A stochastic variable that obeys this expression for the correlation function is often referred to as **white noise**.

Substitution in the expression for $\langle v(t_1) v(t_2) \rangle$ gives

$$\begin{aligned} \langle v(t_1) v(t_2) \rangle &= \langle v(t_1) \rangle \, \langle v(t_2) \rangle + \frac{2D\gamma^2}{m^2}\, {\rm e}^{-\gamma(t_1+t_2)/m} \int_0^{t_1} {\rm d}t' {\rm e}^{2\gamma t'/m} \\ &= \langle v(t_1) \rangle \, \langle v(t_2) \rangle + \frac{D\gamma}{m}\, \left[{\rm e}^{-\gamma(t_1-t_2)/m} - {\rm e}^{-\gamma(t_1+t_2)/m} \right] \end{aligned}$$

For $t_1, t_2 \to \infty$ we can drop the last term, as well as $\langle v(t_1) \rangle$ and $\langle v(t_2) \rangle$ (after all, due to the friction, $\langle v \rangle \to 0$). Hence,

$$\langle v(t_1) v(t_2) \rangle \underset{t_1,t_2\to\infty}{\longrightarrow} \frac{D\gamma}{m}\, {\rm e}^{-\gamma(t_2-t_1)/m}$$

Since we assume the stochastic force to be **stationary**, we can replace $t_2 - t_1$ with Δt, to obtain the more general expression

$$\langle v(t) v(t+\Delta t) \rangle \underset{t\to\infty}{\longrightarrow} \frac{D\gamma}{m}\, {\rm e}^{-\gamma \Delta t/m}$$

As we can see, velocities are correlated, but only for a finite amount of time. The characteristic time scale on which they become uncorrelated is the dissipation time $\tau_{\rm d} = m/\gamma$.

If we now revert back to 3D, we can use the above result to write

$$\langle \vec{v}(t) \cdot \vec{v}(t) \rangle = \langle v^2(t) \rangle \underset{t\to\infty}{\longrightarrow} \frac{3D\gamma}{m}$$

Hence, at late times the velocity dispersion among particles that all started at an identical phase-space point $(\vec{q}_0, \vec{p}_0)$ but at different times becomes

$$\sigma_v^2 = \langle v^2 \rangle - \langle v \rangle^2 = \frac{3D\gamma}{m}$$

where $v = |\vec{v}|$ and we have used that $\langle v \rangle \to 0$. Hence, due to collisions, the average velocity of our subject mass vanishes (friction), while the velocity dispersion asymptotes to a constant value. Since we know that collisions drive the system towards **equipartition** in which the kinetic energy of the subject mass

$$\frac{1}{2} m \langle v^2 \rangle = \frac{3}{2} D\gamma$$

is equal to that of the field particles. If the latter are in local thermal equilibrium, the latter is equal to $\frac{3}{2} k_{\rm B} T$, and we thus infer that

$$\boxed{D\gamma = k_{\rm B} T}$$

This relation, which is known as the **Einstein-Smoluchowski relation**, indicates that the **diffusion coefficient** D and the frictional dissipation γ are tightly related. This is a manifestation of the **fluctuation-dissipation theorem**, which basically states that fluctuating forces cause dissipation (friction).

The Smoluchowski equation: We end this chapter on stochasticity with the Smolochowski equation, which will be the starting point for the derivation of the Fokker-Planck equation in the next chapter. Consider three (arbitrary) times t_1, t_2 and t_3, ordered such that $t_3 > t_2 > t_1$. Consider a realization of an arbitrary random process $x(t)$ starting from x_1 at t_1. Suppose we want to know the conditional probability $P_2(x_3, t_3 | x_1, t_1)$. Since the realization must go through some point x_2 at the intermediate time t_2 we have that

$$P_2(x_3, t_3 | x_1, t_1) = \int P_3(x_3, t_3 | x_2, t_2; x_1, t_1)\, P_2(x_2, t_2 | x_1, t_1)\, {\rm d}x_2$$

This is almost too obvious to be interesting. But, now let's consider the case in which $x(t)$ is **Markovian**. In that case we have that

$$P_3(x_3, t_3 | x_2, t_2; x_1, t_1) = P_2(x_3, t_3 | x_2, t_2) = P_2(x_3, t_3 - t_2 | x_2)$$

where the last step follows from our assumption that the probability distributions are **stationary** (i.e., the value of t_1 is irrelevant). Substitution in the above expression yields

$$\boxed{P_2(x_3, t_3 | x_1) = \int P_2(x_3, t_3 - t_2 | x_2)\, P_2(x_2, t_2 | x_1)\, {\rm d}x_2}$$

which is known as the **Smoluchowski equation** or the **Chapman-Kolmogorov equation**. Note that this equation is only valid for Markovian random processes, and for $t_3 > t_2 > 0$.

CHAPTER 9

The Fokker-Planck Equation

When describing the dynamics of many-body systems, we often want to know the collective influence of many degrees of freedom (this is often called a 'thermal bath', or 'bath' for short) on a single (possibly vectorial) degree of freedom (say the momentum vector $\vec{p}$ of one particular subject mass. The subject can have a variety of interactions with the bath; they can be short-ranged, i.e., van der Waals forces in a gas, or long-ranged, such as the Coulomb forces in a plasma or the gravitational forces acting on stars in a galaxy. Typically, these interactions with the bath cause $\vec{p}$ to be a (Markovian) random process, and thus subject to the Langevin and Smolochowski equations described in the previous chapter. Our goal in this chapter is to derive an equation that describes how the probability distribution for a (Markovian) degree of freedom (typically this will be the velocity or momentum of our subject mass) evolves over time due to being exposed to a stochastic force field arising from the bath's many degrees of freedom. This equation is called the Fokker-Planck equation, which is a powerful differential equation for the evolution of any probability function for any Markovian degree(s) of freedom. In particular, as we will see, it can be used to describe the collision term, $(\partial f/\partial t)_{\rm c}$, in the Boltzmann equation describing a collisional many-body system in which the interactions are long-range.

We start with a derivation for the 1-dimensional case, i.e., we consider a 1-dimensional random variable $x(t)$. The result is then easily generalized to higher dimensional random variables.

Starting from the **Smoluchowski equation** at the end of the previous chapter, and using the following change of notation $t_3 \to t + \Delta t$, $x_3 \to x$, $x_2 \to x - \Delta x$ and $x_1 \to x_0$ we have that a Markovian random process obeys

$$P_2(x, t + \Delta t) = \int \Psi(\Delta x, \Delta t | x - \Delta x, t)\, P_2(x - \Delta x, t|x_0) \mathrm{d}(\Delta x)$$

where we have introduced the **transition probability** $\Psi(\Delta x, \Delta t|x - \Delta x, t)$ which expresses the probability that starting from $x - \Delta x$ at t the random variable undergoes a change Δx in a time step Δt. Note that this is merely a change of notation, since

$$\Psi(\Delta x, \Delta t|x - \Delta x, t) = P_2(x, t + \Delta t|x - \Delta x, t)$$

If we now use a Taylor series expansion for the integrand in the above expression, we obtain that

$$\begin{aligned}
P_2(x,t+\Delta t) &= \int \mathrm{d}(\Delta x) \sum_{n=0}^{\infty} \frac{(-\Delta x)^n}{n!} \frac{\partial^n}{\partial x^n} \left[\Psi(\Delta x, \Delta t|x-\Delta x,t)\, P_2(x-\Delta x,t|x_0)\right]_{x-\Delta x=x} \\
&= \int \mathrm{d}(\Delta x) \sum_{n=0}^{\infty} \frac{(-\Delta x)^n}{n!} \frac{\partial^n}{\partial x^n} \left[\Psi(\Delta x, \Delta t|x,t)\, P_2(x,t|x_0)\right] \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{\partial^n}{\partial x^n} \left[P_2(x,t|x_0) \int \mathrm{d}(\Delta x)\, (\Delta x)^n\, \Psi(\Delta x, \Delta t|x-\Delta x,t)\right] \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{\partial^n}{\partial x^n} \left[\langle (\Delta x)^n \rangle\, P_2(x,t|x_0)\right]
\end{aligned}$$

where in the last step we have used the fact that the integral expresses the expectation value (ensemble average) of $(\Delta x)^n$. Using that the $n=0$ term of the sum is simply equal to $P_2(x,t|x_0)$, and bringing this term to the left-hand side of the equation, we obtain that

$$\begin{aligned}
\frac{\partial P_2(x,t|x_0)}{\partial t} &= \lim_{\Delta t \to 0} \frac{P_2(x,t+\Delta t|x_0) - P_2(x,t|x_0)}{\Delta t} \\
&= \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{\partial^n}{\partial x^n} \left[D^{(n)}(x,t)\, P_2(x,t|x_0)\right]
\end{aligned}$$

where we have defined

$$D^{(n)}(x,t) \equiv \lim_{\Delta t \to 0} \frac{\langle (\Delta x)^n \rangle}{\Delta t}$$

If we only keep the first two terms of the Taylor series expansion (i.e., we assume that Δx is small, such that the higher-order terms can be ignored), then we finally obtain the **Fokker-Planck equation**:

$$\boxed{\frac{\partial P_2}{\partial t} = -\frac{\partial}{\partial x}[D^{(1)} P_2] + \frac{1}{2}\frac{\partial^2}{\partial x^2}[D^{(2)} P_2]}$$

Here $P_2 = P_2(x,t|x_0)$ is to be regarded as a function of the variables x and t with x_0 fixed, i.e., $P_2(x,0|x_0) = \delta(x-x_0)$. As we will see below, this Fokker-Planck (hereafter FP) equation is a generalized diffusion equation for the evolution of P_2.

Validity of Fokker-Planck equation: Before we discuss what this equation tells us, it is important to first ask under what conditions it is valid. Along the way we have made the assumptions that the random process is **stationary** and **Markovian**. In addition, the truncation of the Taylor series implies that we have assumed that Δx is small. We say that we have assumed that we are in the **diffusive limit**. One might be tempted to argue that if one chooses Δt to be sufficiently small, then one is always in the diffusive limit. However, we also need to satisfy the Markovian constraint.

Suppose we want to use the FP equation to describe the diffusion of particles in a many-body system in velocity space, i.e., our random variable is $v(t)$. As we have seen in the previous chapter, $v(t)$ is a random process due to the stochastic forces $\delta F(t)$ that it experiences. Since $\delta F(t)$ is a continuous, smooth (but unpredictable, and thus stochastic) function, it is never really Markovian. However, if we examine $\delta F(t)$ with a time step $\Delta t \gg \tau_{\rm coll}$ then the resulting $v(t)$ will be Markovian. However, on such time scales many of the collisions will result in large δv. Hence, the FP-equation is not very well suited to study how individual particles in a gas diffuse in velocity space. However, if we consider a relatively massive particle (say, a pollen floating in the air), then the velocity changes due to collisions with individual gas atoms will in general be small due to momentum conservation. Hence, the FP equation *can* be used to describe the Brownian motion of massive particles immersed in a fluid.

In the case of a galaxy or globular cluster, each star undergoes many simultaneous interactions with all other stars in the system (i.e., gravity is an unshielded long-range force). Hence, we effectively have that $\tau_{\rm coll} = 0$, and thus that $\delta F(t)$ is to good approximation Markovian, even for very small time steps. Typically, though, there is always a non-zero probability for a very close encounter that results in a large Δv. These encounters are *not* treated by the FP equation. However, as we will see later, the impact of the many weak encounters (those corresponding to large impact parameters and small Δv) is always dominant over the impact of the few strong encounters (those corresponding to small impact parameters and larer Δv), and one thus should not make a large error by ignoring the strong interactions (i.e., the FP equation is able to accurately describe the diffusion that stars undergo due to the fact that the system is not exactly collisionless.

Diffusion in velocity space:
Using that $\Delta v = v(\Delta t) - v$, we have that

$$\begin{aligned} D^{(1)} &= \lim_{\Delta t\to 0} \frac{\langle \Delta v\rangle}{\Delta t} = \lim_{\Delta t\to 0} \frac{1}{\Delta t}\left[\langle v(\Delta t)\rangle - v\right] \\ D^{(2)} &= \lim_{\Delta t\to 0} \frac{\langle (\Delta v)^2\rangle}{\Delta t} = \lim_{\Delta t\to 0} \frac{1}{\Delta t}\left[\langle v(\Delta t)v(\Delta t)\rangle - 2v\langle v(\Delta t)\rangle + v^2\right] \end{aligned}$$

In Chapter 8 we have seen that for white noise we have that

$$\langle v(t)\rangle = v\,\mathrm{e}^{-\gamma t/m}$$

and

$$\langle v(t)v(t)\rangle = \langle v(t)\rangle^2 + \frac{D\gamma}{m}\left(1 - \mathrm{e}^{-2\gamma t/m}\right)$$

Substitution in the above expressions for $D^{(1)}$ and $D^{(2)}$, and using the Taylor series expansion for e^x, yields

$$D^{(1)} = -v\frac{\gamma}{m}\,, \qquad D^{(2)} = 2D\frac{\gamma^2}{m^2}$$

The FP equation for the diffusion in velocity space (subject to a white noise stochastic force) is thus given by

$$\boxed{\frac{\partial P(v,t)}{\partial t} = \frac{1}{m}\left[\gamma\frac{\partial\, vP(v,t)}{\partial v} + \frac{D\gamma^2}{m}\frac{\partial^2 P(v,t)}{\partial v^2}\right] = \frac{1}{\tau_{\rm d}}\frac{\partial}{\partial v}\left[vP(v,t) + \frac{D}{\tau_{\rm d}}\frac{\partial P(v,t)}{\partial v}\right]}$$

where the latter part we expressed in terms of the **dissipation time** $\tau_{\rm d} = m/\gamma$.

Let's see what it does. Suppose our original distribution is a (narrow) Gaussian centered on some velocity $v_0 > 0$ and with a small dispersion σ_v. The first term is proportional to $v\partial P/\partial v$ which is negative for $v < 0$ and $v > v_0$ and positive for $0 < v < v_0$. Hence, this term causes the center of the velocity distribution to move towards zero (i.e., $|v_0| \to 0$), consistent with the friction term γv in front of the term. The second term is proportional to the second derivative $\partial^2 P/\partial v^2$, which is negative for $|v - v_0| < \sigma_v$ and positive otherwise. Hence, the second term causes a 'diffusion' of velocities away from the mean v_0, and thus a broadening of $P(v,t)$.

We can also use the Fokker-Planck equation to examine what the equilibrium velocity distibution looks like. Since at equilibrium $\partial P(v,t)/\partial t = 0$, we see that the equilibrium distribution P satisfies

$$\frac{\partial P}{\partial v} = -\frac{v\tau_{\rm d}}{D}P \quad\Rightarrow\quad P \propto \exp\left(-\frac{mv^2}{2D\gamma}\right)$$

Using the **Einstein-Smoluchowski relation** according to which $D\gamma = k_{\rm B}T$ (see Chapter 8), and requiring normalization, we obtain that

$$P_{\rm eq}(v) = \left(\frac{m}{2\pi k_{\rm B}T}\right)^{1/2} \exp\left(-\frac{mv^2}{2k_{\rm B}T}\right)$$

which we recognize as the **Maxwell-Boltzmann distribution** (in 1D). Hence, collisions (stochastic forces) drive the velocity distribution towards the MW distribution. In fact, we can even do better than this. Suppose that we start all the particles off at $t = 0$ with some fixed velocity v_0, i.e., $P(v, 0) = \delta(v - v_0)$. Then, the full time-dependent solution to the Fokker-Planck equation is

$$P(v,t) = \left(\frac{m}{2\pi k_{\rm B}T(1-{\rm e}^{-2\gamma t/m})}\right)^{1/2} \exp\left[-\frac{m(v-v_0{\rm e}^{-\gamma t/m})^2}{2k_{\rm B}T(1-{\rm e}^{-2\gamma t/m})}\right]$$

This distribution is known as the **Ornstein-Uhlenbeck distribution**. As $t \rightarrow \infty$ it the Maxwell-Boltzmann distribution, but now we can see how the velocity distribution approaches equilibrium!

Extension to multi-dimensional Markov processes: the discussion thus far was restricted to one-dimensional Markov processes $x(t)$. However, it is straightforward to extent this to any n-dimensional Markov process $\vec{x}(t)$, with $\vec{x} = (x_1, x_2, ..., x_n)$. In most case that we encounter in astrophysics n will be either 3, for example when considering the diffusion in 3D velocity space $\vec{v}$, or 6, when considering diffusion in 6D phase space $\vec{w} = (\vec{q}, \vec{p})$. The obvious generalization for the n-dimensional Fokker-Planck equation is given by

$$\boxed{\frac{\partial P}{\partial t} = -\frac{\partial}{\partial x_i}[D_i^{(1)} P_2] + \frac{1}{2}\frac{\partial^2}{\partial x_i \partial x_j}[D_{ij}^{(2)} P_2]}$$

Here the functions $D_i^{(1)}$ and $D_{ij}^{(2)}$ are the obvious generalizations of their 1D equivalents, i.e.,

$$\begin{aligned} D_i^{(1)} &= \lim_{\Delta t\rightarrow 0}\frac{1}{\Delta t}\int {\rm d}^n(\Delta\vec{x})\,(\Delta x)_i\,\Psi(\Delta\vec{x},\Delta t|\vec{x}) = \lim_{\Delta t\rightarrow 0}\frac{\langle(\Delta x)_i\rangle}{\Delta t} \\ D_{ij}^{(2)} &= \lim_{\Delta t\rightarrow 0}\frac{1}{\Delta t}\int {\rm d}^n(\Delta\vec{x})\,(\Delta x)_i\,(\Delta x)_j\,\Psi(\Delta\vec{x},\Delta t|\vec{x}) = \lim_{\Delta t\rightarrow 0}\frac{\langle(\Delta x)_i\,(\Delta x)_j\rangle}{\Delta t} \end{aligned}$$

Recap: The FP equation we have derived thus far allows us to compute the evolution of $P_2(\vec{x}, t|\vec{x}_0)$ for a **stationary, Markov process** $\vec{x}(t)$. In the **diffusive limit**, which implies that we need $\Delta\vec{x}$ on a time scale $\Delta t \sim \tau_{\rm coll}$ to be small, we can truncate the Taylor series expansion of the **transition probability function** at second order, thus giving rise to the FP equation, which only depends on the first and second order **diffusion coefficients** $D_i^{(1)}$ and $D_{ij}^{(2)}$.

Although this is a powerful result for many applications in physics, in some cases we are not interested in the evolution of subject masses that start from a fixed $\vec{x}_0$ at $t = t_0$. Rather, we want to find an equation that describes how the (unconstrained 1-point probability function $P(\vec{x}, t)$ evolves under the influence of a stochastic force field, characterized by a correlation function $\langle \delta F(\vec{x}_1, t)\, \delta F(\vec{x}_2, t + \Delta t)\rangle$. A particular application of this sort, and the one most relevant to ASTR 501, is the **collision term** $\partial f/\partial t|_{\rm c}$ of the **Boltzmann equation** that we derived in Chapter 6.

As we have seen, using the **BBGKY hierarchy**, we have that

$$\frac{{\rm d}f}{{\rm d}t} = \frac{\partial f}{\partial t} + \{f, \mathcal{H}\} = \left(\frac{\partial f}{\partial t}\right)_{\rm c}$$

where

$$\left(\frac{\partial f}{\partial t}\right)_{\rm c} = \int {\rm d}^3\vec{q}_2\, {\rm d}^3\vec{p}_2\, \frac{\partial U(|\vec{q}_1 - \vec{q}_2|)}{\partial \vec{q}_1} \cdot \frac{\partial f^{(2)}}{\partial \vec{p}_1}$$

which depends on the 2-particle DF $f^{(2)}(\vec{q}_1, \vec{q}_2, \vec{p}_1, \vec{p}_2, t)$. Solving this integro-differential equation is extremely difficult. In addition, since it depends on $f^{(2)}$ it is not a closed equation, but part of the full BBGKY hierarchy.

We have seen that <u>if</u> we can assume that the collisions are localized, i.e., are due to a **short-range force** such as the van der Waals force, <u>and</u> we assume **molecular chaos**, <u>then</u> we have that $\partial f/\partial t|_{\rm c} \rightarrow I[f]$, such that the Boltzmann equation becomes closed. We can then obtain moment equations (continuity, momentum and energy) by multiplying this closed Boltzmann equation with v^n ($n = 0, 1$ and 2, respectively) and integrating over momentum space.

But how can we describe $\partial f/\partial t|_{\rm c}$ for a gravitational N-body system (or a charged plasma) in which the forces are **long-range**? As you may expect at this point, the answer is to resort to a FP-like equation, which after all expresses how a probability function evolves under the influence of a stochastic force field. We thus want to derive a FP-like equation to describe $\partial f(\vec{q}, \vec{p}, t)/\partial t$ due to collisions (i.e., due to stochasticity in the force field that arises from finite-N effects). Note that the evolution of $f(\vec{q}, \vec{p}, t)$

due to the *smooth* potential is accounted for by the CBE

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \{f, \mathcal{H}\} = 0 \qquad \rightarrow \qquad \frac{\partial f}{\partial t} = \{\mathcal{H}, f\}$$

We need to modify this by adding a term $\partial f/\partial t|_{\rm c}$ due to the finite-N induced non-smoothness. This is exactly what we have been working towards in this and the previous chapter. However, we need to modify the FP equation derived above such that it describes the evolution of $f(\vec{q}, \vec{p}, t)$ (due to collisionality), rather than the evolution of $P_2(\vec{q}, \vec{p}, t|\vec{q}_0, \vec{p}_0, t_0)$. As it turns out, this is actually quite simple. The reason is that both f and P_2 are probability functions associated with a Markovian processes. In addition, the initial conditions (i.e., $(\vec{q}_0, \vec{p}_0)$ at t_0) played no role in our derivation of the FP equation.

In what follows to simplify the notation we use $(\vec{q}, \vec{p}) \rightarrow \vec{w}$, and we describe the evolution of $f(\vec{w}, t)$ for the 6D stochastic variable $\vec{w}(t)$. Let $\Psi(\Delta\vec{w}, \vec{w})\, \mathrm{d}^6(\Delta\vec{w})\, \Delta t$ be the probability that a particle with phase-space coordinates $\vec{w}$ is 'scattered' to the phase-space volume $\mathrm{d}^6(\Delta\vec{w})$ centered on $\vec{w} + \Delta\vec{w}$ during the (short) time interval Δt. Note that the **transition probability** only accounts for the effects due to the stochastic component of the force field; the phase-space evolution due to the smooth potential is already accounted for as described above.

The distribution function $f(\vec{w}, t)$ at a fixed $\vec{w}$ changes due to collisions in two ways. First of all, particles with phase-space coordinates $\vec{w}, \vec{w} + \mathrm{d}\vec{w}$ are scattered into other parts of phase-space due to collisions, thus giving rise to a **loss term**, i.e., a decreases of $f(\vec{w}, t)$. Secondly, $f(\vec{w}, t)$ increases due to the fact that particles outside of $\vec{w}, \vec{w} + \mathrm{d}\vec{w}$ are scattered into this phase-space volume due to encounters, thus giving rise to a **gain term**. It is easy to see that

$$\begin{aligned}
\text{loss term}: \qquad \left(\frac{\partial f(\vec{w})}{\partial t}\right)_- &= -f(\vec{w}) \int \mathrm{d}^6(\Delta\vec{w})\, \Psi(\Delta\vec{w}, \vec{w})\,, \\
\text{gain term}: \qquad \left(\frac{\partial f(\vec{w})}{\partial t}\right)_+ &= \int \mathrm{d}^6(\Delta\vec{w})\, \Psi(\Delta\vec{w}, \vec{w} - \Delta\vec{w})\, f(\vec{w} - \Delta\vec{w})\,.
\end{aligned}$$

The combination of these loss and gain terms equals the collision term of the Boltzmann equation, i.e.,

$$\boxed{\left(\frac{\partial f}{\partial t}\right)_{\rm c} = \int \mathrm{d}^6(\Delta\vec{w})\, \left[\Psi(\Delta\vec{w}, \vec{w} - \Delta\vec{w})\, f(\vec{w} - \Delta\vec{w}) - \Psi(\Delta\vec{w}, \vec{w})\, f(\vec{w})\right]}$$

This is sometimes called the **Master equation** (not to be confused with the master-moment equation defined in Chapter 7). Upon closer examination, there is a subtle assumption lurking in this expression. We have assumed that $f(\vec{w})$ and $\Psi(\Delta\vec{w},\vec{w})$ are statistically independent, such that the scattering rates can be written as their products. This implies that we have assumed that the distributions of 'field' particles, described by $f(\vec{w})$, and 'subject' masses, baked into $\Psi(\Delta\vec{w},\vec{w})$, are independent. But the collisions should introduce correlations among them....Effectively we have thus once more relied on the assumption of **molecular chaos**, i.e., that by the time a field and subject particle meet for a second encounter the correlation induced by the first encounter has been erased by encounters with other field particle.

If we restrict ourselves, as before, to **weak encounters only**, such that $|\Delta\vec{w}|$ is small, then we can Taylor expand the first term in the master equation and truncate at second-order (i.e., make the Fokker-Planck assumption):

$$\Psi(\Delta\vec{w},\vec{w}-\Delta\vec{w})\,f(\vec{w}-\Delta\vec{w}) = \Psi(\Delta\vec{w},\vec{w})\,f(\vec{w}) - \sum_{i=1}^{6}\Delta w_i\frac{\partial}{\partial w_i}\left[\Psi(\Delta\vec{w},\vec{w})\,f(\vec{w})\right] + \frac{1}{2}\sum_{i,j=1}^{6}\Delta w_i\,\Delta w_j\frac{\partial^2}{\partial w_i\partial w_j}\left[\Psi(\Delta\vec{w},\vec{w})\,f(\vec{w})\right]$$

Substituting this in the master equation, we finally obtain that

$$\boxed{\left(\frac{\partial f}{\partial t}\right)_{\rm c} = -\sum_{i=1}^{6}\frac{\partial}{\partial w_i}\left\{D[\Delta w_i]\,f(\vec{w})\right\} + \frac{1}{2}\sum_{i,j=1}^{6}\frac{\partial^2}{\partial w_i\partial w_j}\left\{D[\Delta w_i\Delta w_j]\,f(\vec{w})\right\}}$$

with

$$\begin{aligned} D[\Delta w_i] &\equiv \int {\rm d}^6(\Delta\vec{w})\,\Delta w_i\,\Psi(\Delta\vec{w},\vec{w}) \\ D[\Delta w_i\Delta w_j] &\equiv \int {\rm d}^6(\Delta\vec{w})\,\Delta w_i\Delta w_j\,\Psi(\Delta\vec{w},\vec{w}) \end{aligned}$$

the **diffusion coefficients** that express the expectation values for a change Δw_i and $\Delta w_i\Delta w_j$ per unit time interval. These coefficients describe the rate at which particles diffuse through phase-space due to collisions. Note that the use of square brackets after the diffusion coefficients is to remind the reader that $D[\Delta w_i\Delta w_j]$ is *not* a function of the variable $\Delta w_i\Delta w_j$. Rather, it is an *average* of $\Delta w_i\Delta w_j$ over $\Delta\vec{w}$ per unit time and a function of the position in phase-space where the average

is taken. And since with this new notation it is immediately evident whether the diffusion coefficien is first of second order, we also removed the superscripts (1) and (2).

The above expression for the collision term $\partial f/\partial t|_{\rm c}$ is a **Fokker-Planck equation**. Substituting this in the Boltzmann equation yields

$$\boxed{\frac{{\rm d}f}{{\rm d}t} = -\sum_{i=1}^{6} \frac{\partial}{\partial w_i} \left\{ D[\Delta w_i]\, f(\vec{w}) \right\} + \frac{1}{2} \sum_{i,j=1}^{6} \frac{\partial^2}{\partial w_i \partial w_j} \left\{ D[\Delta w_i \Delta w_j]\, f(\vec{w}) \right\}}$$

This form of the Fokker-Planck equation is sometimes called **Kramers equation** or the **Chandrasekhar equation**. More often, though, in the astrophysical literature, it is simply referred to as 'the' Fokker-Planck equation.

The virtue of the FP equation is that the **Lagrangian** evolution of the distribution function is entirely described by the first and second-order diffusion coefficients, which are functions only of the phase-space coordinates. Hence, it is a differential equation, rather than an integro-differential equation like the first-order equation of the BBGKY hierarchy, and therefore much more easily solved. As a consequence, the FP equation is the primary tool to describe the evolution of the distribution function of a gravitational system under the influence of collisions.

Weak versus Strong encounters:

As we have mentioned several times, the FP equation is only valid for **weak encounters** (those resulting in a small $\Delta\vec{w}$). In a gravitational system there will always be some encounters, though, for which $\Delta\vec{w}$ will be large. Typically these will be close encounters with a small impact parameter $b < b_{\rm min}$. If the *net* impact of these strong encounters is small compared to that of weak encounters we are justified in ignoring the strong encounters, and the FP equation should be accurate.

In the problem sets you derive that the velocity impuls of a subject mass due to a (high-speed) encounter with a field particle of mass m with impact parameter b and velocity v is given by

$$\Delta v_\perp \simeq \frac{2G\,m}{b\,v}$$

where the perpendicular sign is used to indicate that the velocity impuls is in the direction perpendicular to that of the encounter velocity. This approximation assumes that the deflection angle is small, such that the trajectory of the field particle

wrt the subject mass can be approximated as a straight line. Roughly speaking, this approximation breaks down when $\Delta v_\perp \approx v$, which occurs when

$$b = b_{90} = \frac{2Gm}{v^2}$$

Here the '90' is to indicate that for this impact parameter the deflection angle is equal to 90 degrees. In what follows we set $b_{\rm min} = b_{90}$[4]

The surface density of field particles in a system of size R is roughly of the order of $N/(\pi R^2)$. Hence, when a subject mass crosses the system once it has

$$\frac{{\rm d}N}{{\rm d}b}\,{\rm d}b = \frac{N}{\pi R^2}\cdot 2\pi\, b\,{\rm d}b = \frac{2N}{R^2}\,b\,{\rm d}b$$

encounters with impact parameters in the range b, $b + {\rm d}b$. Each such encounter (assuming weak encounters only, i.e., $b > b_{\rm min}$) produces an impulsive velocity perturbation $\Delta v_\perp(b) \sim 2Gm/bv$. If, for simplicity, we assume that the system is homogeneous then the net $\Delta v_\perp$ per crossing will be approximately zero (the perturbing field particles are randomly distributed in the plane perpendicular to the subject's velocity). However, the mean square velocity change is given by

$$\overline{\Delta v^2} = \int_{b_{\rm min}}^{b_{\rm max}} (\Delta v_\perp)^2(b)\,\frac{{\rm d}N}{{\rm d}b}\,{\rm d}b = 8\,N\,\left(\frac{Gm}{Rv}\right)^2\,\int_{b_{\rm min}}^{b_{\rm max}}\frac{{\rm d}b}{b}$$

An important conclusion from this is that

Each octave in impact parameters contributes equally to gravitational scattering

If we use that the typical velocity is given by

$$v \sim \sqrt{\frac{GM}{R}} = \sqrt{\frac{GNm}{R}}\,,$$

and introduce the **Coulomb logarithm**

$$\ln\Lambda = \ln\left(\frac{b_{\rm max}}{b_{\rm min}}\right)$$

[4]Defining the minimum impact parameter for a weak encounter as that for which the deflection angle is somewhat smaller, say 5 degrees, which may seem more reasonable, does not significantly impact any of what follows.

then we can obtain that

$$\overline{\Delta v^2} = \frac{8}{N} \ln \Lambda \, v^2$$

If we set $b_{\max} = R$ and $b_{\min} = b_{90}$, then, using the above expression for the typical velocity, we have that $\ln \Lambda \simeq \ln N$. We thus have that it takes of the order of $N/(8 \ln N)$ crossing times for the net effect of weak encounters to be such that $(\Delta v_\perp)^2 \sim v^2$. This time is often called the **two-body relaxation time**:

$$\boxed{\tau_{\rm relax} = \frac{N}{8 \ln N} \tau_{\rm cross}}$$

(see also Chapter 2).

For comparison, we now compute how long it takes for **strong encounters** to produce a net effect of that magnitude. In fact, this is trivial: by definition, a strong encounter has $b < b_{90}$, and therefore a single strong encounter suffices to have $(\Delta v_\perp)^2 \geq v^2$. All we need to compute is therefore the collision time for strong encounters. Using that $\tau_{\rm coll} = (n \sigma v)^{-1}$ (see Chapter 2), where the number density $n = 3N/4\pi R^3$ and the cross section $\sigma = \pi (2 b_{90})^2 = 4 \pi b_{90}^2$, we find that

$$\tau_{\rm coll}^{\rm strong} = \frac{1}{3N} \left(\frac{R}{b_{90}} \right)^2 \tau_{\rm cross}$$

Using the expression for b_{90} and the typical velocity v, we have that $R/b_{90} \simeq N/2$, so that

$$\tau_{\rm coll}^{\rm strong} = \frac{N}{12} \tau_{\rm cross} = \frac{2}{3} \ln N \, \tau_{\rm relax}$$

Hence, we see that

The net impact of **weak encounters** is of order $\ln N$ times as important as that of **strong encounters**

Thus, as long as N is large, we can safely ignore strong encounters, and the Fokker-Planck equation accurately describes the evolution of the distribution function due to collisions.

Diffusion coefficients:

Solving the Fokker-Planck equations requires that we can write down expressions for the diffusion coefficients $D[\Delta w_i]$ and $D[\Delta w_i \, \Delta w_j]$ with w_i being a 6D phase-space

vector. Working out the diffusion coeffients in 6D phase-space from first principles is extremely complicated. However, we can drastically simplify matters using the fact that each octave in impact parameter makes an equal contribution to the scattering. If we focus on weak encounters only then the impact parameters to consider run from $b_{\rm min} = b_{90}$ to $b_{\rm max} = R$. We now derive the impact parameter $b_{1/2}$, such that encounters with $b < b_{1/2}$ account for 50 percent of the total $(\Delta v_\perp)^2$. Using that each octave makes an equal contribution we thus have that

$$\ln(b_{1/2}) - \ln(b_{\rm min}) = \ln(b_{\rm max}) - \ln(b_{\rm min})$$

Solving for $b_{1/2}$ and using that $b_{\rm max} = R$ and $b_{\rm max}/b_{\rm min} = \Lambda \sim N$ one trivially finds that

$$b_{1/2} = \frac{R}{\sqrt{N}}$$

Hence, *more than 50 percent of the scattering impact is due to encounters with an impact parameter that is significantly smaller than the mean particle separation* $\lambda_{\rm int} = R/N^{1/3}$! This amazing result implies that we are more than justified in making a **local approximation**, i.e., it is surprisingly accurate to assume that the field particles are an infinite, homogeneous sea with a DF that is equal to the local DF. But there is more; since scattering is mainly a local process, this supports the assumption of molecular chaos (i.e., it is unlikely to have a local encounter with the same particle twice, and even if one does, the memory/correlation from the previous encounter will have been erased in the meantime). It also implies that the encounter time, which is of the order of b/v, is much smaller than the orbital time. Hence, we can ignore any change in position during the encounter (further justifying the local approximation), and focus purely on changes in velocity (a change in velocity will, over time, induce a change in position, which is accounted for by the CBE).

If we choose the canonical phase-space coordinates to be the Cartesian coordinates, $\vec{w} = (\vec{x}, \vec{v})$, and use that we can ignore changes in position, such that $D[\Delta x_i] = D[\Delta x_i \Delta x_j] = D[\Delta x_i \Delta v_j] = 0$, then the collision term in the Fokker-Planck equation simplifies to

$$\boxed{\left(\frac{\partial f}{\partial t}\right)_{\rm c} = -\sum_{i=1}^{3} \frac{\partial}{\partial v_i} \left\{ D[\Delta v_i]\, f(\vec{w}) \right\} + \frac{1}{2} \sum_{i,j=1}^{3} \frac{\partial^2}{\partial v_i \partial v_j} \left\{ D[\Delta v_i \Delta v_j]\, f(\vec{w}) \right\}}$$

Hence, we are left with the task to compute $D[\Delta v_i]$ (3 components) and $D[\Delta v_i \Delta v_j]$ (6 independent components). In the specific case of *local* gravitational encounters

this can be done fairly straightforward. Consider an infinite, homogeneous sea of field particles of mass $m_{\rm a}$ and with a velocity distribution $f_{\rm a}(\vec{v}_{\rm a})$ (note, because of the assumed homogeneity, we can ignore the position dependence). Let m be the mass of our subject(s), whose velocity distribution function is $f(\vec{v})$. Often, but not always, we are interested in the situation where the subject masses are the field particles themselves, in which case $m = m_{\rm a}$ and $f(\vec{v}) = f_{\rm a}(\vec{v}_{\rm a})\delta(\vec{v}_{\rm a} - \vec{v})$. In what follows, though, we consider the more general case in which the subjects can be different from the field particles.

Working out how an encounter with impact parameter b between a field particle and a subject mass impacts the velocity of the latter, and computing the expectation values $\langle \Delta v_i \rangle$ and $\langle \Delta v_i \Delta v_j \rangle$ by integrating over b and the velocity distribution $f(\vec{v})$ of the field particles (see *Galactic Dynamics* by Binney & Tremaine, or *Dynamics and Evolution of Galactic Nuclei* by Merritt for detailed derivations), one obtains

$$\begin{aligned} D[\Delta v_i] = \langle \Delta v_i \rangle &= 4\pi G^2 m_{\rm a}(m + m_{\rm a}) \ln\Lambda \, \frac{\partial}{\partial v_i} h(\vec{v}) \\ D[\Delta v_i \Delta v_j] = \langle \Delta v_i \Delta v_j \rangle &= 4\pi G^2 m_{\rm a}^2 \ln\Lambda \, \frac{\partial^2}{\partial v_i \partial v_j} g(\vec{v}) \end{aligned}$$

where $h(\vec{v})$ and $g(\vec{v})$ are known as the **Rosenbluth potentials**, given by

$$\begin{aligned} h(\vec{v}) &\equiv \int {\rm d}^3\vec{v}_{\rm a} \frac{f_{\rm a}(\vec{v}_{\rm a})}{|\vec{v} - \vec{v}_{\rm a}|} \\ g(\vec{v}) &\equiv \int {\rm d}^3\vec{v}_{\rm a} f_{\rm a}(\vec{v}_{\rm a}) \, |\vec{v} - \vec{v}_{\rm a}| \end{aligned}$$

Here we have assumed that Λ is sufficiently large, such that $\ln(1+\Lambda^2) = 2\ln\Lambda$. Note that the expression the Coulomb logarithm is slightly different than before due to the fact that we consider the general case in which subject and field particles have different masses, i.e.,

$$\Lambda \equiv \frac{b_{\rm max}}{b_{\rm min}} = \frac{Rv^2}{G(m + m_{\rm a})}$$

If we assume that the velocity distributions are **isotropic**, such that $f(\vec{v}) \to f(v)$ and $f_{\rm a}(\vec{v}_{\rm a}) \to f_{\rm a}(v_{\rm a})$, then the above expressions for the diffusion coefficients significantly

simplify to

$$\begin{aligned} D[\Delta v_\parallel] &= -16\pi^2 G^2 m_{\rm a}(m+m_{\rm a})\,\ln\Lambda\,\mathcal{E}_2(v) \\ D[(\Delta v_\parallel)^2] &= \frac{32}{3}\pi^2 G^2 m_{\rm a}^2\,\ln\Lambda\,v\,\left[\mathcal{E}_4(v)+\mathcal{F}_1(v)\right] \\ D[(\Delta v_\perp)^2] &= \frac{32}{3}\pi^2 G^2 m_{\rm a}^2\,\ln\Lambda\,v\,\left[3\mathcal{E}_2(v)-\mathcal{E}_4(v)+2\mathcal{F}_1(v)\right] \end{aligned}$$

where

$$\begin{aligned} \mathcal{E}_n(v) &= \int_0^v \left(\frac{v_{\rm a}}{v}\right)^n f_{\rm a}(v_{\rm a})\mathrm{d}v_{\rm a} \\ \mathcal{F}_n(v) &= \int_v^\infty \left(\frac{v_{\rm a}}{v}\right)^n f_{\rm a}(v_{\rm a})\mathrm{d}v_{\rm a} \end{aligned}$$

As a final aside; once the diffusion coefficients in velocity space are known it is fairly straightforward to derive related diffusion coefficients. As an example,

$$\begin{aligned} D[\Delta E] &= \frac{1}{2}\langle(\vec{v}+\Delta\vec{v})^2-\vec{v}^2\rangle = \langle\Delta v\cdot\vec{v}\rangle+\langle\Delta\vec{v}\cdot\Delta\vec{v}\rangle \\ &= v\,D[(\Delta v)_\parallel]+\frac{1}{2}D[((\Delta\vec{v})_\parallel)^2]+\frac{1}{2}D[((\Delta\vec{v})_\perp)^2] \end{aligned}$$

and, along similar lines, one finds that

$$D[(\Delta E)^2] = v^2 D[(\Delta\vec{v})_\parallel)^2]$$

The two-body relaxation time:

Now that we have derived expressions for the diffusion coefficients we can derive a more accurate, more local expression for the **relaxation time**. The latter is loosely defined as the time scale on which the cumulative effect of two-body encounters becomes significant for a typical particle. A meaningful way to quantify this is as follows:

$$\tau_{\rm relax} = \frac{v_{\rm rms}^2}{D[(\Delta v_\parallel)^2]}$$

We want to derive an expression for the characteristic relaxation time for a population of identical particles (i.e., $m = m_{\rm a}$). For simplicity, we assume that the velocity distribution of the field particles is isotropic and Maxwellian, with a one-dimensional

velocity dispersion σ. Using that $v_{\rm rms}^2 = \sigma^2$ and assuming that a typical particle is moving with a speed equal to the rms velocity of the field particles, i.e., $v = \sqrt{3}\sigma$, then it is an easy exercise to show that one finds that

$$\boxed{\tau_{\rm relax} = 0.34 \frac{\sigma^3}{G^2\, m\, \rho\, \ln\Lambda}}$$

Note that σ and ρ are *local* properties here. Substituting typical values for the Solar neighborhood ($\sigma = 30\,{\rm km\,s^{-1}}$, $\rho = 0.04\,{\rm M_\odot\,pc^{-3}}$, $m = 1\,{\rm M_\odot}$) yields $\tau_{\rm relax} = 6 \times 10^{14}\,{\rm yr}(\ln\Lambda/18.5)^{-1}$ indicating that the impact of stellar encounters in the Solar neighborhood is entirely unimportant; i.e., to very good approximation we may consider the Solar neighborhood to be collisionless.

Summary Flowcharts:

Figures 8 and 9 on the next two pages summarize the material covered in Part II of these lecture notes. In particular, Fig. 8 shows a flowchart for the dynamical equations describing many-body systems, starting from **Liouville's theorem**, it shows how, via the **BBGKY hierarchy**, one arrives at the **collisionless Boltzmann equation** (CBE) for collisionless fluids, at the (collisional) **Boltzmann equation** for a collisional fluid in which the inter-particle forces are short-range (this relies on the assumption of **molecular chaos** which closes the first-order equation of the BBGKY hierarchy), and finally at the **Fokker-Planck equation** for a collisional system in which the inter-particle forces are long-range.

Fig. 9 lists three fundamental equations used to describe the dynamics of many-body systems[5]. It also shows how the Boltzmann equations (both collisional and collisionless) can be integrated over velocity space to give rise to the master-moment equations (using collisional invariants), from which one derives the continuity, momentum and energy equations. For collisionless fluids the resulting momentum equations are known as the Jeans equations, while for a collisional fluid they are the Navier-Stokes equations, which reduce to the Euler equations for in inviscid fluid.

[5]One equation is missing from this list, which is the Lenard-Balescu equation to be discussed later in Part V

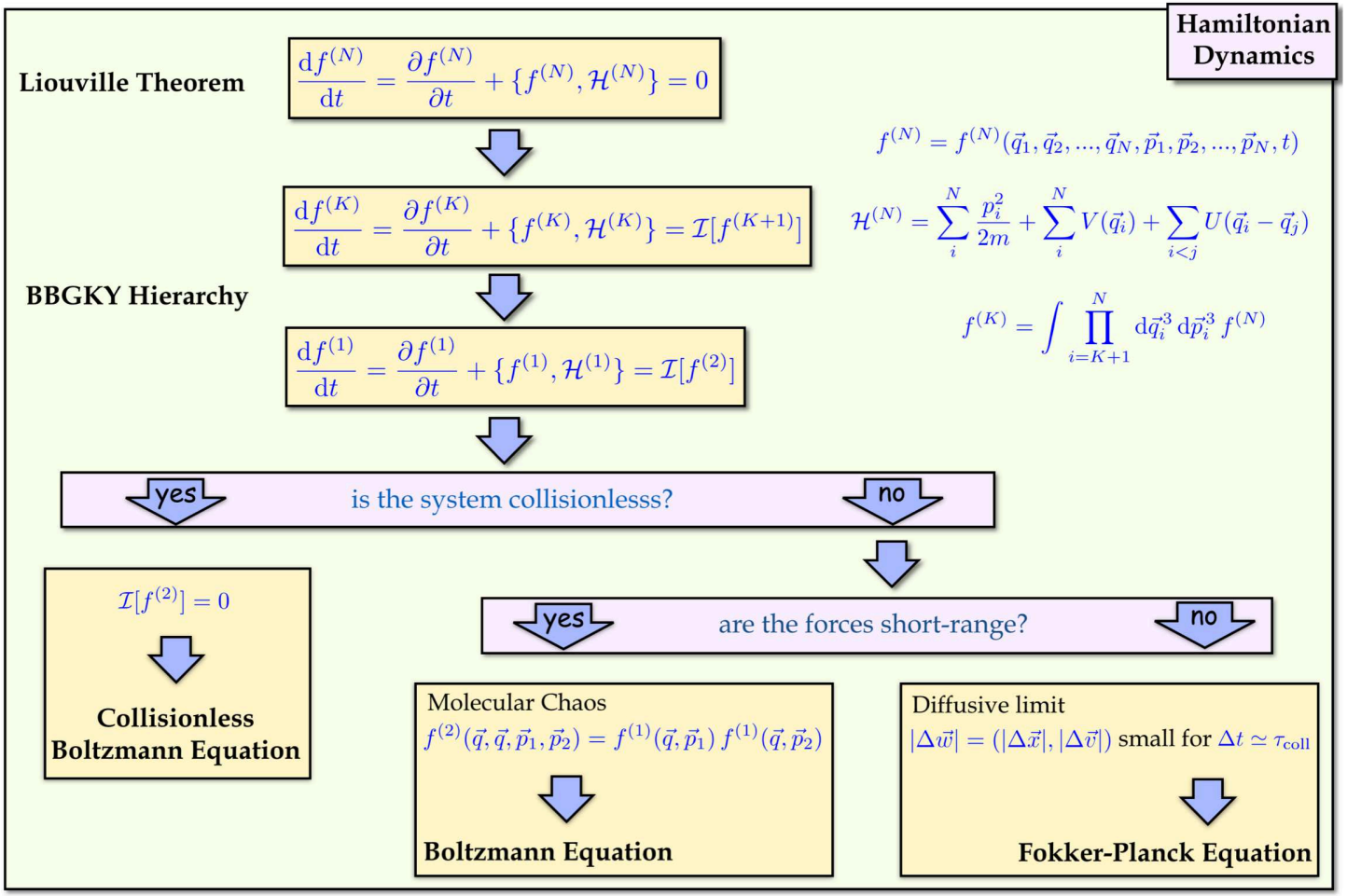

Hamiltonian Dynamics
Liouville Theorem
$\frac{\mathrm{d}f^{(N)}}{\mathrm{d}t} = \frac{\partial f^{(N)}}{\partial t} + \{f^{(N)}, \mathcal{H}^{(N)}\} = 0$
$f^{(N)} = f^{(N)}(\vec{q}_1, \vec{q}_2, ..., \vec{q}_N, \vec{p}_1, \vec{p}_2, ..., \vec{p}_N, t)$
$\mathcal{H}^{(N)} = \sum_i^N \frac{p_i^2}{2m} + \sum_i^N V(\vec{q}_i) + \sum_{i<j} U(\vec{q}_i - \vec{q}_j)$
BBGKY Hierarchy
$\frac{\mathrm{d}f^{(K)}}{\mathrm{d}t} = \frac{\partial f^{(K)}}{\partial t} + \{f^{(K)}, \mathcal{H}^{(K)}\} = \mathcal{I}[f^{(K+1)}]$
$f^{(K)} = \int \prod_{i=K+1}^{N} \mathrm{d}\vec{q}_i^{\,3}\, \mathrm{d}\vec{p}_i^{\,3}\, f^{(N)}$
$\frac{\mathrm{d}f^{(1)}}{\mathrm{d}t} = \frac{\partial f^{(1)}}{\partial t} + \{f^{(1)}, \mathcal{H}^{(1)}\} = \mathcal{I}[f^{(2)}]$
yes
is the system collisionlesss?
no
$\mathcal{I}[f^{(2)}] = 0$
Collisionless Boltzmann Equation
yes
are the forces short-range?
no
Molecular Chaos
$f^{(2)}(\vec{q}, \vec{q}, \vec{p}_1, \vec{p}_2) = f^{(1)}(\vec{q}, \vec{p}_1)\, f^{(1)}(\vec{q}, \vec{p}_2)$
Boltzmann Equation
Diffusive limit
$|\Delta\vec{w}| = (|\Delta\vec{x}|, |\Delta\vec{v}|)$ small for $\Delta t \simeq \tau_{\mathrm{coll}}$
Fokker-Planck Equation

Figure 8:

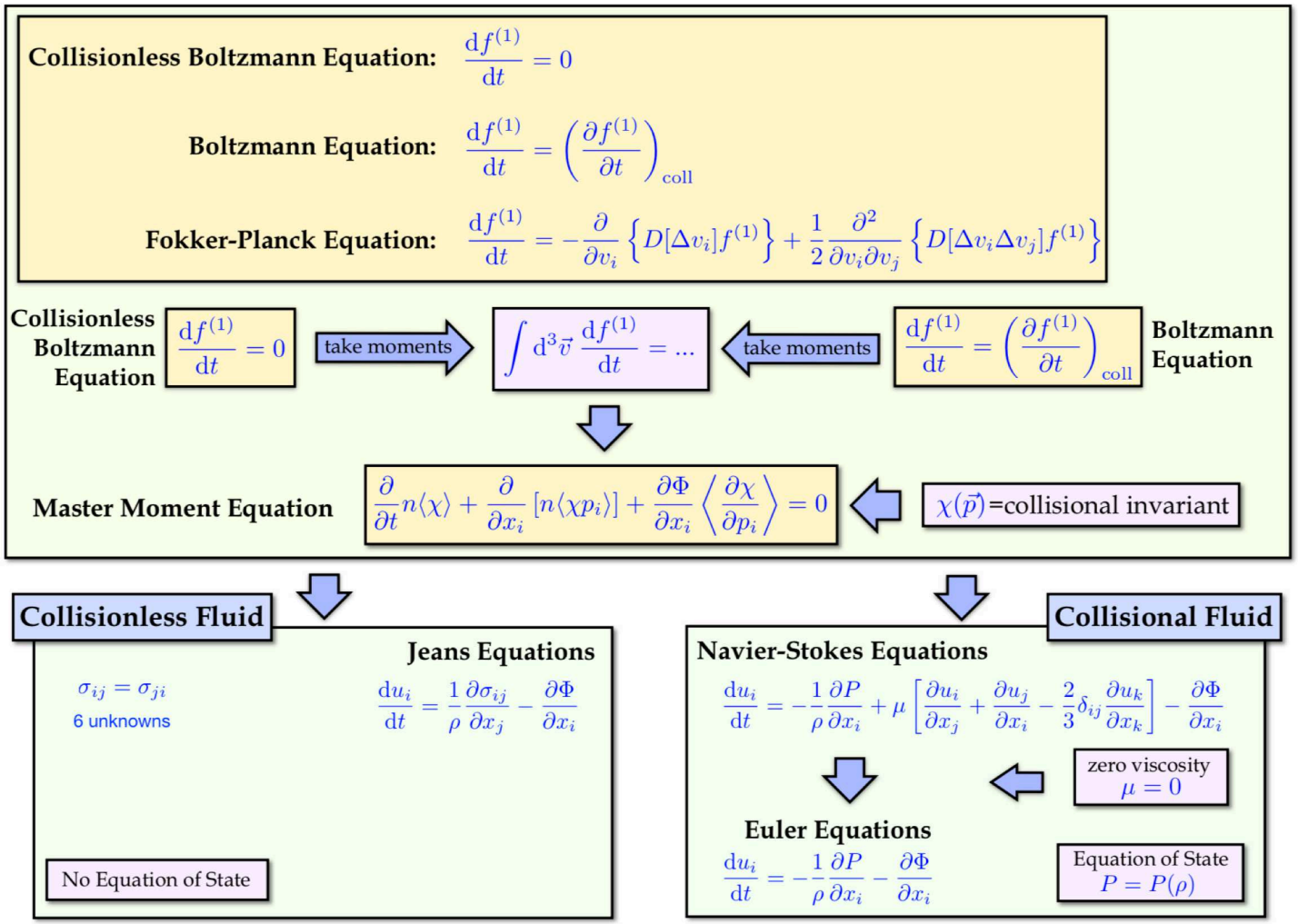

Collisionless Boltzmann Equation: $\frac{\mathrm{d}f^{(1)}}{\mathrm{d}t} = 0$
Boltzmann Equation: $\frac{\mathrm{d}f^{(1)}}{\mathrm{d}t} = \left(\frac{\partial f^{(1)}}{\partial t}\right)_{\mathrm{coll}}$
Fokker-Planck Equation: $\frac{\mathrm{d}f^{(1)}}{\mathrm{d}t} = -\frac{\partial}{\partial v_i}\left\{D[\Delta v_i]f^{(1)}\right\} + \frac{1}{2}\frac{\partial^2}{\partial v_i \partial v_j}\left\{D[\Delta v_i \Delta v_j]f^{(1)}\right\}$
Collisionless Boltzmann Equation
$\frac{\mathrm{d}f^{(1)}}{\mathrm{d}t} = 0$
take moments
$\int \mathrm{d}^3\vec{v}\,\frac{\mathrm{d}f^{(1)}}{\mathrm{d}t} = \ldots$
take moments
$\frac{\mathrm{d}f^{(1)}}{\mathrm{d}t} = \left(\frac{\partial f^{(1)}}{\partial t}\right)_{\mathrm{coll}}$
Boltzmann Equation
Master Moment Equation
$\frac{\partial}{\partial t}n\langle\chi\rangle + \frac{\partial}{\partial x_i}\left[n\langle\chi p_i\rangle\right] + \frac{\partial \Phi}{\partial x_i}\left\langle\frac{\partial \chi}{\partial p_i}\right\rangle = 0$
$\chi(\vec{p})$=collisional invariant
Collisionless Fluid
Jeans Equations
$\sigma_{ij} = \sigma_{ji}$
6 unknowns
$\frac{\mathrm{d}u_i}{\mathrm{d}t} = \frac{1}{\rho}\frac{\partial \sigma_{ij}}{\partial x_j} - \frac{\partial \Phi}{\partial x_i}$
No Equation of State
Collisional Fluid
Navier-Stokes Equations
$\frac{\mathrm{d}u_i}{\mathrm{d}t} = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} + \mu\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\frac{\partial u_k}{\partial x_k}\right] - \frac{\partial \Phi}{\partial x_i}$
zero viscosity
$\mu = 0$
Euler Equations
$\frac{\mathrm{d}u_i}{\mathrm{d}t} = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} - \frac{\partial \Phi}{\partial x_i}$
Equation of State
$P = P(\rho)$

Figure 9:

Part III: Hydro-Dynamics

Hydrodynamics is the study of gases and liquids which are many-body systems in which the inter-particle forces are short-range. In this Part III of the lecture notes we develop the equations used to liquids and gases, and consider a number of astrophysical applications.

Fluid dynamics is a rich topic, and one could easily devote an entire course to it. The following chapters therefore only scratch the surface of this rich topic. Readers who want to get more indepth information are referred to the following excellent textbooks

- *The Physics of Fluids and Plasmas* by A. Choudhuri
- *Modern Fluid Dynamics for Physics and Astrophysics* by O.Regev et al.
- *The Physics of Astrophysics II. Gas Dynamics* by F. Shu
- *Principles of Astrophysical Fluid Dynamics* by C. Clarke & B. Carswell
- *Modern Classical Physics* by K.Thorne & R. Blandford

CHAPTER 10

Introduction to Hydrodynamics

Hydrodynamics is the study of liquids and neutral gases. In this chapter we introduce some relevant nomenclature.

Compressibility: The main difference between a **liquid** and a **gas** is with regard to their compressibility. A gas is **compressible** and will completely fill the volume available to it. A liquid, on the other hand, is (to good approximation) **incompressible**, which means that a liquid of given mass occupies a given volume.

NOTE: Although a gas is said to be compressible, many gaseous *flows* (and virtually all astrophysical flows) are **incompressible**. When the gas is in a container, you can easily compress it with a piston, but if you move your hand (sub-sonically) through the air, the gas adjust itself to the perturbation in an incompressible fashion (it moves out of the way at the **speed of sound**). The small compression at your hand propagates forward at the speed of sound (**sound wave**) and disperses the gas particles out of the way. In astrophysics we rarely encounter containers, and **subsonic** gas flow is often treated (to good approximation) as being incompressible.

If a *flow* is incompressible then $\nabla \cdot \vec{u} = 0$.
If a *substance* is incompressible then $\mathrm{d}\rho/\mathrm{d}t = 0$.

Ideal (Perfect) Fluids and Ideal Gases:

As we discuss in more detail in the next Chapter, the resistance of fluids to shear distortions is called **viscosity**, which is a microscopic property of the fluid that depends on the nature of its constituent particles, and on thermodynamic properties such as temperature. Fluids are also **conductive**, in that the microscopic collisions between the constituent particles cause heat conduction through the fluid. In many fluids encountered in astrophysics, the viscosity and conduction are very small. *An ideal fluid, also called a perfect fluid, is a fluid with zero viscosity and zero conduction.*

NOTE: An ideal (or perfect) fluid should NOT be confused with an ideal or perfect gas, which is defined as a gas in which the pressure is solely due to the kinetic motions of the constituent particles. As shown in Appendix I, and as you have probably seen

before, this implies that the pressure can be written as $P = n\,k_{\rm B}\,T$, with n the particle number density, $k_{\rm B}$ the Boltzmann constant, and T the temperature.

Fluid Elements & the Macroscopic Continuum Approach:
In the macroscopic approach, the fluid is treated as a **continuum**. It is often useful to think of this continuum as 'made up' of **fluid elements** (FE). These are small fluid volumes that nevertheless contain many particles, that are significantly larger than the mean-free path of the particles, and for which one can define local hydrodynamical variables such as density, pressure and temperature. The requirements are:

1. the FE needs to be much smaller than the characteristic scale in the problem, which is the scale over which the hydrodynamical quantities Q change by an order of magnitude, i.e.
$$l_{\rm FE} \ll l_{\rm scale} \sim \frac{Q}{\nabla Q}$$

2. the FE needs to be sufficiently large that fluctuations due to the finite number of particles ('discreteness noise') can be neglected, i.e.,
$$n\,l_{\rm FE}^3 \gg 1$$
where n is the number density of particles.

3. the FE needs to be sufficiently large that it 'knows' about the local conditions through collisions among the constituent particles, i.e.,
$$l_{\rm FE} \gg \lambda_{\rm mfp}$$

The ratio of the mean-free path, $\lambda_{\rm mfp}$, to the characteristic scale, $l_{\rm scale}$ is known as the **Knudsen number:** ${\rm Kn} = \lambda_{\rm mfp}/l_{\rm scale}$. Fluids typically have ${\rm Kn} \ll 1$; if not, then one is not justified in using the continuum approach to fluid dynamics, and one is forced to resort to a more statistical approach based on kinetic theory (i.e., using the **Boltzmann equation**).

Fluid Dynamics: Eulerian *vs.* **Lagrangian Formalism:**
One distinguishes two different formalisms for treating fluid dynamics:

- **Eulerian Formalism:** in this formalism one solves the fluid equations 'at fixed positions': the evolution of a quantity Q is described by the local (or partial, or Eulerian) derivative $\partial Q/\partial t$. An Eulerian hydrodynamics code is a 'grid-based code', which solves the hydro equations on a fixed grid, or using an adaptive grid, which refines resolution where needed. The latter is called **Adaptive Mesh Refinement** (AMR).

- **Lagrangian Formalism:** in this formalism one solves the fluid equations 'comoving with the fluid'. The evolution of a quantity Q is described by the 'substantial' (or 'Lagrangian') derivative $\mathrm{d}Q/\mathrm{d}t$ (sometimes written as $\mathrm{D}Q/\mathrm{D}t$). A Lagrangian hydrodynamics code is a 'particle-based code', which solves the hydro equations per simulation particle. Since it needs to smooth over neighboring particles in order to compute quantities such as the fluid density, it is called **Smoothed Particle Hydrodynamics** (SPH).

To derive an expression for the **substantial derivative** $\mathrm{d}Q/\mathrm{d}t$, realize that $Q = Q(t, x, y, z)$. When the fluid element moves, the scalar quantity Q experiences a change

$$\mathrm{d}Q = \frac{\partial Q}{\partial t}\,\mathrm{d}t + \frac{\partial Q}{\partial x}\,\mathrm{d}x + \frac{\partial Q}{\partial y}\,\mathrm{d}y + \frac{\partial Q}{\partial z}\,\mathrm{d}z$$

Dividing by $\mathrm{d}t$ yields

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \frac{\partial Q}{\partial t} + \frac{\partial Q}{\partial x}\,u_x + \frac{\partial Q}{\partial y}\,u_y + \frac{\partial Q}{\partial z}\,u_z$$

where we have used that $\mathrm{d}x/\mathrm{d}t = u_x$, which is the x-component of the fluid velocity $\vec{u}$, etc. Hence we have that

$$\boxed{\frac{\mathrm{d}Q}{\mathrm{d}t} = \frac{\partial Q}{\partial t} + \vec{u}\cdot\nabla Q}$$

Using a similar derivation, but now for a vector quantity $\vec{A}(\vec{x}, t)$, it is straightforward to show that

$$\boxed{\frac{\mathrm{d}\vec{A}}{\mathrm{d}t} = \frac{\partial \vec{A}}{\partial t} + (\vec{u}\cdot\nabla)\,\vec{A}}$$

which, in index-notation, is written as

$$\frac{\mathrm{d}A_i}{\mathrm{d}t} = \frac{\partial A_i}{\partial t} + u_j \frac{\partial A_i}{\partial x_j}$$

Another way to derive the above relation between the Eulerian and Lagrangian derivatives, is to think of $\mathrm{d}Q/\mathrm{d}t$ as

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \lim_{\delta t \to 0} \left[\frac{Q(\vec{x} + \delta\vec{x}, t + \delta t) - Q(\vec{x}, t)}{\delta t} \right]$$

Using that

$$\vec{u} = \lim_{\delta t \to 0} \left[\frac{\vec{x}(t + \delta t) - \vec{x}(t)}{\delta t} \right] = \frac{\delta \vec{x}}{\delta t}$$

and

$$\nabla Q = \lim_{\delta \vec{x} \to 0} \left[\frac{Q(\vec{x} + \delta\vec{x}, t) - Q(\vec{x}, t)}{\delta \vec{x}} \right]$$

it is straightforward to show that this results in the same expression for the substantial derivative as above.

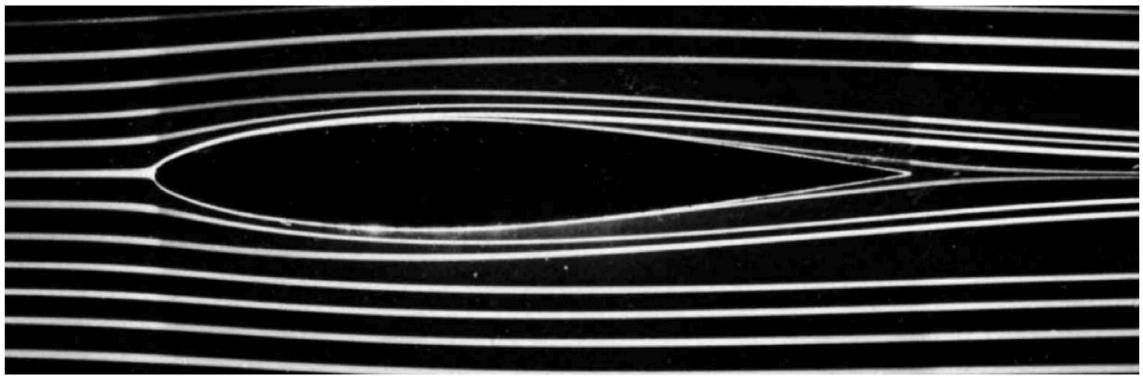


Figure 10: *Streaklines showing laminar flow across an airfoil; made by injecting dye at regular intervals in the flow*

Kinematic Concepts: Streamlines, Streaklines and Particle Paths:

In fluid dynamics it is often useful to distinguish the following kinematic constructs:

- **Streamlines:** curves that are instantaneously tangent to the velocity vector of the flow. Streamlines show the direction a massless fluid element will travel in at any point in time.
- **Streaklines:** the locus of points of all the fluid particles that have passed continuously through a particular spatial point in the past. Dye steadily injected into the fluid at a fixed point extends along a streakline.
- **Particle paths:** (aka pathlines) are the trajectories that individual fluid elements follow. The direction the path takes is determined by the streamlines of the fluid at each moment in time.

Only if the flow is **steady**, which means that all partial time derivatives (i.e., $\partial\vec{u}/\partial t = \partial\rho/\partial t = \partial P/\partial t$) vanish, will streamlines be identical to streaklines be identical to particle paths. For a non-steady flow, they will differ from each other.

Fluid Dynamics: closure:

The starting point for our development of macroscopic hydrodynamics are the moment equations of the Boltzmann equation. In Chapter 7, starting from the **master-moment equation**, we obtained the continuity equation,

$$\boxed{\frac{\partial\rho}{\partial t} + \frac{\partial\rho u_i}{\partial x_i} = 0\,,}$$

the momentum equations,

$$\boxed{\frac{\partial u_j}{\partial t} + u_i\frac{\partial u_j}{\partial x_i} = \frac{1}{\rho}\frac{\partial\sigma_{ij}}{\partial x_i} - \frac{\partial\Phi}{\partial x_j}\,,}$$

and the energy equation,

$$\boxed{\frac{\partial}{\partial t}\left[\rho\left(\frac{u^2}{2}+\varepsilon\right)\right] = -\frac{\partial}{\partial x_k}\left[\rho\left(\frac{u^2}{2}+\varepsilon\right)u_k - \sigma_{jk}u_j + \rho\langle w_k\frac{1}{2}w^2\rangle\right] - \rho u_k\frac{\partial\Phi}{\partial x_k}\,.}$$

Here $\vec{u} = \langle\vec{v}\rangle$ is the 'bulk' (or 'streaming' velocity), $\sigma_{ij} \equiv -\rho\langle w_i w_j\rangle$ is the **stress tensor**, and $\varepsilon = \frac{1}{2}\langle w^2\rangle$ is the **specific internal energy**. Here and throughout Part III, the angle brackets $\langle\cdot\rangle$ can be interpreted as an average over the fluid element.

Together with the **Poisson equation**,

$$\boxed{4\pi G\rho = \nabla^2\Phi}$$

this is a set of 6 equations (the momentum equations is a set of 3 equations, one for each direction) that contain a total of 15 unknowns: ρ, Φ, u_i, ε, σ_{ij} and $\langle w_i \frac{1}{2}w^2\rangle$. Note that i and j run from 1 to 3, and that the stress tensor is manifestly symmetric (thus contributing a total of 6 unknowns). Clearly, this is *not* a closed set.

In the next Chapter, we will show that the stress tensor can be written in terms of two fluid properties, the (thermodynamic) **pressure**, P, and the (shear) **viscosity**, μ, while the vector $\langle w_i \frac{1}{2}w^2\rangle$ is specified by a single property, the **conductivity**, $\mathcal{K}$. These three fluid properties obey **constitutive equations**, which ultimately allow us to achieve closure; i.e., to have as many equations as unknowns.

CHAPTER 11

Tranport Mechanisms & Constitutive Equations

In the previous chapter we summarized the **moment equations** obtained from the **Boltzmann equation**. If we momentarily ignore the gravitational potential, we have

1 continuum equation	relating ρ and $\vec{u}$
3 momentum equations	relating ρ, $\vec{u}$ and σ_{ij}
1 energy equation	relating ρ, $\vec{u}$, ε, σ_{ij} and F_i

Here $\sigma_{ij} = -\rho\langle w_i w_j\rangle$ is the **stress tensor** and $F_i \equiv \rho\langle w_i \frac{1}{2}w^2\rangle$ is a vector, the physical interpretation of which will become clear shortly. Our goal in this chapter is to find expressions for σ_{ij} and F_i that allow the moment equations to become a closed set of macroscopic equations of hydrodynamics. Note that both σ_{ij} and F_i depend on the microscopic random velocities $w_i \equiv v_i - \langle v_i\rangle$, and are thus dependent on the random collisions. Recall that the angle-brackets indicate an average over the fluid elements, which are large compared to the mean free path; i.e., there are many collisions within a fluid element, and our goal to develop macroscopic equations of hydrodynamics is to determine the 'macroscopic outcome' of the many, many collisions. Collisions will drive the system towards thermal equilibrium, in which the (local) velocity distribution is Maxwell-Boltzmann. In addition, if there are gradients in density, velocity or internal energy, collisions will work towards erasing these gradients.

• When there are gradients in the streaming velocity, $\vec{u}$, in the direction perpendicular to $\vec{u}$ (**shear**) then the collisions among neighboring fluid elements give rise to a **net transport of momentum**. The collisions drive the system towards equilibrium, i.e., towards no shear. Hence, the collisions act as a resistance to shear, which is called **viscosity**. See Fig. 11 for an illustration.

• When there are gradients in temperature (i.e., in specific internal energy), then the collisions give rise to a **net transport of energy**. Again, the collisions drive the system towards equilibrium, in which the gradients vanish, and the rate at which the fluid can erase a non-zero ∇T is called the **(thermal) conductivity**, $\mathcal{K}$.

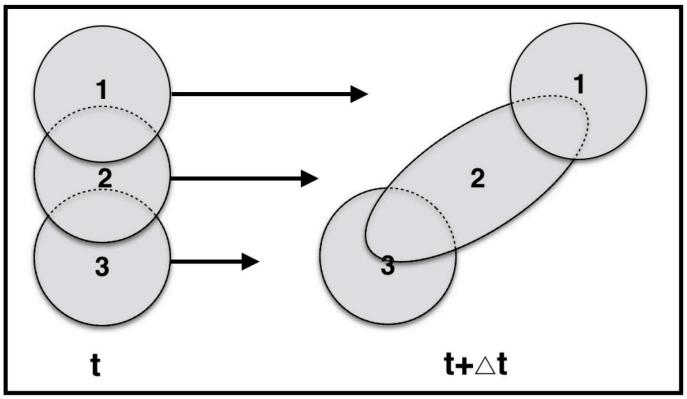


Figure 11: *Illustration of origin of viscosity and shear stress. Three neighboring fluids elements (1, 2 and 3) have different streaming velocities, $\vec{u}$. Due to the microscopic motions and collisions (characterized by a non-zero mean free path), there is a net* **transfer of momentum** *from the faster moving fluid elements to the slower moving fluid elements. This net transfer of momentum will tend to erase the shear in $\vec{u}(\vec{x})$, and therefore manifests itself as a shear-resistance, known as* **viscosity**. *Due to the transfer of momentum, the fluid elements deform; in our figure, 1 transfers linear momentum to the top of 2, while 3 extracts linear momentum from the bottom of 2. Consequently, fluid element 2 is sheared as depicted in the figure at time $t+\Delta t$. From the perspective of fluid element 2, some internal force (from within its boundaries) has exerted a* **shear-stress** *on its bounding surface.*

The transport of momentum and internal energy (heat) due to microscopic collisions among the constituent atoms or molecules are examples of **transport mechanisms**. Their efficiency depends on the interparticle forces (i.e., the effective cross section of the collisions). Typically, 'transport' is more efficient for smaller cross sections (i.e., a shorter-range force), as a smaller (effective) cross section, σ_{eff}, implies a larger mean free path, λ_{mfp}.

Transport mechanisms can be treated rigorously using the **Chapman-Enskog expansion**, which is valid as long as the collisions are well separated in space and time (i.e., collision duration is much smaller than the time in between collisions), *and* the **Knudsen number** is small ($\text{Kn} = \lambda_{\text{mfp}}/l_{\text{scale}} \ll 1$ with l_{scale} the characteristic scale over which macroscopic fluid properties change). In the Chapman-Enskog expansion one expands the distribution function (DF) as $f = f^{(0)} + \epsilon f^{(1)} + \epsilon^2 f^{(2)} + ...$ Here $f^{(0)}$ is the equilibrium Maxwell-Boltzmann distribution, while $\epsilon = \text{Kn} \ll 1$. Substituting this expansion in the Boltzmann equation, and equating orders of ϵ yields a hierarchy of equations (one for each order of ϵ). The zeroth-order expression yields the Euler equations for an ideal (inviscid, zero conduction) fluid. The first-order expression

yields the Navier-stokes and energy equations, but with σ_{ij} and F_i expressed in forms that allow for closure of the macroscopic equations of hydrodynamics. The detailed derivation of these expressions using Chapman-Enskog expansion is extremely complicated, and outside of the scope of this course. Interested readers should consult the classical monographs *"Statistical Mechanics"* by K. Huang (highly recommended), or *"The Mathematical Theory of Non-uniform Gases"* by S. Chapman and T. Cowling.

In what follows we shall use a more heuristic approach to obtain useful expressions for σ_{ij} and F_i. Let's start with

$$F_i = \rho \langle w_i \frac{1}{2} w^2 \rangle$$

Since $\langle w_i \rangle$ is zero, the only way for F_i to be non-zero is for w^2 to have a gradient over the fluid element. And since $\varepsilon = \frac{1}{2}\langle w^2 \rangle$, this means that we need to have a gradient in the specific, internal energy, which is basically a measure for heat (i.e., as we will see, $\varepsilon \propto T$). With that interpretation we see that F_i is a measure for how internal energy density is transported in direction i due to microscopic motions (collisions). That is what we call the **conductive heat flux**, which experimentally is known to scale proportional to the temperature gradient; i.e., we suspect that

$$\boxed{F_i = \rho \langle w_i \frac{1}{2} w^2 \rangle = -\mathcal{K} \frac{\partial T}{\partial x_i} \qquad \Rightarrow \qquad \vec{F}_{\rm cond} = -\mathcal{K} \nabla T}$$

with $\mathcal{K}$ the **coefficient of conductivity** (or 'conductivity' for short). This is indeed what a more elaborate treatment based on the Chapman-Enskog expansion yields. In addition, the latter shows that

$$\mathcal{K} = k_1 \frac{c_{\rm V}}{\sigma_{\rm eff}} \sqrt{m k_{\rm B} T}$$

with k_1 a numerical constant of order unity, $c_{\rm V}$ the specific heat capacity (i.e., per unit mass), and $\sigma_{\rm eff}$ the effective cross section for the inter-particle forces (typically this is comparable to the diameter of the atoms or molecules that make up the gas).

The stress tensor: In order to obtain an expression for the stress tensor, we first realize that $\rho \langle w_i w_j \rangle$ has the form of a ram pressure due to the microscopic motions. Imagine injecting a surface with normal vector $\hat{n}$ inside a fluid element at $\vec{x}$. Let the surface be at rest with respect to the fluid element (i.e., it has the same $\vec{u}$ as the

fluid element itself). The stress tensor is related to the **stress** $\vec{\Sigma}(\vec{x}, \hat{n})$ acting on the surface located at $\vec{x}$ according to

$$\Sigma_i(\hat{n}) = \sigma_{ij}\, n_j$$

Here $\Sigma_i(\hat{n})$ is the i-component of the stress acting on a surface with normal $\hat{n}$, whose j-component is given by n_j. Hence, in general the stress will not necessarily be along the normal to the surface, and it is useful to decompose the stress in a **normal stress**, which is the component of the stress along the normal to the surface, and a **shear stress**, which is the component along the tangent to the surface.

To see that fluid elements in general are subjected to shear stress, imagine injecting a small, spherical blob (a fluid element) of dye in a flow (i.e., a river). If the only stress to which the blob is subject is normal stress, the only thing that can happen to the blob is an overall compression or expansion. However, from experience we know that the blob of dye will shear into an extended, 'spaghetti'-like feature; hence, the blob is clearly subjected to shear stress, and this shear stress is obvisouly related to another tensor called the **deformation tensor**

$$T_{ij} = \frac{\partial u_i}{\partial x_j}$$

which describes the (local) shear in the fluid flow.

Since $\partial u_i/\partial x_j = 0$ in a **static** fluid ($\vec{u}(\vec{x}) = 0$), we see that in a static fluid the stress tensor can only depend on the normal stress, which we call the **pressure**.

Pascal's law for hydrostatistics: In a **static** fluid, there is no preferred direction, and hence the (normal) stress has to be isotropic:

$$\text{static fluid} \quad \Longleftrightarrow \quad \sigma_{ij} = -P\,\delta_{ij}$$

The minus sign is a consequence of the sign convention of the stress.

Sign Convention: The stress $\vec{\Sigma}(\vec{x}, \hat{n})$ acting at location $\vec{x}$ on a surface with normal $\hat{n}$, is exerted *by* the fluid on the side of the surface to which the normal points, *on* the fluid from which the normal points. In other words, *a positive stress results in compression*. Hence, in the case of pure, normal pressure, we have that $\Sigma = -P$.

Viscous Stress Tensor: The expression for the stress tensor in the case of a static fluid motivates us to write in general

$$\boxed{\sigma_{ij} = -P\,\delta_{ij} + \tau_{ij}}$$

where we have introduced a new tensor, τ_{ij}, which is known as the **viscous stress tensor**, or the **deviatoric stress tensor**.

Since the deviatoric stress tensor is only non-zero in the presence of shear in the fluid flow, this suggests that in full generality

$$\tau_{ij} = T_{ijkl}\,\frac{\partial u_k}{\partial x_l}$$

where T_{ijkl} is a proportionality tensor of **rank** four. In what follows we derive an expression for T_{ijkl}. We start by noting that since σ_{ij} is symmetric, we also have that τ_{ij} will be symmetric. Hence, we expect that the above dependence can only involve the **symmetric** component of the deformation tensor, $\mathcal{T}_{kl} = \partial u_k/\partial x_l$. Hence, it is useful to split the deformation tensor in its **symmetric** and **anti-symmetric** components:

$$\frac{\partial u_i}{\partial x_j} = e_{ij} + \xi_{ij}$$

where

$$\begin{aligned} e_{ij} &= \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right] \\ \xi_{ij} &= \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right] \end{aligned}$$

The symmetric part of the deformation tensor, e_{ij}, is called the **rate of strain tensor**, while the anti-symmetric part, ξ_{ij}, expresses the **vorticity** $\vec{w} \equiv \nabla \times \vec{u}$ in the velocity field, i.e., $\xi_{ij} = -\frac{1}{2}\,\varepsilon_{ijk}\,w_k$ (here ε_{ijk} is the Levi-Civita symbol; see Appendix G). Note that one can always find a coordinate system for which e_{ij} is diagonal. The axes of that coordinate frame indicate the **eigendirections** of the strain (compression or stretching) on the fluid element.

In terms of the relation between the viscous stress tensor, τ_{ij}, and the deformation tensor, $\mathcal{T}_{kl}$, there are a number of properties that are important.

- **Locality:** the $\tau_{ij} - \mathcal{T}_{kl}$-relation is said to be **local** if the stress tensor is only a function of the deformation tensor and the local values of thermodynamic state functions like temperature.

- **Homogeneity:** the $\tau_{ij} - \mathcal{T}_{kl}$-relation is said to be **homogeneous** if it is everywhere the same. The viscous stress tensor may depend on location $\vec{x}$ only insofar as $\mathcal{T}_{ij}$ or the thermodynamic state functions depend on $\vec{x}$. This distinguishes a fluid from a solid, in which the stress tensor depends on the stress itself.

- **Isotropy:** the $\tau_{ij} - \mathcal{T}_{kl}$-relation is said to be **isotropic** if it has no preferred direction.

- **Linearity:** the $\tau_{ij} - \mathcal{T}_{kl}$-relation is said to be **linear** if the relation between the stress and rate-of-strain is linear. This is equivalent to saying that τ_{ij} does not depend on $\nabla^2\vec{u}$ or higher-order derivatives.

A fluid that is local, homogeneous and isotropic is called a **Stokesian** fluid. A Stokesian fluid that is linear is called a **Newtonian** fluid. Experiments have shown that most fluids are Newtonian to good approximation. Hence, in what follows we will assume that our astrophysical fluids of interest are Newtonian, unless specifically stated otherwise. For a Newtonian fluid, it can be shown (using linear algebra) that the most general form of our proportionality tensor is given by

$$T_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)$$

where λ and μ are scalar quantities, and δ_{ij} is the Kronecker delta function.

Substitution in the expression relating the deviatoric stress tensor and the deformation tensor yields that, for a **Newtonian** fluid,

$$\tau_{ij} = 2\mu e_{ij} + \lambda e_{kk}\delta_{ij}$$

Here $e_{kk} = \mathrm{Tr}(e) = \partial u_k/\partial x_k = \nabla\cdot\vec{u}$ (Einstein summation convention).

Thus far we have derived that the stress tensor, σ_{ij}, which in principle has 6 unknowns, can be reduced to a function of three unknowns only (P, μ, λ) as long as the fluid is **Newtonian**. We now focus on these three scalars in more detail, starting with the pressure P. To be exact, P is the **thermodynamic equilibrium pressure**, and is normally computed thermodynamically from some equation of state,

$P = P(\rho, T)$. Another way to define the pressure is purely 'mechanical' as arising from the microscopic, translational motion of the particles, i.e., as

$$P_{\rm m} = -\frac{1}{3}\,{\rm Tr}(\sigma_{ij}) = -\frac{1}{3}\,(\sigma_{11} + \sigma_{22} + \sigma_{33})$$

where we have used the subscript 'm' to indicate that this is the **mechanical** pressure. Using that

$$\sigma_{ij} = -P\,\delta_{ij} + 2\,\mu\,e_{ij} + \lambda\,e_{kk}\,\delta_{ij}$$

we thus obtain that

$$P_{\rm m} = P - \eta\,\nabla\cdot\vec{u}$$

where $\eta = \frac{2}{3}\mu + \lambda$. Typically (but not always, see below), we have that $P_{\rm m} = P$, which implies that $\lambda = -\frac{2}{3}\mu$; i.e., λ and μ are not independent. This allows us to write the stress tensor as

$$\boxed{\sigma_{ij} = -P\delta_{ij} + \mu\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\frac{\partial u_k}{\partial x_k}\right]}$$

Using the Chapman-Enskog expansion one obtains exactly the same expression. The parameter μ is called the coefficient of **shear viscosity**, and according to the Chapman-Enskog expansion we have that

$$\mu = k_2\,\frac{1}{\sigma_{\rm eff}}\,\sqrt{mk_{\rm B}T} = \frac{k_2}{k_1}\frac{\mathcal{K}}{c_{\rm V}}$$

with k_2 another numerical constant of order unity.

Constitutive equations: We have shown that (for a Newtonian fluid), the stress tensor, despite being a symmetric tensor of rank two, only has two independent quantities: P and μ, while the conductive flux is characterized by only a single quantity, $\mathcal{K}$. Most importantly, these are not 'unknowns' but rather depend on other macroscopic quantities via what are called **constitutive relations**. For the pressure this is the equation of state $P = P(\rho, T)$ (see Appendix I for a detailed treatment), while both the viscosity and conductivity only depend on temperature, i.e., $\mu \propto T^{1/2}$ and $\mathcal{K} \propto T^{1/2}$. Note that the $\mu \propto T^{1/2}$ scaling implies that viscosity *increases* with temperature. This only holds for gases! For liquids we know from experience that viscosity *decreases* with increasing temperature (think of honey).

Since in astrophysics we are mainly concerned with gas, $\mu \propto T^{1/2}$ will be a good approximation for most of what follows.

The second viscosity: There is one complication to the story outlined above. We assumed that the thermodynamic pressure, P, is equal to the mechanical pressure $P_{\rm m}$. This, however, is only true if the fluid, in equilibrium, has reached **equipartition** of energy among **all** its degrees of freedom, including (in the case of molecules) rotational and vibrations degrees of freedom. For a fluid of monoatoms (ideal gas), this is always the case. However, if the fluid consists of particles with internal degrees of freedom (e.g., molecules) and has just undergone a large volumetric change (i.e., during a shock) then there can be a lag between the time the translational motions reach equilibrium and the time when the system reaches full equipartition in energy among all degrees of freedom. In these (rare) circumstances, we have that

$$\eta = \frac{P_{\rm m} - P}{\nabla \cdot \vec{u}}$$

is non-zero. In that case the expression for the stress tensor becomes

$$\boxed{\sigma_{ij} = -P\delta_{ij} + \mu\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\frac{\partial u_k}{\partial x_k}\right] + \eta\,\delta_{ij}\frac{\partial u_k}{\partial x_k}}$$

Note that there is now an extra term, proportional to η which is called the coefficient of **bulk viscosity**, also known as the **second viscosity**. Typically, though, the bulk viscosity can safely be ignored. This only leaves the shear viscosity μ, which describes the ability of the fluid to resist shear stress via momentum transport resulting from collisions and the non-zero mean free path of the particles.

CHAPTER 12

Hydrodynamic Equations

Having derived expressions for the **stress tensor** $\sigma_{ij} = -\rho\langle w_i w_j\rangle$ and the **conductive flux** $F_{\mathrm{cond},i} = \rho\langle w_i \frac{1}{2}w^2\rangle$, we are now ready to write down the macroscopic equations of hydrodynamics. Our starting point are the **moment equations** of the **Boltzmann equation** that we derived in Chapter 7. For completeness, here they are once more (in Eulerian index form):

Continuity Eq.	$\dfrac{\partial \rho}{\partial t} + \dfrac{\partial \rho u_i}{\partial x_i} = 0$
Momentum Eqs.	$\dfrac{\partial u_i}{\partial t} + u_j \dfrac{\partial u_i}{\partial x_j} = \dfrac{1}{\rho}\dfrac{\partial \sigma_{ij}}{\partial x_j} - \dfrac{\partial \Phi}{\partial x_i}$
Energy Eq.	$\dfrac{\partial}{\partial t}\left[\rho\left(\dfrac{u^2}{2}+\varepsilon\right)\right] = -\dfrac{\partial}{\partial x_k}\left[\rho\left(\dfrac{u^2}{2}+\varepsilon\right) u_k - \sigma_{jk}u_j + \rho\langle w_k \dfrac{1}{2}w^2\rangle\right] - \rho u_k \dfrac{\partial \Phi}{\partial x_k}$

All we need to do at this point is substitute the expressions for σ_{ij} and $\rho\langle w_k \frac{1}{2}w^2\rangle$ that we derived in the previous chapter. We now discuss the resulting equations in some detail:

Continuity Equation: this equation expresses mass conservation. This is clear from the Eulerian form, which shows that changing the density at some fixed point in space requires a converging, or diverging, mass flux at that location. In Lagrangian vector form, the continuity equation is given by

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \frac{\partial \rho}{\partial t} + \vec{u}\cdot\nabla\rho = -\rho\,\nabla\cdot\vec{u}$$

which shows that the density of a fluid element (moving with the flow) can only change if the flow is **compressible** (i.e., the flow has non-zero divergence).

If a flow is incompressible, then $\nabla \cdot \vec{u} = 0$ everywhere, and we thus have that $\mathrm{d}\rho/\mathrm{d}t = 0$ (i.e., the density of each fluid element is fixed in time as it moves with the flow).

If a fluid is incompressible, than $\mathrm{d}\rho/\mathrm{d}t = 0$ and we see that the flow is divergence free ($\nabla \cdot \vec{u} = 0$), which is also called **solenoidal**.

> Students should become familiar with switching between the **Eulerian** and **Lagrangian** versions of the equations, and between the **vector notation** and the **index notation**. The latter is often easier to work with. When writing down the index versions, make sure that each term carries the same index, and make use of the Einstein summation convention.

Momentum Equations: substituting the full expression for the stress tensor in the momentum equations yields:

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = \\ -\frac{1}{\rho}\frac{\partial P}{\partial x_i} + \frac{1}{\rho}\frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\frac{\partial u_k}{\partial x_k}\right)\right] + \frac{1}{\rho}\frac{\partial}{\partial x_i}\left(\eta\frac{\partial u_k}{\partial x_k}\right) - \frac{\partial \Phi}{\partial x_i}$$

These are the **Navier-Stokes** equations (in Lagragian index form) in all their glory, containing both the **shear viscosity** term and the **bulk viscosity** term.

Note that μ and η are usually functions of density and temperature so that they have spatial variations. However, it is common to assume that these are suficiently small so that μ and η can be treated as constants, in which case they can be taken outside the differentials. In what follows we will make this assumption as well.

In Lagrangian vector form the Navier-Stokes equations become

$$\rho\frac{\mathrm{d}\vec{u}}{\mathrm{d}t} = -\nabla P + \mu\nabla^2\vec{u} + \left(\eta + \frac{1}{3}\mu\right)\nabla(\nabla\cdot\vec{u}) - \rho\nabla\Phi$$

If we ignore the bulk viscosity ($\eta = 0$) then this reduces to

$$\boxed{\frac{\mathrm{d}\vec{u}}{\mathrm{d}t} = -\frac{\nabla P}{\rho} + \nu\left[\nabla^2\vec{u} + \frac{1}{3}\nabla(\nabla\cdot\vec{u})\right] - \nabla\Phi}$$

where we have introduced the **kinetic viscosity** $\nu \equiv \mu/\rho$. This is the form of the Navier-Stokes equation most commonly encountered. It expresses that the fluid

element can be accelerated by pressure gradients, by gradients in the gravitational potential (i.e., by an external gravitational force acting on the fluid element), and by viscosity. Basically these momentum equations are nothing but Newton's $\vec{F} = m\vec{a}$ applied to a fluid element.

The $\nabla(\nabla \cdot \vec{u})$ component of the viscosity term is only significant in the case of flows with **variable compression** (i.e., viscous dissipation of accoustic waves or shocks). As discussed in Chapter 10, astrophysical flows are typically incompressible, and this term can thus typically be ignored. That leaves the $\nu\nabla^2\vec{u}$ term. This term quantifies **viscous momentum diffusion**, and describes how, in the presence of shear, momentum 'diffuses' across fluid elements.

It is interesting to point out that this innocent-looking diffuse term dramatically changes the character of the equation. The main reason is that it introduces a higher spatial derivative. Hence, additional boundary conditions are required to solve the equations. When solving problems with solid boundaries (not common in astrophysics), this condition is typically that the *tangential* (or shear) velocity at the boundary vanishes ("no-slip boundary condition"). Although this may sound ad hoc, it is supported by observation; for example, the blades of a fan collect dust.

Recall that when writing the Navier-Stokes equation in **Eulerian** form, we have that $\mathrm{d}\vec{u}/\mathrm{d}t \rightarrow \partial\vec{u}/\partial t + \vec{u}\cdot\nabla\vec{u}$. It is often useful to rewrite this extra term using the vector calculus identity (see Appendix A)

$$\vec{u}\cdot\nabla\vec{u} = \nabla\left(\frac{\vec{u}\cdot\vec{u}}{2}\right) + (\nabla\times\vec{u})\times\vec{u}$$

Hence, for an **irrotational flow** (i.e., a flow for which $\nabla\times\vec{u} = 0$), we have that $\vec{u}\cdot\nabla\vec{u} = \frac{1}{2}\nabla u^2$, where $u \equiv |\vec{u}|$.

Energy Equation: The energy equation written above, with $\rho\langle w_k \frac{1}{2}w^2\rangle$ replaced by $F_{\mathrm{cond},k}$ is obtained from the Boltzmann equation, which in turn derives from the Liouville theorem (and thus Hamiltonian dynamics) via the BBGKY hierarchy. The assumption of Hamiltonian dynamics implies that we have assumed that two-body collisions are non-dissipative. In reality, though, the atoms and molecules in a gas have internal degrees of freedom that can absorb kinetic energy from the collisions (excitation) and then radiate it away (spontaneous decay). In addition, the atoms and molecules can also absorb incoming photons, and, due to collisional deexcitation or photo-ionization transfer some of that energy into kinetic energy (heat). As long as the time scale on which these radiative processes change the internal energy of

the gas is long compared to the hydrodynamical time scales, we can simply add their impact by adding the **net volumetric cooling rate**,

$$\mathcal{L} = \rho \frac{\mathrm{d}Q}{\mathrm{d}t} = \mathcal{C} - \mathcal{H}$$

to the rhs of the energy equation. Here Q is the thermodynamic heat, and $\mathcal{C}$ and $\mathcal{H}$ are the net volumetric cooling and heating rates, respectively.

We also want to recast the above energy equation in a form that more directly describes the evolution of the **internal energy**, ε. This is obtained by subtracting u_i times the **Navier-Stokes equation** in conservative, Eulerian form from the energy equation derived above as follows

The Navier-Stokes equation in Eulerian index form is

$$\frac{\partial u_i}{\partial t} + u_k \frac{\partial u_i}{\partial x_k} = \frac{1}{\rho}\frac{\partial \sigma_{ik}}{\partial x_k} - \frac{\partial \Phi}{\partial x_i}$$

Using the **continuity equation**, this can be rewritten in the so-called **conservation form** as

$$\frac{\partial \rho u_i}{\partial t} + \frac{\partial}{\partial x_k}\left[\rho u_i u_k - \sigma_{ik}\right] = -\rho \frac{\partial \Phi}{\partial x_i}$$

Next we multiply this equation with u_i. Using that

$$\begin{aligned} u_i \frac{\partial \rho u_i}{\partial t} &= \frac{\partial \rho u^2}{\partial t} - \rho u_i \frac{\partial u_i}{\partial t} = \frac{\partial}{\partial t}\left[\rho \frac{u^2}{2}\right] + \frac{\partial}{\partial t}\left[\rho \frac{u^2}{2}\right] - \rho u_i \frac{\partial u_i}{\partial t} \\ &= \frac{\partial}{\partial t}\left[\rho \frac{u^2}{2}\right] + \frac{\rho}{2}\frac{\partial u^2}{\partial t} + \frac{u^2}{2}\frac{\partial \rho}{\partial t} - \rho u_i \frac{\partial u_i}{\partial t} \\ &= \frac{\partial}{\partial t}\left[\rho \frac{u^2}{2}\right] + \frac{u^2}{2}\frac{\partial \rho}{\partial t} \end{aligned}$$

where we have used that $\partial u^2/\partial t = 2u_i \partial u_i/\partial t$. Similarly, we have that

$$\begin{aligned} u_i \frac{\partial}{\partial x_k}\left[\rho u_i u_k\right] &= \frac{\partial}{\partial x_k}\left[\rho \frac{u^2}{2} u_k\right] + \frac{\partial}{\partial x_k}\left[\rho \frac{u^2}{2} u_k\right] - \rho u_i u_k \frac{\partial u_i}{\partial x_k} \\ &= \frac{\partial}{\partial x_k}\left[\rho \frac{u^2}{2} u_k\right] + \frac{\rho}{2} u_k \frac{\partial u^2}{\partial x_k} + \frac{u^2}{2}\frac{\partial \rho u_k}{\partial x_k} - \rho u_i u_k \frac{\partial u_i}{\partial x_k} \\ &= \frac{\partial}{\partial x_k}\left[\rho \frac{u^2}{2} u_k\right] + \frac{u^2}{2}\frac{\partial \rho u_k}{\partial x_k} \end{aligned}$$

Combining the above two terms, and using the **continuity equation** to dispose of the two terms containing the factor $u^2/2$, the Navier-Stokes equation in conservation form multiplied by u_i becomes

$$\frac{\partial}{\partial t}\left[\rho\frac{u^2}{2}\right] + \frac{\partial}{\partial x_k}\left[\rho\frac{u^2}{2}u_k\right] = u_i\frac{\partial \sigma_{ik}}{\partial x_k} - \rho u_i\frac{\partial \Phi}{\partial x_i}$$

Subtracting this from the energy equation at the beginning of this chapter (with the radiative term added) yields

$$\frac{\partial}{\partial t}\left(\rho\varepsilon\right) + \frac{\partial}{\partial x_k}\left(\rho\varepsilon u_k\right) = -P\frac{\partial u_k}{\partial x_k} + \mathcal{V} - \frac{\partial F_{\mathrm{cond},k}}{\partial x_k} - \mathcal{L}$$

Note the minus sign in front of $\mathcal{L}$, which expresses that net cooling results in a loss of internal energy. Here

$$\mathcal{V} \equiv \tau_{ik}\frac{\partial u_i}{\partial x_k}$$

is the **rate of viscous dissipation**, describing the rate at which heat is added to the internal energy budget via viscous conversion of ordered motion ($\vec{u}$) to disordered energy in random particle motions ($\vec{w}$).

Finally, using that the left-hand side of the above energy equation can be written as

$$\frac{\partial}{\partial t}\left(\rho\varepsilon\right) + \frac{\partial}{\partial x_k}\left(\rho\varepsilon u_k\right) = \rho\,\frac{\partial \varepsilon}{\partial t} + \varepsilon\,\frac{\partial \rho}{\partial t} + \varepsilon\,\frac{\partial \rho u_k}{\partial x_k} + \rho u_k\,\frac{\partial \varepsilon}{\partial x_k} = \rho\left(\frac{\partial \varepsilon}{\partial t} + u_k\frac{\partial \varepsilon}{\partial x_k}\right)$$

where in the second step we have used the continuity equation, we obtain the energy equation in Lagrangian vector form as

$$\boxed{\rho\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} = -P\,\nabla\cdot\vec{u} - \nabla\cdot\vec{F}_{\mathrm{cond}} + \mathcal{V} - \mathcal{L}}$$

This equation shows that the internal energy of a fluid element (a measure of its temperature or 'heat') can change due to (i) adiabatic compression or expansion of the fluid element, (ii) a local divergence in the conductive heat flux, (iii) viscous dissipation, or (iv) radiation (net cooling or heating). Note that (i), (ii) and (iv) can either cause an increase or a decrease in the internal specific energy, while viscous dissipation can only cause an increase in ε.

Hydrodynamic equations for an ideal fluid:

As mentioned in Chapter 10, an ideal (or perfect) fluid is a fluid without viscosity and without conductivity. Formally, ideal fluids are also incompressible (i.e., $\mathrm{d}\rho/\mathrm{d}t = 0$) and irrotational ($\nabla \times \vec{u} = 0$, i.e., flow is laminar w.o. turbulence), but throughout these lecture notes we adopt the somewhat weaker definition of an inviscid, non-conductive fluid.

The hydrodynamic equations for an ideal fluid simply follow from the more general equations derived above by simply setting $\nu = \vec{F}_{\rm cond} = 0$. The momentum equations for an ideal fluid (i.e., the equivalent of the Navier-Stokes equations) are known as the **Euler equations**. The table below lists the full set of hydrodynamic equations for ideal fluids, in both Eulerian and Lagrangian form.

	Lagrangian	**Eulerian**
Continuity Eq:	$\dfrac{\mathrm{d}\rho}{\mathrm{d}t} = -\rho\,\nabla\cdot\vec{u}$	$\dfrac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{u}) = 0$
Momentum Eqs:	$\dfrac{\mathrm{d}\vec{u}}{\mathrm{d}t} = -\dfrac{\nabla P}{\rho} - \nabla\Phi$	$\dfrac{\partial\vec{u}}{\partial t} + (\vec{u}\cdot\nabla)\,\vec{u} = -\dfrac{\nabla P}{\rho} - \nabla\Phi$
Energy Eq:	$\dfrac{\mathrm{d}\varepsilon}{\mathrm{d}t} = -\dfrac{P}{\rho}\nabla\cdot\vec{u} - \dfrac{\mathcal{L}}{\rho}$	$\dfrac{\partial\varepsilon}{\partial t} + \vec{u}\cdot\nabla\varepsilon = -\dfrac{P}{\rho}\,\nabla\cdot\vec{u} - \dfrac{\mathcal{L}}{\rho}$

Hydrodynamic equations for an ideal, neutral fluid in gravitational field

In astrophysics, numerical hydrodynamical simulations rarely if ever account for viscosity and/or conductivity; in other words, they assume an ideal fluid and solve the equations listed above. For numerical reasons, it is advantageous to write the hydro equations in **conservative form**. Let $A(\vec{x},t)$ be some state variable of the fluid (either scalar or vector). The evolution equation for A is said to be in conservative form if

$$\frac{\partial A}{\partial t} + \nabla\cdot\vec{F}(A) = S$$

Here $\vec{F}(A)$ describes the appropriate flux of A and S describes the various sources and/or sinks of A. The continuity, momentum and energy equations for an ideal fluid in conservative form are:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0$$

$$\frac{\partial \rho \vec{u}}{\partial t} + \nabla \cdot \overline{\overline{\mathbf{\Pi}}} = -\rho \nabla \Phi$$

$$\frac{\partial E}{\partial t} + \nabla \cdot \left[(E + P)\, \vec{u} \right] = \rho \frac{\partial \Phi}{\partial t} - \mathcal{L}$$

Here

$$\overline{\overline{\mathbf{\Pi}}} = \rho\, \vec{u} \otimes \vec{u} + P$$

is the **momentum flux density tensor** (of rank 2), and

$$E = \rho \left(\frac{1}{2} u^2 + \Phi + \varepsilon \right)$$

is the **energy density**.

NOTE: In the expression for the momentum flux density tensor $\vec{A} \otimes \vec{B}$ is the **tensor product** of $\vec{A}$ and $\vec{B}$ defined such that $(\vec{A} \otimes \vec{B})_{ij} = a_i\, b_j$ (see Appendix A). Hence, the index-form of the momentum flux density tensor is simply $\Pi_{ij} = \rho\, u_i\, u_j + P \delta_{ij}$, with δ_{ij} the Kronecker delta function. Note that this expression is ONLY valid for an ideal fluid.

Note also that whereas there is no source or sink term for the density, gradients in the gravitational field act as a source of momentum, while its time-variability can cause an increase or decrease in the energy density of the fluid. Another source/sink term for the energy density is radiation (emission or absorption of photons).

Poisson equation: If a hydrodynamic fluid is governed by self-gravity (as opposed to, is placed in an external gravitational field), then one needs to complement the hydrodynamical equations (continuity, momentum and energy) with the **Poisson equation:**

$$\nabla^2 \Phi = 4\pi G \rho$$

Constitutive equations: Closure of the set of macroscopic equations of hydrodynamics requires additional constitutive relations. As discussed in Chapter 11, both

the viscosity and conductivity scale with temperature as $T^{1/2}$, with proportionality constants that are best determined experimentally. The more important constitutive relation is the **equation-of-state** (EoS), relating the (hydrodynamical) pressure to one or more other macroscopic quantities. Typically, these are the density and the internal energy or temperature. In particular, for an ideal gas (i.e., a fluid that obeys the ideal gas law), we have that

$$P = \frac{k_{\rm B}\,T}{\mu\,m_{\rm p}}\,\rho\,, \qquad\qquad \varepsilon = \frac{1}{\gamma-1}\,\frac{k_{\rm B}\,T}{\mu\,m_{\rm p}}$$

Here μ is the mean molecular weight of the fluid in units of the proton mass, $m_{\rm p}$, and γ is the adiabatic index, which is often taken to be 5/3 as appropriate for a mono-atomic gas (see Appendix I for details).

• If the EoS is of the form $P = P(\rho)$, the EoS is said to be **barotropic**. In the case of a barotropic fluid, the continuity equation, the momentum equations and the EoS form a closed set, and the energy equation is not needed. There are two barotropic EoS that are encountered frequently in astrophysics: the **isothermal** EoS, which describes a fluid for which cooling and heating always balance each other to maintain a constant temperature, and the **adiabatic** EoS, in which there is no net heating or cooling (other than adiabatic heating or cooling due to the compression or expansion of volume, i.e., the $P\,{\rm d}V$ work). We will discuss these cases in more detail later in the course.

• If a barotropic EoS is of the form $P \propto \rho^{\Gamma}$, the EoS is said to be **polytropic**, with Γ the polytropic index. Isothermal and adiabatic equations of state are both polytropic. Whereas the former has $\Gamma = 1$, the latter has $\Gamma = \gamma = 5/3$.

SUMMARY: The Equations of Gravitational, Radiative Hydrodynamics

Continuity Eq.	$\dfrac{\mathrm{d}\rho}{\mathrm{d}t} = -\rho\,\nabla\cdot\vec{u}$
Momentum Eqs.	$\rho\,\dfrac{\mathrm{d}\vec{u}}{\mathrm{d}t} = -\nabla P + \mu\left[\nabla^2\vec{u} + \dfrac{1}{3}\nabla(\nabla\cdot\vec{u})\right] - \rho\,\nabla\Phi$
Energy Eq.	$\rho\,\dfrac{\mathrm{d}\varepsilon}{\mathrm{d}t} = -P\,\nabla\cdot\vec{u} - \nabla\cdot\vec{F}_{\rm cond} - \mathcal{L} + \mathcal{V}$
Poisson Eq.	$\nabla^2\Phi = 4\pi G\rho$
Dissipation	$\mathcal{V} \equiv \tau_{ik}\dfrac{\partial u_i}{\partial x_k}\,, \qquad \tau_{ik} = \mu\left[\dfrac{\partial u_i}{\partial x_k} + \dfrac{\partial u_k}{\partial x_i} - \dfrac{2}{3}\delta_{ik}\nabla\cdot\vec{u}\right]$
Conduction	$\vec{F}_{\rm cond} = \rho\langle\vec{w}\dfrac{1}{2}w^2\rangle = -\mathcal{K}\,\nabla T$
Radiation	$\mathcal{L} \equiv \mathcal{C} - \mathcal{H}$
Constitutive Eqs.	$P = P(\rho,\varepsilon)\,, \qquad \mu = \mu(T) \propto T^{1/2}\,, \qquad \mathcal{K} = \mathcal{K}(T) \propto T^{1/2}$

CHAPTER 13

Vorticity & Circulation

Vorticity: The vorticity of a flow is defined as the curl of the velocity field:

$$\boxed{\textbf{vorticity}: \qquad \vec{w} = \nabla \times \vec{u}}$$

It is a **microscopic** measure of rotation (vector) at a given point in the fluid, which can be envisioned by placing a paddle wheel into the flow. If it spins about its axis at a rate Ω, then $w = |\vec{w}| = 2\Omega$.

Circulation: The circulation around a closed contour C is defined as the line integral of the velocity along that contour:

$$\boxed{\textbf{circulation}: \qquad \Gamma_C = \oint_C \vec{u} \cdot \mathrm{d}\vec{l} = \int_S \vec{w} \cdot \mathrm{d}\vec{S}}$$

where S is an *arbitrary* surface bounded by C. The circulation is a **macroscopic** measure of rotation (scalar) for a finite area of the fluid.

Irrotational fluid: An irrotational fluid is defined as being curl-free; hence, $\vec{w} = 0$ and therefore $\Gamma_C = 0$ for any C.

Vortex line: a line that points in the direction of the vorticity vector. Hence, a vortex line relates to $\vec{w}$, as a streamline relates to $\vec{u}$ (cf. Chapter 10).

Vortex tube: a bundle of vortex lines. The circularity of a curve C is proportional to the number of vortex lines that thread the enclosed area.

In an **inviscid** fluid the vortex lines/tubes move **with** the fluid: a vortex line anchored to some fluid element remains anchored to that fluid element.

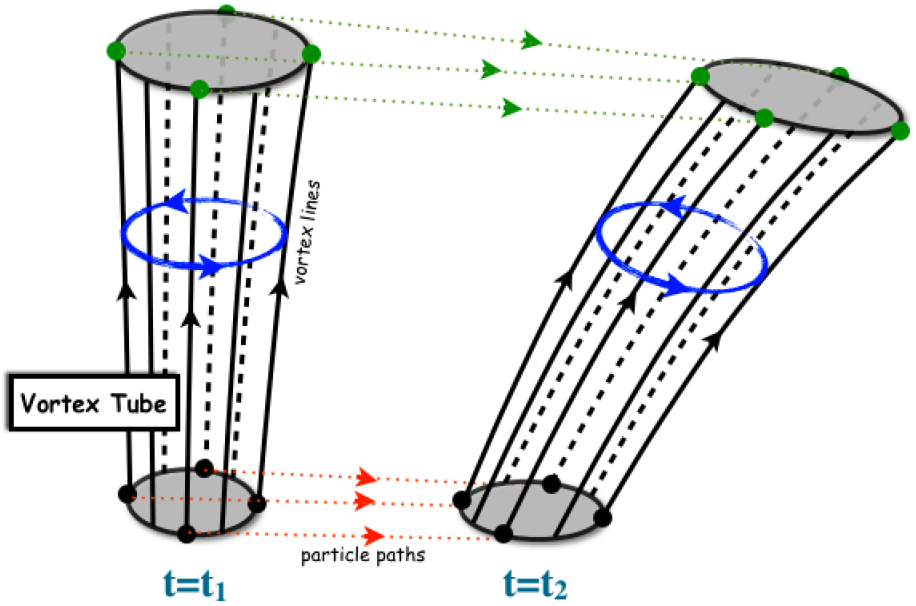


Figure 12: *Evolution of a vortex tube. Solid dots correspond to fluid elements. Due to the shear in the velocity field, the vortex tube is stretched and tilted. However, as long as the fluid is inviscid and barotropic Kelvin's circularity theorem assures that the circularity is conserved with time. In addition, since vorticity is divergence-free ('solenoidal'), the circularity along different cross sections of the same vortex-tube is the same.*

Vorticity equation: The Navier-Stokes momentum equations, in the absence of bulk viscosity, in Eulerian vector form, are given by

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\nabla)\,\vec{u} = -\frac{\nabla P}{\rho} - \nabla\Phi + \nu\left[\nabla^2\vec{u} + \frac{1}{3}\nabla(\nabla\cdot\vec{u})\right]$$

Using the vector identity $(\vec{u}\cdot\nabla)\,\vec{u} = \frac{1}{2}\nabla u^2 + (\nabla\times\vec{u})\times\vec{u} = \nabla(u^2/2) - \vec{u}\times\vec{w}$ allows us to rewrite this as

$$\frac{\partial \vec{u}}{\partial t} - \vec{u}\times\vec{w} = -\frac{\nabla P}{\rho} - \nabla\Phi - \frac{1}{2}\nabla u^2 + \nu\left[\nabla^2\vec{u} + \frac{1}{3}\nabla(\nabla\cdot\vec{u})\right]$$

If we now take the curl on both sides of this equation, and we use that curl(grad S) = 0 for any scalar field S, and that $\nabla\times(\nabla^2\vec{A}) = \nabla^2(\nabla\times\vec{A})$, we obtain the **vorticity equation**:

$$\boxed{\frac{\partial \vec{w}}{\partial t} = \nabla \times (\vec{u} \times \vec{w}) - \nabla \times \left(\frac{\nabla P}{\rho} \right) + \nu \nabla^2 \vec{w}}$$

To write this in Lagrangian form, we first use that $\nabla \times (S\,\vec{A}) = \nabla S \times \vec{A} + S\,(\nabla \times \vec{A})$ [see Appendix A] to write

$$\nabla \times (\frac{1}{\rho}\,\nabla P) = \nabla(\frac{1}{\rho}) \times \nabla P + \frac{1}{\rho}(\nabla \times \nabla P) = \frac{\rho \nabla(1) - 1 \nabla \rho}{\rho^2} \times \nabla P = \frac{\nabla P \times \nabla \rho}{\rho^2}$$

where we have used, once more, that $\mathrm{curl}(\mathrm{grad}\, S) = 0$. Next, using the vector identities from Appendix A, we write

$$\nabla \times (\vec{w} \times \vec{u}) = \vec{w}(\nabla \cdot \vec{u}) - (\vec{w} \cdot \nabla)\vec{u} - \vec{u}(\nabla \cdot \vec{w}) + (\vec{u} \cdot \nabla)\vec{w}$$

The third term vanishes because $\nabla \cdot \vec{w} = \nabla \cdot (\nabla \times \vec{u}) = 0$. Hence, using that $\partial \vec{w}/\partial t + (\vec{u} \cdot \nabla)\vec{w} = \mathrm{d}\vec{w}/\mathrm{d}t$ we finally can write the **vorticity equation in Lagrangian form:**

$$\boxed{\frac{\mathrm{d}\vec{w}}{\mathrm{d}t} = (\vec{w} \cdot \nabla)\vec{u} - \vec{w}(\nabla \cdot \vec{u}) + \frac{\nabla \rho \times \nabla P}{\rho^2} + \nu \nabla^2 \vec{w}}$$

This equation describes how the vorticity of a fluid element evolves with time. We now describe the various terms of the *rhs* of this equation in turn:

- $(\vec{w} \cdot \nabla)\vec{u}$: This term represents the **stretching** and **tilting** of vortex tubes due to velocity gradients. To see this, we pick $\vec{w}$ to be pointing in the z-direction. Then

  $$(\vec{w} \cdot \nabla)\vec{u} = w_z \frac{\partial \vec{u}}{\partial z} = w_z \frac{\partial u_x}{\partial z}\vec{e}_x + w_z \frac{\partial u_y}{\partial z}\vec{e}_y + w_z \frac{\partial u_z}{\partial z}\vec{e}_z$$

  The first two terms on the *rhs* describe the tilting of the vortex tube, while the third term describes the stretching.

- $\vec{w}(\nabla \cdot \vec{u})$: This term describes **stretching** of vortex tubes due to flow **compressibility**. This term is zero for an incompressible fluid or flow ($\nabla \cdot \vec{u} = 0$). Note that, again under the assumption that the vorticity is pointing in the z-direction,

$$\vec{w}(\nabla \cdot \vec{u}) = w_z \left[\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \right] \vec{e}_z$$

- $(\nabla\rho \times \nabla P)/\rho^2$: This is the **baroclinic** term. It describes the production of vorticity due to a misalignment between pressure and density gradients. This term is zero for a **barotropic** EoS: if $P = P(\rho)$ the pressure and density gradiens are parallel so that $\nabla P \times \nabla \rho = 0$. Obviously, this baroclinic term also vanishes for an incompressible fluid ($\nabla\rho = 0$) or for an isobaric fluid ($\nabla P = 0$). The baroclinic term is responsible, for example, for creating vorticity in pyroclastic flows (see Fig. 13).

- $\nu\nabla^2\vec{w}$: This term describes the **diffusion** of vorticity due to **viscosity**, and is obviously zero for an inviscid fluid ($\nu = 0$). Typically, viscosity generates/creates vorticity at a bounding surface: due to the *no-slip* boundary condition shear arises giving rise to vorticity, which is subsequently diffused into the fluid by the viscosity. In the interior of a fluid, no new vorticity is generated; rather, viscosity diffuses and dissipates vorticity.

- $\nabla \times \vec{F}$: There is a fifth term that can create vorticity, which however does not appear in the vorticity equation above. The reason is that we assumed that the only external force is gravity, which is a conservative force and can therefore be written as the gradient of a (gravitational) potential. More generally, though, there may be non-conservative, external body forces present, which would give rise to a $\nabla \times \vec{F}$ term in the rhs of the vorticity equation. An example of a non-conservative force creating vorticity is the **Coriolis force**, which is responsible for creating hurricanes.

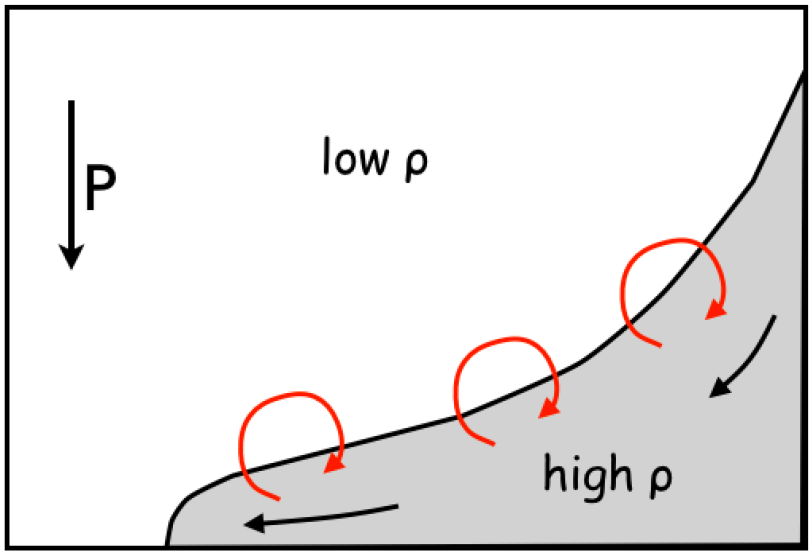


Figure 13: *The baroclinic creation of vorticity in a pyroclastic flow. High density fluid flows down a mountain and shoves itself under lower-density material, thus creating non-zero baroclinicity.*

Using the definition of **circulation**, it can be shown (here without proof) that

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}t} = \int_S \left[\frac{\partial \vec{w}}{\partial t} + \nabla \times (\vec{w} \times \vec{u})\right] \cdot \mathrm{d}\vec{S}$$

Using the **vorticity equation**, this can be rewritten as

$$\boxed{\frac{\mathrm{d}\Gamma}{\mathrm{d}t} = \int_S \left[\frac{\nabla\rho \times \nabla P}{\rho^2} + \nu\nabla^2\vec{w} + \nabla \times \vec{F}\right] \cdot \mathrm{d}\vec{S}}$$

where, for completeness, we have added in the contribution of an external force $\vec{F}$ (which vanishes if $\vec{F}$ is conservative). Using Stokes' Curl Theorem (see Appendix B) we can also write this equation in a line-integral form as

$$\boxed{\frac{\mathrm{d}\Gamma}{\mathrm{d}t} = -\oint \frac{\nabla P}{\rho} \cdot \mathrm{d}\vec{l} + \nu \oint \nabla^2\vec{u} \cdot \mathrm{d}\vec{l} + \oint \vec{F} \cdot \mathrm{d}\vec{l}}$$

which is the form that is more often used.

NOTE: By comparing the equations expressing $\mathrm{d}\vec{w}/\mathrm{d}t$ and $\mathrm{d}\Gamma/\mathrm{d}t$ it is clear that the stretching and tilting terms present in the equation describing $\mathrm{d}\vec{w}/\mathrm{d}t$, are absent in the equation describing $\mathrm{d}\Gamma/\mathrm{d}t$. This implies that stretching and tilting changes the vorticity, but keeps the circularity invariant. This is basically the first theorem of Helmholtz described below.

Kelvin's Circulation Theorem: The number of vortex lines that thread any element of area that moves with the fluid (i.e., the circulation) remains unchanged in time for an inviscid, barotropic fluid, in the absence of non-conservative forces.

The proof of **Kelvin's Circulation Theorem** is immediately evident from the above equation, which shows that $\mathrm{d}\Gamma/\mathrm{d}t = 0$ if the fluid is both **inviscid** ($\nu = 0$), **barotropic** ($P = P(\rho) \Rightarrow \nabla\rho \times \nabla P = 0$), and there are no non-conservative forces ($\vec{F} = 0$).

We end this chapter on vorticity and circulation with the three theorems of Helmholtz, which hold in the absence of non-conservative forces (i.e., $\vec{F} = 0$).

Helmholtz Theorem 1: The **strength** of a vortex tube, which is defined as the circularity of the circumference of any cross section of the tube, is constant along its length. This theorem holds for **any** fluid, and simply derives from the fact that the vorticity field is **divergence-free** (we say **solenoidal**): $\nabla \cdot \vec{w} = \nabla \cdot (\nabla \times \vec{u}) = 0$. To see this, use Gauss' divergence theorem to write that

$$\int_V \nabla \cdot \vec{w}\, \mathrm{d}V = \int_S \vec{w} \cdot \mathrm{d}^2 S = 0$$

Here V is the volume of a subsection of the vortex tube, and S is its bounding surface. Since the vorticity is, by definition, perpendicular to S along the sides of the tube, the only non-vanishing components to the surface integral come from the areas at the top and bottom of the vortex tube; i.e.

$$\int_S \vec{w} \cdot \mathrm{d}^2 \vec{S} = \int_{A_1} \vec{w} \cdot (-\hat{n})\, \mathrm{d}A + \int_{A_2} \vec{w} \cdot \hat{n}\, \mathrm{d}A = 0$$

where A_1 and A_2 are the areas of the cross sections that bound the volume V of the vortex tube. Using Stokes' curl theorem, we have that

$$\int_A \vec{w} \cdot \hat{n} \, \mathrm{d}A = \oint_C \vec{u} \cdot \mathrm{d}\vec{l}$$

Hence we have that $\Gamma_{C_1} = \Gamma_{C_2}$ where C_1 and C_2 are the curves bounding A_1 and A_2, respectively.

Helmholtz Theorem 2: A vortex line cannot end in a fluid. Vortex lines and tubes must appear as closed loops, extend to infinity, or start/end at solid boundaries.

Helmholtz Theorem 3: A barotropic, inviscid fluid that is initially irrotational will remain irrotational in the absence of rotational (i.e., non-conservative) external forces. Hence, such a fluid does not and cannot create vorticity (except across curved shocks, see Chapter 11).

The proof of Helmholtz' third theorem is straightforward. According to Kelvin's circulation theorem, a barotropic, inviscid fluid has $\mathrm{d}\Gamma/\mathrm{d}t = 0$ everywhere. Hence,

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}t} = \int_S \left[\frac{\partial \vec{w}}{\partial t} + \nabla \times (\vec{w} \times \vec{u}) \right] \cdot \mathrm{d}^2 \vec{S} = 0$$

Since this has to hold for any S, we have that $\partial \vec{w}/\partial t = \nabla \times (\vec{u} \times \vec{w})$. Hence, if $\vec{w} = 0$ initially, the vorticity remains zero for ever.

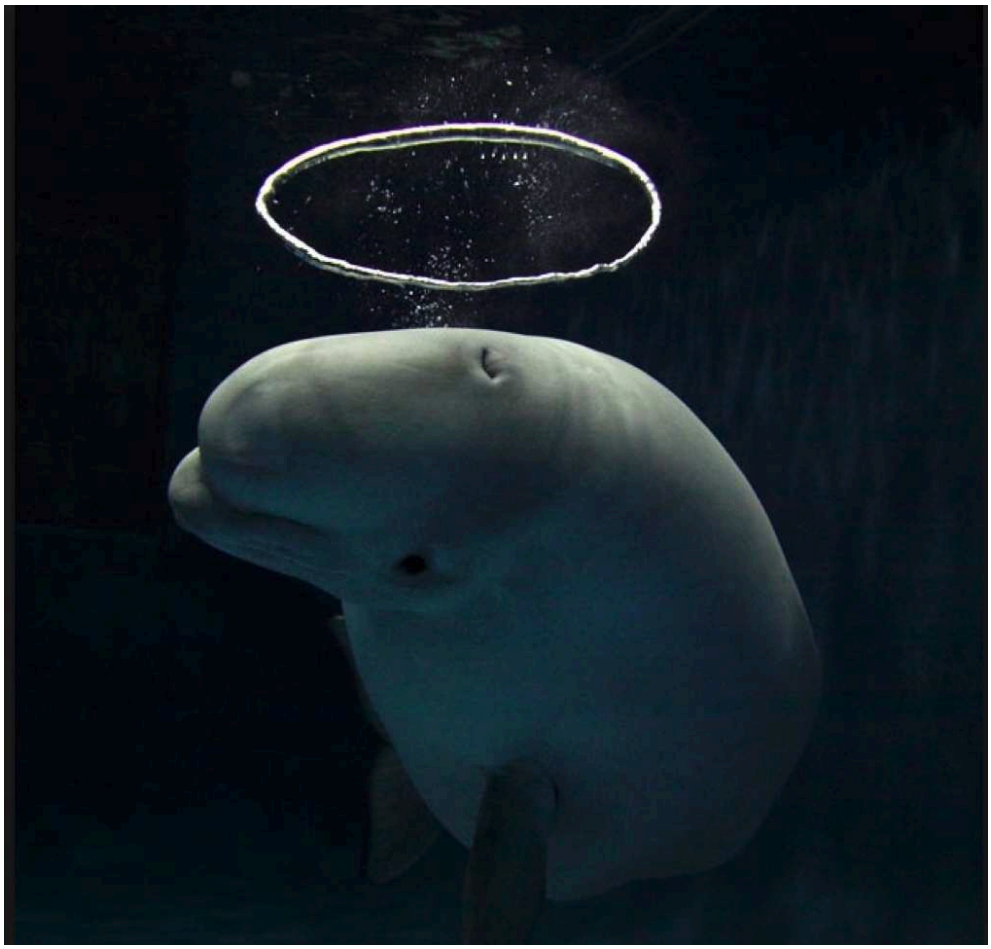


Figure 14: *A beluga whale demonstrating Kelvin's circulation theorem and Helmholtz' second theorem by producing a closed vortex tube under water, made out of air.*

CHAPTER 14

Hydrostatics and Steady Flows

Having derived all the relevant equations for hydrodynamics, we now start examining several specific flows. Since a fully general solution of the Navier-Stokes equation is (still) lacking (this is one of the seven Millenium Prize Problems, a solution of which will earn you $1,000,000), we can only make progress if we make several assumptions.

We start with arguably the simplest possible flow, namely 'no flow'. This is the area of **hydrostatics** in which $\vec{u}(\vec{x},t) = 0$. And since we seek a static solution, we also must have that all $\partial/\partial t$-terms vanish. Finally, in what follows we shall also ignore radiative processes (i.e., we set $\mathcal{L} = 0$).

Applying these restrictions to the continuity, momentum and energy equations (see box at the end of Chapter 5) yields the following two non-trivial equations:

$$\nabla P = -\rho \nabla \Phi$$

$$\nabla \cdot \vec{F}_{\rm cond} = 0$$

The first equation is the well known equation of **hydrostatic equilibrium**, stating that the gravitational force is balanced by pressure gradients, while the second equation states that in a static fluid the **conductive flux** needs to be divergence-free.

To further simplify matters, let's assume (i) spherical symmetry, and (ii) a barotropic equation of state, i.e., $P = P(\rho)$.

The equation of hydrostatic equilibrium now reduces to

$$\frac{{\rm d}P}{{\rm d}r} = -\frac{G\,M(r)\,\rho(r)}{r^2}$$

In addition, if the gas is self-gravitating (such as in a star) then we also have that

$$\boxed{\frac{\mathrm{d}M}{\mathrm{d}r} = 4\pi\rho(r)\, r^2}$$

For a barotropic EoS this is a closed set of equations, and the density profile can be solved for (given proper boundary conditions). Of particular interest in astrophysics, is the case of a **polytropic EoS**: $P \propto \rho^{\Gamma}$, where Γ is the polytropic index. Note that $\Gamma = 1$ and $\Gamma = \gamma$ for isothermal and adiabatic equations of state, respectively. A spherically symmetric, polytropic fluid in HE is called a polytropic sphere.

Lane-Emden equation: Upon substituting the **polytropic EoS** in the equation of **hydrostatic equilibrium** and using the **Poisson equation**, one obtains a single differential equation that completely describes the structure of the polytropic sphere, known as the **Lane-Emden equation**:

$$\frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right) = -\theta^n$$

Here $n = 1/(\Gamma - 1)$ is related to the polytropic index (in fact, confusingly, some texts refer to n as the polytropic index),

$$\xi = \left(\frac{4\pi G\rho_{\mathrm{c}}}{\Phi_0 - \Phi_{\mathrm{c}}}\right)^{1/2} r$$

is a dimensionless radius,

$$\theta = \left(\frac{\Phi_0 - \Phi(r)}{\Phi_0 - \Phi_{\mathrm{c}}}\right)$$

with Φ_{c} and Φ_0 the values of the gravitational potential at the center ($r = 0$) and at the surface of the star (where $\rho = 0$), respectively. The density is related to θ according to $\rho = \rho_{\mathrm{c}}\theta^n$ with ρ_{c} the central density.

Solutions to the Lane-Emden equation are called **polytropes of index** n. In general, the Lane-Emden equation has to be solved numerically subject to the boundary conditions $\theta = 1$ and $\mathrm{d}\theta/\mathrm{d}\xi = 0$ at $\xi = 0$. Analytical solutions exist, however, for $n = 0$, 1, and 5. Examples of polytropes are stars that are supported by **degeneracy pressure**. For example, a non-relativistic, degenerate equation of state has $P \propto \rho^{5/3}$

(see Appendix I) and is therefore describes by a polytrope of index $n = 3/2$. In the relativistic case $P \propto \rho^{4/3}$ which results in a polytrope of index $n = 3$.

Another polytrope that is often encountered in astrophysics is the **isothermal sphere**, which has $P \propto \rho$ and thus $n = \infty$. It has $\rho \propto r^{-2}$ at large radii, which implies an infinite total mass. If one truncates the isothermal sphere at some radius and embeds it in a medium with external pressure (to prevent the sphere from expanding), it is called a **Bonnor-Ebert sphere**, which is a structure that is frequently used to describe molecular clouds.

Stellar Structure: stars are gaseous spheres in hydrostatic equilibrium (except for radial pulsations, which may be considered perturbations away from HE). The structure of stars is therefore largely governed by the above equation.

However, in general the **equation of state** is of the form $P = P(\rho, T, \{X_i\})$, where $\{X_i\}$ is the set of the abundances of all emements i. The temperature structure of a star and its abundance ratios are governed by **nuclear physics** (which provides the source of energy) and the various **heat transport mechanisms**.

Heat transport in stars: Typically, ignoring abundance gradients, stars have the equation of state of an ideal gas, $P = P(\rho, T)$. This implies that the equations of stellar structure need to be complemented by an equation of the form

$$\boxed{\frac{\mathrm{d}T}{\mathrm{d}r} = F(r)}$$

Since T is a measure of the internal energy, the rhs of this equation describes the **heat flux**, $F(r)$.

The main heat transport mechanisms in a star are:

- conduction
- convection
- radiation

Note that the fourth heat transport mechanism, advection, is not present in the case of hydrostatic equilibrium, because $\vec{u} = 0$.

Recall from Chapter 4 that the **thermal conductivity** $\mathcal{K} \propto (k_{\rm B}\,T)^{1/2}/\sigma$ where σ is the collisional cross section. Using that $k_{\rm B}T \propto v^2$ and that the mean-free path of the particles is $\lambda_{\rm mfp} = 1/(n\sigma)$, we have that

$$\mathcal{K} \propto n\,\lambda_{\rm mfp}\,v$$

with v the thermal, microscopic velocity of the particles (recall that $\vec{u} = 0$). Since radiative heat transport in a star is basically the conduction of photons, and since $c \gg v_{\rm e}$ and the mean-free part of photons is much larger than that of electrons (after all, the cross section for Thomson scattering, $\sigma_{\rm T}$, is much smaller than the typical cross section for Coulomb interactions), we have that in stars *radiation is a far more efficient heat transport mechanism than conduction*. An exception are relativistic, degenerate cores, for which $v_{\rm e} \sim c$ and photons and electrons have comparable mean-free paths.

Convection: convection only occurs if the **Schwarzschild Stability Criterion** is violated, which happens when the temperature gradient $\mathrm{d}T/\mathrm{d}r$ becomes too large (i.e., larger than the temperature gradient that would exist if the star was adiabatic; see Chapter 18). If that is the case, convection always dominates over radiation as the most efficient heat transport mechanism. In general, as a rule of thumb, more massive stars are *more radiative* and *less convective*.

Trivia: On average it takes ~ 200.000 years for a photon created at the core of the Sun in nuclear burning to make its way to the Sun's photosphere; from there it only takes ~ 8 minutes to travel to the Earth.

Hydrostatic Mass Estimates: Now let us consider the case of an ideal gas, for which

$$P = \frac{k_{\rm B}T}{\mu m_{\rm p}}\rho\,,$$

but this time the gas is not self-gravitating; rather, the gravitational potential may be considered 'external'. A good example is the ICM; the hot gas that permeates clusters. From the EoS we have that

$$\begin{aligned}\frac{\mathrm{d}P}{\mathrm{d}r} &= \frac{\partial P}{\partial \rho}\frac{\mathrm{d}\rho}{\mathrm{d}r} + \frac{\partial P}{\partial T}\frac{\mathrm{d}T}{\mathrm{d}r} = \frac{P}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}r} + \frac{P}{T}\frac{\mathrm{d}T}{\mathrm{d}r} \\ &= \frac{P}{r}\left[\frac{r}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}r} + \frac{r}{T}\frac{\mathrm{d}T}{\mathrm{d}r}\right] = \frac{P}{r}\left[\frac{\mathrm{d}\ln\rho}{\mathrm{d}\ln r} + \frac{\mathrm{d}\ln T}{\mathrm{d}\ln r}\right]\end{aligned}$$

Substitution of this equation in the equation for Hydrostatic equilibrium (HE) yields

$$\boxed{M(r) = -\frac{k_\mathrm{B}\,T(r)\,r}{\mu m_\mathrm{p} G}\left[\frac{\mathrm{d}\ln\rho}{\mathrm{d}\ln r} + \frac{\mathrm{d}\ln T}{\mathrm{d}\ln r}\right]}$$

This equation is often used to measure the 'hydrostatic' mass of a galaxy cluster; X-ray measurements can be used to infer $\rho(r)$ and $T(r)$ (after deprojection, which is analytical in the case of spherical symmetry). Substitution of these two radial dependencies in the above equation then yields an estimate for the cluster's mass profile, $M(r)$. Note, though, that this mass estimate is based on three crucial assumptions: (i) sphericity, (ii) hydrostatic equilibrium, and (iii) an ideal-gas EoS. Clusters typically are not spherical, often are turbulent (such that $\vec{u} \neq 0$, violating the assumption of HE), and can have significant contributions from non-thermal pressure due to magnetic fields, cosmic rays and/or turbulence. Including these non-thermal pressure sources the above equation becomes

$$M(r) = -\frac{k_\mathrm{B}\,T(r)\,r}{\mu m_\mathrm{p} G}\left[\frac{\mathrm{d}\ln\rho}{\mathrm{d}\ln r} + \frac{\mathrm{d}\ln T}{\mathrm{d}\ln r} + \frac{P_\mathrm{nt}}{P_\mathrm{th}}\frac{\mathrm{d}\ln P_\mathrm{nt}}{\mathrm{d}\ln r}\right]$$

were P_nt and P_th are the non-thermal and thermal contributions to the total gas pressure. Unfortunately, it is extremely difficult to measure P_nt reliably, which is therefore often ignored. This may result in systematic biases of the inferred cluster mass (typically called the 'hydrostatic mass').

Solar Corona: As a final example of a hydrostatic problem in astrophysics, consider the problem of constructing a static model for the Solar corona.

The Solar corona is a large, spherical region of hot ($T \sim 10^6$K) plasma extending well beyond its photosphere. Let's assume that the heat is somehow (magnetic reconnection?) produced in the lower layers of the corona, and try to infer the density, temperature and pressure profiles under the assumption of hydrostatic equilibrium.

We have the boundary condition of the temperature at the base, which we assume to be $T_0 = 3 \times 10^6$K, at a radius of $r = r_0 \sim R_\odot \simeq 6.96 \times 10^{10}$ cm. The mass of the corona is negligble, and we therefore have that

$$\begin{aligned} \frac{\mathrm{d}P}{\mathrm{d}r} &= -\frac{G\ \mathrm{M}_\odot}{r^2}\ \frac{\mu m_\mathrm{p}}{k_\mathrm{B}}\ \frac{P}{T} \\ \frac{\mathrm{d}}{\mathrm{d}r}\left(\mathcal{K}\, r^2\, \frac{\mathrm{d}T}{\mathrm{d}r}\right) &= 0 \end{aligned}$$

where we have used the ideal gas EoS to substitute for ρ. As we have seen above $\mathcal{K} \propto n \lambda_\mathrm{mfp} T^{1/2}$. In a plasma one furthermore has that $\lambda_\mathrm{mfp} \propto n^{-1}\, T^2$, which implies that $\mathcal{K} \propto T^{5/2}$. Hence, the second equation can be written as

$$r^2 T^{5/2} \frac{\mathrm{d}T}{\mathrm{d}r} = \text{constant}$$

which implies

$$T = T_0\ \left(\frac{r}{r_0}\right)^{-2/7}$$

Note that this equation satisfies our boundary condition, and that $T_\infty = \lim_{r\to\infty} T(r) = 0$. Substituting this expression for T in the HE equation yields

$$\frac{\mathrm{d}P}{P} = -\frac{G\ \mathrm{M}_\odot\, \mu m_\mathrm{p}}{k_\mathrm{B} T_0\, r_0^{2/7}}\ \frac{\mathrm{d}r}{r^{12/7}}$$

Solving this ODE under the boundary condition that $P = P_0$ at $r = r_0$ yields

$$P = P_0\ \exp\left[-\frac{7}{5}\ \frac{G\ \mathrm{M}_\odot\, \mu m_\mathrm{p}}{k_\mathrm{B} T_0\, r_0}\ \left\{\left(\frac{r}{r_0}\right)^{-5/7} - 1\right\}\right]$$

Note that

$$\lim_{r\to\infty} P = P_0\ \exp\left[+\frac{7}{5}\ \frac{G\ \mathrm{M}_\odot\, \mu m_\mathrm{p}}{k_\mathrm{B} T_0\, r_0}\right] \neq 0$$

Hence, you need an external pressure to confine the corona. Well, that seems OK, given that the Sun is embedded in an ISM, whose pressure we can compute taking

characteristic values for the warm phase ($T \sim 10^4$K and $n \sim 1\,\mathrm{cm}^{-3}$). Note that the other phases (cold and hot) have the same pressure. Plugging in the numbers, we find that

$$\frac{P_\infty}{P_{\rm ISM}} \sim 10 \frac{\rho_0}{\rho_{\rm ISM}}$$

Since $\rho_0 \gg \rho_{\rm ISM}$ we thus infer that the ISM pressure falls short, by orders of magnitude, to be able to confine the corona....

As first inferred by Parker in 1958, the correct implicication of this puzzling result is that a hydrostatic corona is impossible; instead, Parker made the daring suggestion that there should be a **solar wind**, which was observationally confirmed a few years later.

Having addressed hydrostatics ('no flow'), we now consider the next simplest flow; **steady flow**, which is characterised by $\vec{u}(\vec{x},t) = \vec{u}(\vec{x})$. For steady flow $\partial\vec{u}/\partial t = 0$, and fluid elements move along the **streamlines** (see Chapter 10).

Using the vector identity $(\vec{u}\cdot\nabla)\,\vec{u} = \frac{1}{2}\nabla u^2 + (\nabla\times\vec{u})\times\vec{u} = \nabla(u^2/2) - \vec{u}\times\vec{w}$, allows us to write the Navier-Stokes equation for a steady flow of ideal fluid as

$$\boxed{\nabla\left(\frac{u^2}{2}+\Phi\right) + \frac{\nabla P}{\rho} - \vec{u}\times\vec{w} = 0}$$

This equation is known as **Crocco's theorem**. In order to write this in a more 'useful' form, we first proceed to demonstrate that $\nabla P/\rho$ can be written in terms of the gradients of the **specific enthalpy**, h, and the **specific entropy**, s:

The **enthalpy**, H, is a measure for the total energy of a thermodynamic system that includes the **internal** energy, U, and the amount of energy required to make room for it by displacing its environment and establishing its volume and pressure:

$$H = U + PV$$

The differential of the enthalpy can be written as

$$\mathrm{d}H = \mathrm{d}U + P\,\mathrm{d}V + V\,\mathrm{d}P$$

Using the first law of thermodynamics, according to which $\mathrm{d}U = \mathrm{d}Q - P\mathrm{d}V$, and the second law of thermodynamics, according to which $\mathrm{d}Q = T\mathrm{d}S$, we can rewrite this as

$$\mathrm{d}H = T\,\mathrm{d}S + V\,\mathrm{d}P$$

which, in specific form, becomes

$$\mathrm{d}h = T\mathrm{d}s + \frac{\mathrm{d}P}{\rho}$$

(i.e., we have $s = S/m$). This relation is one of the **Gibbs relations** frequently encountered in thermodynamics. NOTE: for completeness, we point out that this expression ignores changes in the chemical potential (see Appendix K).

The above expression for $\mathrm{d}h$ implies that

$$\boxed{\frac{\nabla P}{\rho} = \nabla h - T\,\nabla s}$$

(for a formal proof, see at the end of this chapter). Now recall from the previous chapter on vorticity that the **baroclinic term** is given by

$$\nabla \times \left(\frac{\nabla P}{\rho}\right) = \frac{\nabla\rho \times \nabla P}{\rho^2}$$

Using the above relation, and using that the curl of the gradient of a scalar vanishes, we can rewrite this baroclinic term as $\nabla \times (T\,\nabla s)$. This implies that *one can create vorticity by creating a gradient in (specific) entropy*! One way to do this, which is one of the most important mechanisms for creating vorticity in astrophysics, is via **curved shocks**; when an irrotational, isentropic fluid comes across a curved shock, different streamlines will experience a different jump in entropy (Δs will depend on the angle under which you cross the shock). Hence, in the post-shocked gas there will be a gradient in entropy, and thus vorticity.

Intermezzo: isentropic vs. adiabatic

We consider a flow to be **isentropic** if it conserves (specific) entropy, which implies that $\mathrm{d}s/\mathrm{d}t = 0$. Note that an ideal fluid is a fluid without dissipation (viscosity) and conduction (heat flow). Hence, any flow of ideal fluid is isentropic. A fluid is said to be isentropic if $\nabla s = 0$. A process is said to be **adiabatic** if $\mathrm{d}Q/\mathrm{d}t = 0$. Note that, according to the second law of thermodynamics, $T\mathrm{d}S \geq \mathrm{d}Q$. Equality only holds for a **reversible** process; in other words, only if a process is adiabatic and reversible do we call it isentropic. An irreversible, adiabatic process, therefore, can still create entropy.

Using the momentum equation for a steady, ideal fluid, and substituting $\nabla P/\rho \rightarrow \nabla h - T\,\nabla s$, we obtain

$$\boxed{\nabla B = T\,\nabla s + \vec{u} \times \vec{w}}$$

where we have introduced the **Bernoulli function**

$$\boxed{B \equiv \frac{u^2}{2} + \Phi + h = \frac{u^2}{2} + \Phi + \varepsilon + P/\rho}$$

which obviously is a measure of energy. The above equation is sometimes referred to as **Crocco's theorem**. It relates entropy gradients to vorticity and gradients in the Bernoulli function.

Let's investigate what happens to the Bernoulli function for an ideal fluid in a steady flow. Since we are in a steady state we have that

$$\frac{\mathrm{d}B}{\mathrm{d}t} = \frac{\partial B}{\partial t} + \vec{u} \cdot \nabla B = \vec{u} \cdot \nabla B$$

Next we use that

$$\begin{aligned} \vec{u} \cdot \nabla B &= \vec{u} \cdot T\nabla s + \vec{u} \cdot (\vec{u} \times \vec{w}) \\ &= T\vec{u} \cdot \nabla s = 0 \end{aligned}$$

Here we have used that the cross-product of $\vec{u}$ and $\vec{w}$ is perpendicular to $\vec{u}$, and that in an ideal fluid $\vec{u} \cdot \nabla s = 0$. The latter follow from the fact that in an ideal fluid $\mathrm{d}s/\mathrm{d}t = 0$, and the fact that $\mathrm{d}s/\mathrm{d}t = \partial s/\partial t + \vec{u} \cdot \nabla s$. Since all $\partial/\partial t$ terms vanish for a steady flow, we see that $\vec{u} \cdot \nabla s = 0$ for a steady flow of ideal fluid. In words, if gradients in the Bernoulli function are present in a steady, ideal fluid, flow can only be perpendicular to those gradient. And as a consequence, we thus also have that

$$\boxed{\frac{\mathrm{d}B}{\mathrm{d}t} = 0}$$

Hence, *in a steady flow of ideal fluid, the Bernoulli function is conserved*. Using the definition of the Bernoulli function we can write this as

$$\frac{\mathrm{d}B}{\mathrm{d}t} = \vec{u} \cdot \frac{\mathrm{d}\vec{u}}{\mathrm{d}t} + \frac{\mathrm{d}\Phi}{\mathrm{d}t} + T\frac{\mathrm{d}s}{\mathrm{d}t} + \frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}t} = 0$$

Since $\mathrm{d}s/\mathrm{d}t = 0$ for an ideal fluid, we have that if the flow is such that the gravitational potential along the flow doesn't change significantly (such that $\mathrm{d}\Phi/\mathrm{d}t \simeq 0$), we find that

$$\boxed{\vec{u} \cdot \frac{\mathrm{d}\vec{u}}{\mathrm{d}t} = -\frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}t}}$$

This is known as **Bernoulli's theorem**, and states that as the speed of a steady flow increases, the internal pressure of the ideal fluid must decrease. Applications of Bernoulli's theorem discussed in class include the shower curtain and the pitot tube (a flow measurement device used to measure fluid flow velocity).

Potential flow: The final flow to consider in this chapter is potential flow. Consider an irrotational flow, which satisfies $\vec{w} \equiv \nabla \times \vec{u} = 0$ everywhere. This implies that there is a scalar function, $\phi_u(x)$, such that $\vec{u} = \nabla \phi_u$, which is why $\phi_u(x)$ is called the **velocity potential**. The corresponding flow $\vec{u}(\vec{x})$ is called potential flow.

If the fluid is ideal (i.e., $\nu = \mathcal{K} = 0$), and barotropic or isentropic, such that the flow fluid has vanishing baroclinicity, then **Kelvin's circulation theorem** assures that the flow will remain irrotational throughout (no vorticity can be created), provided that all forces acting on the fluid are conservative.

If the fluid is **incompressible**, in addition to being irrotational, then we have that both the curl and the divergence of the velocity field vanish. This implies that

$$\nabla \cdot \vec{u} = \nabla^2 \phi_u = 0$$

This is the well known **Laplace equation**, familiar from electrostatics. Mathematically, this equation is of the elliptic PDE type which requires well defined boundary conditions in order for a solution to both exist and be unique. A classical case of potential flow is the flow around a solid body placed in a large fluid volume. In this case, an obvious boundary condition is the one stating that the velocity component perpendicular to the surface of the body at the body (assumed at rest) is zero. This is called a *Neumann* boundary condition and is given by

$$\frac{\partial \phi_u}{\partial n} = \vec{n} \cdot \nabla \phi_u = 0$$

with $\vec{n}$ the normal vector. The Laplace equation with this type of boundary condition constitutes a well-posed problem with a unique solution. An example of potential flow around a solid body is shown in Fig. 10 in Chapter 10. We will not examine any specific examples of potential flow, as this means having to solve a Laplace equation, which is purely a mathematical exercise. We end, though, by pointing out that real fluids are never perfectly inviscid (ideal fluids don't exist). And any flow past a surface involves a boundary layer inside of which viscosity creates vorticity (due to no-slip boundary condition, which states that the tangential velocity at the surface of the body must vanish). Hence, potential flow can never fully describe the flow around a solid body; otherwise one would run into **d'Alembert's paradox** which is that steady potential flow around a body exerts zero force on the body; in other words, it costs no energy to move a body through the fluid at constant speed. We know from everyday experience that this is indeed not true. The solution to the

paradox is that viscosity created in the boundary layer, and subsequently dissipated, results in friction.

Although potential flow around an object can thus never be a full description of the flow, in many cases, the boundary layer is very thin, and away from the boundary layer the solutions of potential flow still provide an accurate description of the flow.

As promised in the text, we end this chapter by demonstrating that

$$\mathrm{d}h = T\,\mathrm{d}s + \frac{\mathrm{d}P}{\rho} \qquad \Longleftrightarrow \qquad \nabla h = T\,\nabla s + \frac{\nabla P}{\rho}$$

To see this, use that the natural variables of h are the specific entropy, s, and the pressure P. Hence, $h = h(s, P)$, and we thus have that

$$\mathrm{d}h = \frac{\partial h}{\partial s}\,\mathrm{d}s + \frac{\partial h}{\partial P}\,\mathrm{d}P$$

From a comparison with the previous expression for $\mathrm{d}h$, we see that

$$\frac{\partial h}{\partial s} = T\,, \qquad\qquad \frac{\partial h}{\partial P} = \frac{1}{\rho}$$

which allows us to derive

$$\begin{aligned}
\nabla h &= \frac{\partial h}{\partial x}\,\vec{e}_x + \frac{\partial h}{\partial y}\,\vec{e}_y + \frac{\partial h}{\partial z}\,\vec{e}_z \\
&= \left(\frac{\partial h}{\partial s}\,\frac{\partial s}{\partial x} + \frac{\partial h}{\partial P}\,\frac{\partial P}{\partial x}\right)\vec{e}_x + \left(\frac{\partial h}{\partial s}\,\frac{\partial s}{\partial y} + \frac{\partial h}{\partial P}\,\frac{\partial P}{\partial y}\right)\vec{e}_y + \left(\frac{\partial h}{\partial s}\,\frac{\partial s}{\partial z} + \frac{\partial h}{\partial P}\,\frac{\partial P}{\partial z}\right)\vec{e}_z \\
&= \frac{\partial h}{\partial s}\left(\frac{\partial s}{\partial x}\,\vec{e}_x + \frac{\partial s}{\partial y}\,\vec{e}_y + \frac{\partial s}{\partial z}\,\vec{e}_z\right) + \frac{\partial h}{\partial P}\left(\frac{\partial P}{\partial x}\,\vec{e}_x + \frac{\partial P}{\partial y}\,\vec{e}_y + \frac{\partial P}{\partial z}\,\vec{e}_z\right) \\
&= T\nabla s + \frac{1}{\rho}\nabla P
\end{aligned}$$

which completes our proof.

CHAPTER 15

Viscous Flow and Accretion Flow

As we have seen in our discussion on potential flow in the previous chapter, realistic flow past an object always involves a boundary layer in which viscosity results in vorticity. Even if the viscosity of the fluid is small, the no-slip boundary condition typically implies a region where the shear is substantial, and viscocity thus manifests itself.

In this chapter we examine two examples of **viscous flow**. We start with a well-known example from engineering, known as **Poiseuille-Hagen flow** through a pipe. Although not really an example of astrophysical flow, it is a good illustration of how viscosity manifests itself as a consequence of the **no-slip boundary condition**. The second example that we consider is viscous flow in a thin accretion disk. This flow, which was first worked out in detail in a famous paper by Shakura & Sunyaev in 1973, is still used today to describe accretion disks in AGN and around stars.

Pipe Flow: Consider the steady flow of an incompressible viscous fluid through a circular pipe of radius $R_{\rm pipe}$ and lenght L. Let ρ be the density of the fluid as it flows through the pipe, and let $\nu = \mu/\rho$ be its **kinetic viscosity**. Since the flow is incompressible, we have that fluid density will be ρ throughout. If we pick a Cartesian coordinate system with the z-axis along the symmetry axis of the cylinder, then the velocity field of our flow is given by

$$\vec{u} = u_z(x, y, z)\, \vec{e}_z$$

In other words, $u_x = u_y = 0$.

Starting from the **continuity equation**

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{u} = 0$$

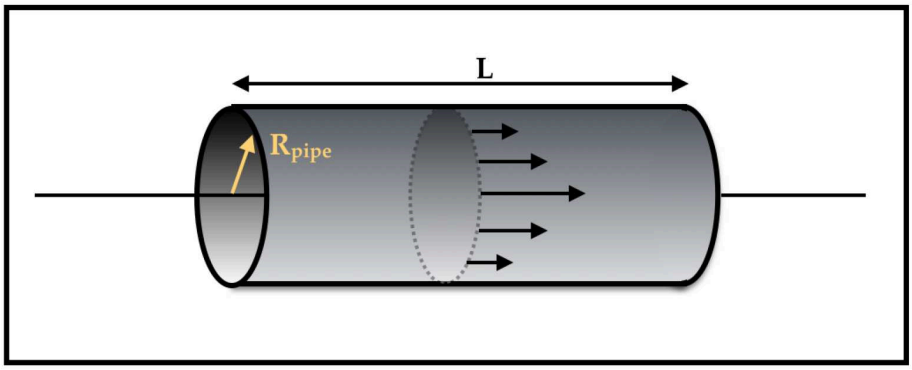


Figure 15: *Poiseuille-Hagen flow of a viscous fluid through a pipe of radius* R_{pipe} *and lenght* L.

and using that all partial time-derivatives of a steady flow vanish, we obtain that

$$\frac{\partial \rho u_x}{\partial x} + \frac{\partial \rho u_y}{\partial y} + \frac{\partial \rho u_z}{\partial z} = 0 \qquad \Rightarrow \qquad \frac{\partial u_z}{\partial z} = 0$$

where we have used that $\partial\rho/\partial z = 0$ because of the incompressibility of the flow. Hence, we can update our velocity field to be $\vec{u} = u_z(x, y)\,\vec{e}_z$.

Next we write down the **momentum equations** for a steady, incompressible flow, which are given by

$$(\vec{u} \cdot \nabla)\vec{u} = -\frac{\nabla P}{\rho} + \nu \nabla^2 \vec{u} - \nabla \Phi$$

In what follows we assume the pipe to be perpendicular to $\nabla\Phi$, so that we may ignore the last term in the above expression. For the x- and y- components of the momentum equation, one obtains that $\partial P/\partial x = \partial P/\partial y = 0$. For the z-component, we instead have

$$u_z \frac{\partial u_z}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial z} + \nu \nabla^2 u_z$$

Combining this with our result from the continuity equation, we obtain that

$$\boxed{\frac{1}{\rho}\frac{\partial P}{\partial z} = \nu \nabla^2 u_z}$$

Next we use that $\partial P/\partial z$ cannot depend on z; otherwise u_z would depend on z, but according to the continuity equation $\partial u_z/\partial z = 0$. This means that the pressure

gradient in the z-direction must be constant, which we write as $-\Delta P/L$, where ΔP is the pressure different between the beginning and end of the pipe, and the minus sign us used to indicate the the fluid pressure declines as it flows throught the pipe.

Hence, we have that

$$\nabla^2 u_z = -\frac{\Delta P}{\rho\,\nu\,L} = \text{constant}$$

At this point, it is useful to switch to **cylindrical coordinates**, (R, θ, z), with the z-axis as before. Because of the symmetries involved, we have that $\partial/\partial\theta = 0$, and thus the above expression reduces to

$$\frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}R}\left(R\frac{\mathrm{d}u_z}{\mathrm{d}R}\right) = -\frac{\Delta P}{\rho\,\nu\,L}$$

(see Appendix D). Rewriting this as

$$\mathrm{d}u_z = -\frac{1}{2}\frac{\Delta P}{\rho\,\nu\,L}\,R\,\mathrm{d}R$$

and integrating from R to R_{pipe} using the no-slip boundary condition that $u_z(R_{\text{pipe}}) = 0$, we finally obtain the flow solution

$$\boxed{u_z(R) = \frac{\Delta P}{4\rho\,\nu\,L}\,\left[R_{\text{pipe}}^2 - R^2\right]}$$

This solution is called **Poiseuille flow** or **Poiseuille-Hagen flow**.

As is evident from the above expression, for a given pressure difference ΔP, the flow speed $u \propto \nu^{-1}$ (i.e., a more viscous fluid will flow slower). In addition, for a given fluid viscosity, applying a larger pressure difference ΔP results in a larger flow speed ($u \propto \Delta P$).

Now let us compute the amount of fluid that flows through the pipe per unit time:

$$\dot{M} = 2\pi \int_0^{R_{\text{pipe}}} \rho\,u_z(R)\,R\,\mathrm{d}R = \frac{\pi}{8}\,\frac{\Delta P}{\nu\,L}\,R_{\text{pipe}}^4$$

Note the strong dependence on the pipe radius; this makes it clear that a clogging of the pipe has a drastic impact on the mass flow rate (relevant for both arteries and oil-pipelines). The above expression also gives one a relatively easy method to measure

the viscosity of a fluid: take a pipe of known $R_{\rm pipe}$ and L, apply a pressure difference ΔP across the pipe, and measure the mass flow rate, $\dot{M}$; the above expression allows one to then compute ν.

The Poiseuille velocity flow field has been experimentally confirmed, *but only for slow flow*! When $|\vec{u}|$ gets too large (i.e., ΔP is too large), then the flows becomes irregular in time and space; turbulence develops and $|\vec{u}|$ drops due to the enhanced drag from the turbulence. This will be discussed in more detail in Chapter 16.

Accretion Disks: We now move to a viscous flow that is more relevant for astrophysics; accretion flow. Consider a thin accretion disk surrounding an accreting object of mass $M_\bullet \gg M_{\rm disk}$ (such that we may ignore the disk's self-gravity). Because of the symmetries involved, we adopt **cylindrical coordinates**, (R, θ, z), with the z-axis perpendicular to the disk. We also have that $\partial/\partial\theta$ is zero, and we set $u_z = 0$ throughout.

We expect u_θ to be the main velocity component, with a small u_R component representing the radial accretion flow. We also take the flow to be **incompressible**.

Let's start with the **continuity equation**, which in our case reads

$$\frac{\partial \rho}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}\left(R\,\rho\,u_R\right) = 0$$

(see Appendix D for how to express the divergence in cylindrical coordinates).

Next up is the **Navier-Stokes equations**. For now, we only consider the θ-component, which is given by

$$\begin{aligned}\frac{\partial u_\theta}{\partial t} \;+\; & u_R\frac{\partial u_\theta}{\partial R} + \frac{u_\theta}{R}\frac{\partial u_\theta}{\partial \theta} + u_z\frac{\partial u_\theta}{\partial z} + \frac{u_R\,u_\theta}{R} = -\frac{1}{\rho}\frac{\partial P}{\partial \theta} \\ & + \nu\left[\frac{\partial^2 u_\theta}{\partial R^2} + \frac{1}{R^2}\frac{\partial^2 u_\theta}{\partial \theta^2} + \frac{\partial^2 u_\theta}{\partial z^2} + \frac{1}{R}\frac{\partial u_\theta}{\partial R} + \frac{2}{R^2}\frac{\partial u_R}{\partial \theta} - \frac{u_\theta}{R^2}\right] + \frac{\partial \Phi}{\partial \theta}\end{aligned}$$

NOTE: There are several terms in the above expression that may seem 'surprising'. The important thing to remember in writing down the equations in curvi-linear

coordinates is that operators can also act on unit-direction vectors. For example, the θ-component of $\nabla^2\vec{u}$ is NOT $\nabla^2 u_\theta$. That is because the operator ∇^2 acts on $u_R\vec{e}_R + u_\theta\vec{e}_\theta + u_z\vec{e}_z$, and the directions of $\vec{e}_R$ and $\vec{e}_\theta$ depend on position! The same holds for the convective operator $(\vec{u}\cdot\nabla)\,\vec{u}$. The full expressions for both cylindrical and spherical coordinates are written out in Appendix D.

Setting all the terms containing $\partial/\partial\theta$ and/or u_z to zero, the Navier-Stokes equation simplifies considerably to

$$\rho\left[\frac{\partial u_\theta}{\partial t} + u_R\frac{\partial u_\theta}{\partial R} + \frac{u_R\,u_\theta}{R}\right] = \mu\left[\frac{\partial^2 u_\theta}{\partial R^2} + \frac{\partial^2 u_\theta}{\partial z^2} + \frac{1}{R}\frac{\partial u_\theta}{\partial R} - \frac{u_\theta}{R^2}\right]$$

where we have replaced the **kinetic viscosity**, ν, with $\mu = \nu\rho$.

Integrating over z and writing

$$\int_{-\infty}^{\infty}\rho\,\mathrm{d}z = \Sigma$$

where Σ is the surface density, as well as neglecting variation of ν, u_R and u_θ with z (a reasonable approximation), the continuity and Navier-Stokes equation become

$$\begin{aligned}\frac{\partial\Sigma}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}\left(R\,\Sigma\,u_R\right) &= 0\\ \Sigma\left(\frac{\partial u_\theta}{\partial t} + u_R\frac{\partial u_\theta}{\partial R} + \frac{u_R\,u_\theta}{R}\right) &= \mathcal{F}(\mu,R)\end{aligned}$$

where $\mathcal{F}(\mu,R)$ describes the various **viscous terms**.

Next we multiply the continuity equation by Ru_θ which we can then write as

$$\frac{\partial(\Sigma\,R\,u_\theta)}{\partial t} - \Sigma\frac{\partial(Ru_\theta)}{\partial t} + \frac{\partial(\Sigma\,R\,u_R\,u_\theta)}{\partial R} - R\,\Sigma\,u_R\frac{\partial u_\theta}{\partial R} = 0$$

Adding this to R times the Navier-Stokes equation, and rearranging terms, yields

$$\frac{\partial(\Sigma\,R\,u_\theta)}{\partial t} + \frac{\partial(\Sigma\,R\,u_R\,u_\theta)}{\partial R} + \Sigma\,u_R\,u_\theta = \mathcal{G}(\mu,R)$$

where $\mathcal{G}(\mu,R) = R\mathcal{F}(\mu)$. Next we introduce the **angular frequency** $\Omega \equiv u_\theta/R$ which allows us to rewrite the above expression as

$$\boxed{\frac{\partial(\Sigma\,R^2\,\Omega)}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}\left(\Sigma\,R^3\,\Omega\,u_R\right) = \mathcal{G}(\mu,R)}$$

Note that $\Sigma\, R^2\, \Omega = \Sigma\, R\, u_\theta$ is the *angular momentum per unit area*. Hence the above equation describes the evolution of angular momentum in the accretion disk. It is also clear, therefore, that $\mathcal{G}(\mu, R)$ must describe the **viscous torque** on the disk material, per unit surface area. To derive an expression for it, recall that

$$\mathcal{G}(\mu, R) = R \int \mathrm{d}z\, \mu \left[\frac{\partial^2 u_\theta}{\partial R^2} + \frac{1}{R} \frac{\partial u_\theta}{\partial R} - \frac{u_\theta}{R^2} \right]$$

where we have ignored the $\partial^2 u_\theta / \partial z^2$ term which is assumed to be small. Using that $\mu = \nu\rho$ and that μ is independent of R and z (this is an assumption that underlies the Navier-Stokes equation from which we started) we have that

$$\mathcal{G}(\mu, R) = \nu\, R\, \Sigma \left[\frac{\partial^2 u_\theta}{\partial R^2} + \frac{1}{R} \frac{\partial u_\theta}{\partial R} - \frac{u_\theta}{R^2} \right]$$

Next we use that $u_\theta = \Omega\, R$ to write

$$\frac{\partial u_\theta}{\partial R} = \Omega + R \frac{\mathrm{d}\Omega}{\mathrm{d}R}$$

Substituting this in the above expression for $\mathcal{G}(\mu, R)$ yield

$$\mathcal{G}(\mu, R) = \nu\, \Sigma \left[R^2 \frac{\mathrm{d}^2\Omega}{\mathrm{d}R^2} + 3R \frac{\mathrm{d}\Omega}{\mathrm{d}R} \right] = \frac{1}{R} \frac{\partial}{\partial R} \left(\nu\, \Sigma\, R^3 \frac{\mathrm{d}\Omega}{\mathrm{d}R} \right)$$

Substituting this expression for the viscous torque in the evolution equation for the angular momentum per unit surface density, we finally obtain the full set of equations that govern our thin accretion disk:

$$\frac{\partial}{\partial t} \left(\Sigma\, R^2\, \Omega \right) + \frac{1}{R} \frac{\partial}{\partial R} \left(\Sigma\, R^3\, \Omega\, u_R \right) = \frac{1}{R} \frac{\partial}{\partial R} \left(\nu\, \Sigma\, R^3 \frac{\mathrm{d}\Omega}{\mathrm{d}R} \right)$$

$$\frac{\partial \Sigma}{\partial t} + \frac{1}{R} \frac{\partial}{\partial R} \left(R\, \Sigma\, u_R \right) = 0$$

$$\Omega = \left(\frac{G\, M_\bullet}{R^3} \right)^{1/2}$$

These three equations describe the dynamics of a thin, viscous accretion disk. The third equation indicates that we assume that the fluid is in Keplerian motion around the accreting object of mass $M_\bullet$. As discussed further below, this is a reasonable assumption as long as the accretion disk is thin.

Note that the non-zero u_R results in a **mass inflow rate**

$$\dot{M}(R) = -2\pi\Sigma\, R\, u_R$$

(a positive u_R reflects outwards motion).

Now let us consider a **steady accretion disk**. This implies that $\partial/\partial t = 0$ and that $\dot{M}(R) = \dot{M} \equiv \dot{M}_\bullet$ (the mass flux is constant throughout the disk, otherwise $\partial\Sigma/\partial t \neq 0$). In particular, the continuity equation implies that

$$R\,\Sigma\, u_R = C_1$$

Using the above expression for the mass inflow rate, we see that

$$\boxed{C_1 = -\frac{\dot{M}_\bullet}{2\pi}}$$

Similarly, for the Navier-Stokes equation, we have that

$$\Sigma\, R^3\, \Omega\, u_R - \nu\, \Sigma\, R^3\, \frac{\mathrm{d}\Omega}{\mathrm{d}R} = C_2$$

Using the boundary condition that at the radius of the accreting object, $R_\bullet$, the disk material must be dragged into rigid rotation (a no-slip boundary condition), which implies that $\mathrm{d}\Omega/\mathrm{d}R = 0$ at $R = R_\bullet$, we obtain that

$$\boxed{C_2 = R_\bullet^2\, \Omega_\bullet\, C_1 = -\frac{\dot{M}_\bullet}{2\pi}\, (G\, M_\bullet\, R_\bullet)^{1/2}}$$

Substituting this in the above expression, and using that

$$\frac{\mathrm{d}\Omega}{\mathrm{d}R} = \frac{\mathrm{d}}{\mathrm{d}R}\left(\frac{G\, M_\bullet}{R^3}\right)^{1/2} = -\frac{3}{2}\,\frac{\Omega}{R}$$

we have that

$$\begin{aligned}\nu\,\Sigma &= -\frac{\dot{M}_\bullet}{2\pi}\left[R^2\Omega-(G\,M_\bullet\,R_\bullet)^{1/2}\right]\left(R^3\frac{\mathrm{d}\Omega}{\mathrm{d}R}\right)^{-1}\\ &= +\frac{\dot{M}_\bullet}{3\pi}\left[1-\left(\frac{R_\bullet}{R}\right)^{1/2}\right]\end{aligned}$$

This shows that *the mass inflow rate and kinetic viscosity depend linearly on each other*.

The gravitational energy lost by the inspiraling material is converted into heat. This is done through **viscous dissipation**: viscosity robs the disk material of angular momentum which in turn causes it to spiral in.

We can work out the **rate of viscous dissipation** using

$$\mathcal{V}=\pi_{ij}\,\frac{\partial u_i}{\partial x_j}$$

where we have that the **viscous stress tensor** is

$$\pi_{ij}=\mu\left[\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}-\frac{2}{3}\delta_{ij}\frac{\partial u_k}{\partial x_k}\right]$$

(see Chapter 11). Note that the last term in the above expression vanishes because the fluid is incompressible, such that

$$\mathcal{V}=\mu\left[\left(\frac{\partial u_i}{\partial x_j}\right)^2+\frac{\partial u_j}{\partial x_i}\,\frac{\partial u_i}{\partial x_j}\right]$$

(remember to apply the Einstein summation convention here!).

In our case, using that $\partial/\partial\theta=\partial/\partial z=0$ and that $u_z=0$, the only surviving terms are

$$\mathcal{V}=\mu\left[\left(\frac{\partial u_R}{\partial R}\right)^2+\left(\frac{\partial u_\theta}{\partial R}\right)^2+\frac{\partial u_R}{\partial R}\,\frac{\partial u_R}{\partial R}\right]=\mu\left[2\left(\frac{\partial u_R}{\partial R}\right)^2+\left(\frac{\partial u_\theta}{\partial R}\right)^2\right]$$

If we make the reasonable assumption that $u_R \ll u_\theta$, we can ignore the first term, such that we finally obtain

$$\boxed{\mathcal{V} = \mu \left(\frac{\partial u_\theta}{\partial R}\right)^2 = \mu R^2 \left(\frac{\mathrm{d}\Omega}{\mathrm{d}R}\right)^2}$$

which expresses the viscous dissipation *per unit volume*. Note that $\partial u_\theta/\partial R = \Omega + R\,\mathrm{d}\Omega/\mathrm{d}R$. Hence, even in a solid body rotation ($\mathrm{d}\Omega/\mathrm{d}R = 0$) there is a radial derivative of u_θ. However, when $\mathrm{d}\Omega/\mathrm{d}R = 0$ there is no velocity shear in the disk, which shows that the Ω term cannot contribute to the viscous dissipation rate.

As before, we now proceed by integrating over the z-direction, to obtain

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \int \mu\, R^2 \left(\frac{\mathrm{d}\Omega}{\mathrm{d}R}\right)^2 \mathrm{d}z = \nu\,\Sigma\, R^2 \left(\frac{\mathrm{d}\Omega}{\mathrm{d}R}\right)^2$$

Using our expression for $\nu\Sigma$ derived above, we can rewrite this as

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\dot{M}_\bullet}{3\pi}\, R^2 \left[1 - \left(\frac{R_\bullet}{R}\right)^{1/2}\right] \left(\frac{\mathrm{d}\Omega}{\mathrm{d}R}\right)^2$$

Using once more that $\mathrm{d}\Omega/\mathrm{d}R = -(3/2)\Omega/R$, and integrating over the entire disk yields the accretion luminosity of a thin accretion disk:

$$\boxed{L_{\rm acc} \equiv 2\pi \int_{R_\bullet}^{\infty} \frac{\mathrm{d}E}{\mathrm{d}t}\, R\, \mathrm{d}R = \frac{G\, M_\bullet\, \dot{M}_\bullet}{2\, R_\bullet}}$$

To put this in perspective, realize that the gravitational energy of mass m at radius $R_\bullet$ is $G\, M_\bullet m/\, R_\bullet$. Thus, $L_{\rm acc}$ is exactly half of the gravitational energy lost due to the inflow. This obviously begs the question where the other half went...The answer is simple; it is stored in kinetic energy at the 'boundary' radius $R_\bullet$ of the accreting flow.

We end our discussion on accretion disks with a few words of caution. First of all, our entire derivation is only valid for a <u>thin</u> accretion disk. In a thin disk, the

pressure in the disk must be small (otherwise it would puff up). This means that the $\partial P/\partial R$ term in the R-component of the Navier-Stokes equation is small compared to $\partial\Phi/\partial R = GM/R^2$. This in turn implies that the gas will indeed be moving on Keplerian orbits, as we have assumed. If the accretion disk is thick, the situation is much more complicated, something that will not be covered in this course.

Finally, let us consider the time scale for accretion. As we have seen above, the energy loss rate per unit surface area is

$$\nu\,\Sigma\,R^2\left(\frac{\mathrm{d}\Omega}{\mathrm{d}R}\right)^2 = \frac{9}{4}\nu\Sigma\,\frac{G\,M_\bullet}{R^3}$$

We can compare this with the gravitation potential energy of disk material per unit surface area, which is

$$E = \frac{G\,M_\bullet\Sigma}{R}$$

This yields an accretion time scale

$$\boxed{t_{\rm acc} \equiv \frac{E}{\mathrm{d}E/\mathrm{d}t} = \frac{4}{9}\frac{R^2}{\nu} \sim \frac{R^2}{\nu}}$$

Too estimate this time-scale, we first estimate the molecular viscosity. Recall that $\nu \propto \lambda_{\rm mfp} v$ with v a typical velocity of the fluid particles. In virtually all cases encountered in astrophysics, we have that the size of the accretion disk, R, is many, many orders of magnitude larger than $\lambda_{\rm mfp}$. As a consequence, the corresponding $t_{\rm acc}$ easily exceeds the Hubble time!

The conclusion is that **molecular viscosity** is way too small to result in any significant accretion in objects of astrophysical size. Hence, other source of viscosity are required, which is a topic of ongoing discussion in the literature. Probably the most promising candidates are turbulence (in different forms), and the magneto-rotational instability (MRI). Given the uncertainties involved, it is common practice to simply write

$$\nu = \alpha\,\frac{P}{\rho}\,\frac{1}{R}\left(\frac{\mathrm{d}\Omega}{\mathrm{d}R}\right)^{-1}$$

where α is a 'free parameter'. A thin accretion disk modelled this way is often called an **alpha-accretion disk**. If you wonder what the origin is of the above expression;

Figure 16: *Image of the central region of NGC 4261 taken with the Hubble Space Telescope. It reveals a* $\sim 100pc$ *scale disk of dust and gas, which happens to be perpendicular to a radio jet that emerges from this galaxy. This is an alledged 'accretion disk' supplying fuel to the central black hole in this galaxy. This image was actually analyzed by the author as part of his thesis.*

it simply comes from assuming that the only non-vanishing off-diagonal term of the stress tensor is taken to be αP (where P is the value along the diagonal of the stress tensor).

CHAPTER 16

Turbulence

Non-linearity: The Navier-Stokes equation is non-linear. This non-linearity arises from the convective (material) derivative term

$$\vec{u} \cdot \nabla \vec{u} = \frac{1}{2} \nabla u^2 - \vec{u} \times \vec{w}$$

which describes the "inertial acceleration" and is ultimately responsible for the origin of the **chaotic character** of many flows and of **turbulence**. Because of this non-linearity, we cannot say whether a solution to the Navier-Stokes equation with nice and smooth initial conditions will remain nice and smooth for all time (at least not in 3D).

Laminar flow: occurs when a fluid flows in parallel layers, without lateral mixing (no cross currents perpendicular to the direction of flow). It is characterized by high momentum diffusion and low momentum convection.

Turbulent flow: is characterized by chaotic and stochastic property changes. This includes low momentum diffusion, high momentum convection, and rapid variation of pressure and velocity in space and time.

The Reynold's number: In order to gauge the importance of viscosity for a fluid, it is useful to compare the ratio of the inertial acceleration $(\vec{u} \cdot \nabla \vec{u})$ to the viscous acceleration $(\nu \left[\nabla^2 \vec{u} + \frac{1}{3} \nabla (\nabla \cdot \vec{u})\right])$. This ratio is called the Reynold's number, $\mathcal{R}$, and can be expressed in terms of the typical velocity scale $U \sim |\vec{u}|$ and length scale $L \sim 1/\nabla$ of the flow, as

$$\boxed{\mathcal{R} = \left| \frac{\vec{u} \cdot \nabla \vec{u}}{\nu \left[\nabla^2 \vec{u} + \frac{1}{3} \nabla (\nabla \cdot \vec{u})\right]} \right| \sim \frac{U^2/L}{\nu U/L^2} = \frac{U\,L}{\nu}}$$

If $\mathcal{R} \gg 1$ then viscosity can be ignored (and one can use the Euler equations to describe the flow). However, if $\mathcal{R} \ll 1$ then viscosity is important.

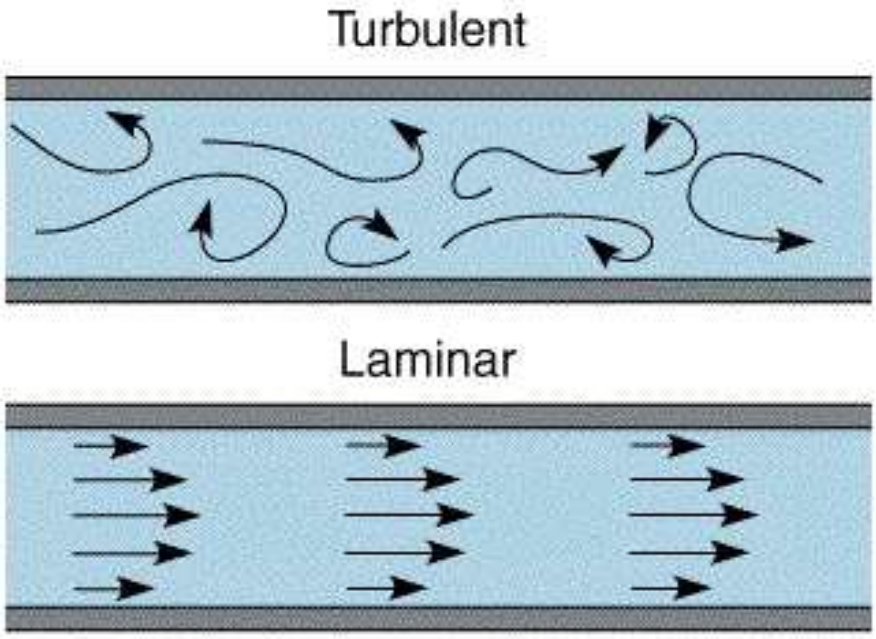


Figure 17: *Illustration of laminar vs. turbulent flow.*

Similarity: Flows with the same Reynold's number are similar. This is evident from rewriting the Navier-Stokes equation in terms of the following dimensionless variables

$$\tilde{u} = \frac{\vec{u}}{U} \qquad \tilde{x} = \frac{\vec{x}}{L} \qquad \tilde{t} = t\,\frac{U}{L} \qquad \tilde{p} = \frac{P}{\rho\,U^2} \qquad \tilde{\Phi} = \frac{\Phi}{U^2} \qquad \tilde{\nabla} = L\,\nabla$$

This yields (after multiplying the Navier-Stokes equation with L/U^2):

$$\boxed{\frac{\partial \tilde{u}}{\partial \tilde{t}} + \tilde{u}\cdot\tilde{\nabla}\tilde{u} + \tilde{\nabla}\tilde{p} + \tilde{\nabla}\tilde{\Phi} = \frac{1}{\mathcal{R}}\left[\tilde{\nabla}^2\tilde{u} + \frac{1}{3}\tilde{\nabla}(\tilde{\nabla}\cdot\tilde{u})\right]}$$

which shows that the form of the solution depends only on $\mathcal{R}$. This principle is extremely powerful as it allows one to making scale models (i.e., when developing airplanes, cars etc). NOTE: the above equation is only correct for an incompressible fluid, i.e., a fluid that obeys $\nabla\rho = 0$. If this is not the case the term $\tilde{P}\,(\nabla\rho/\rho)$ needs to be added at the rhs of the equation, braking its scale-free nature.

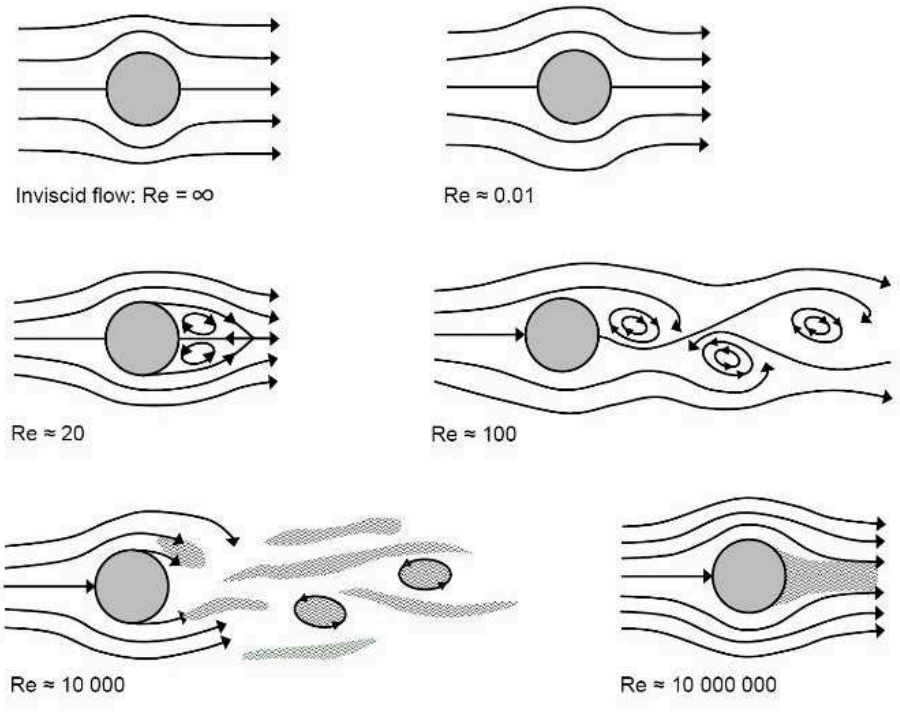


Figure 18: *Illustration of flows at different Reynolds number.*

As a specific example, consider fluid flow past a cylinder of diameter L:

- $\mathcal{R} \ll 1$: "creeping flow". In this regime the flow is viscously dominated and (nearly) symmetric upstream and downstream. The inertial acceleration $(\vec{u} \cdot \nabla \vec{u})$ can be neglected, and the flow is (nearly) time-reversible.
- $\mathcal{R} \sim 1$: Slight asymmetry develops
- $10 \leq \mathcal{R} \leq 41$: Separation occurs, resulting in two counter-rotating votices in the wake of the cylinder. The flow is still steady and laminar, though.
- $41 \leq \mathcal{R} \leq 10^3$: "von Kármán vortex street"; **unsteady laminar flow** with counter-rotating vortices shed periodically from the cylinder. Even at this stage the flow is still 'predictable'.
- $\mathcal{R} > 10^3$: vortices are unstable, resulting in a **turbulent wake** behind the cylinder that is 'unpredictable'.

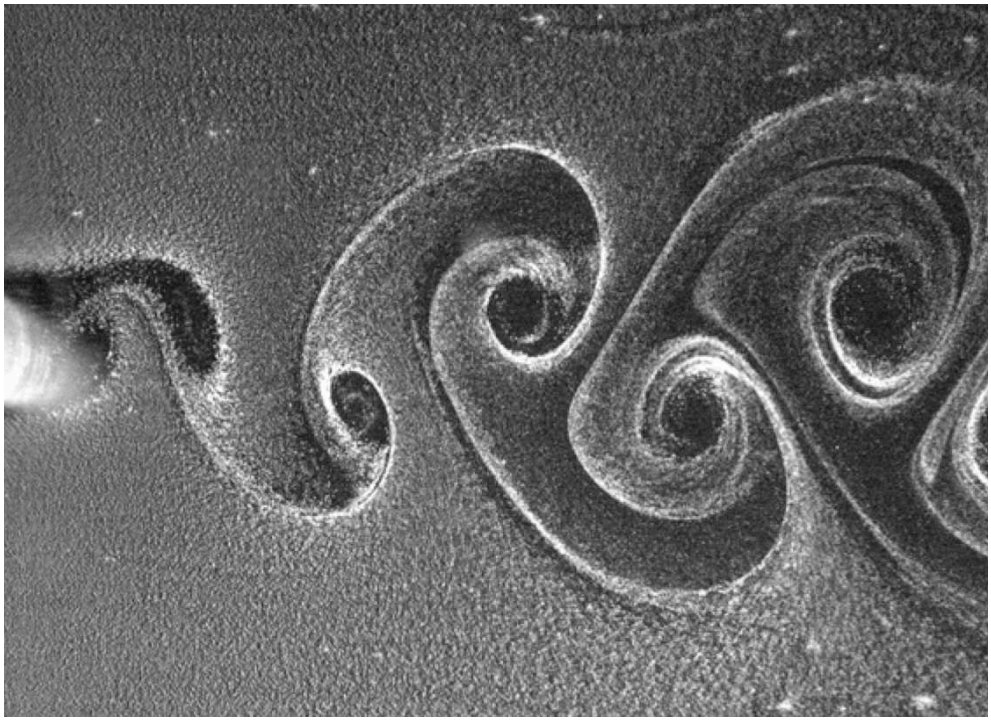


Figure 19: *The image shows the von Kármán Vortex street behind a 6.35 mm diameter circular cylinder in water at Reynolds number of 168. The visualization was done using hydrogen bubble technique.* `Credit:` *Sanjay Kumar & George Laughlin, Department of Engineering, The University of Texas at Brownsville*

The following movie shows a $\mathcal{R} = 250$ flow past a cylinder. Initially one can witness separation, and the creation of two counter-rotating vortices, which then suddenly become 'unstable', resulting in the von Kármán vortex street:

`http://www.youtube.com/watch?v=IDeGDFZSYo8`

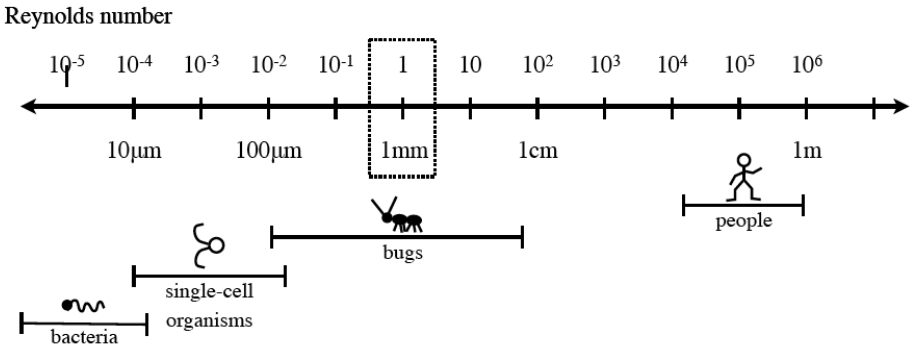


Figure 20: *Typical Reynolds numbers for various biological organisms. Reynolds numbers are estimated using the length scales indicated, the rule-of-thumb in the text, and material properties of water.*

Locomotion at Low-Reynolds number: Low Reynolds number corresponds to high kinetic visocisity *for a given U and L.* In this regime of 'creeping flow' the flow past an object is (nearly) time-reversible. Imagine trying to move (swim) in a highly viscous fluid (take honey as an example). If you try to do so by executing time-symmetric movements, you will not move. Instead, you need to think of a symmetry-breaking solution. Nature has found many solutions for this problem. If we make the simplifying "rule-of-thumb" assumption that an animal of size L meters moves roughly at a speed of $U = L$ meters per second (yes, this is very, very rough, but an ant *does* move close to 1 mm/s, and a human at roughly 1 m/s), then we have that $\mathcal{R} = UL/\nu \simeq L^2/\nu$. Hence, with respect to a fixed substance (say water, for which $\nu \sim 10^{-2}\text{cm}^2/s$), smaller organisms move at lower Reynolds number (effectively in a fluid of higher viscosity). Scaling down from a human to bacteria and single-cell organisms, the motion of the latter in water has $\mathcal{R} \sim 10^{-5} - 10^{-2}$. Understanding the locomotion of these organisms is a fascinating sub-branch of bio-physics.

Boundary Layers: Even when $\mathcal{R} \gg 1$, viscosity always remains important in thin boundary layers adjacent to any solid surface. This boundary layer must exist in order to satisfy the **no-slip boundary condition**. If the Reynolds number exceeds a critical value, the boundary layer becomes turbulent. Turbulent layes and their associated turbulent wakes exert a much bigger drag on moving bodies than their laminar counterparts.

Momentum Diffusion & Reynolds stress: This gives rise to an interesting phenomenon. Consider flow through a pipe. If you increase the viscosity (i.e., decrease $\mathcal{R}$), then it requires a larger force to achieve a certain flow rate (think of how much harder it is to push honey through a pipe compared to water). However, this trend is not monotonic. For sufficiently low viscosity (large $\mathcal{R}$), one finds that the trend reverses, and that is becomes harder again to push the fluid through the pipe. This is a consequence of turbulence, which causes **momentum diffusion** within the flow, which acts very much like viscosity. However, this momentum diffusion is not due to the **viscous stress tensor**, τ_{ij}, but rather to the **Reynolds stress tensor** R_{ij}. To understand the 'origin' of the Reynolds stress tensor,consider the following:

For a turbulent flow, $\vec{u}(t)$, it is advantageous to decompose each component of $\vec{u}$ into a 'mean' component, $\bar{u}_i$, and a 'fluctuating' component, u'_i, according to

$$u_i = \bar{u}_i + u'_i$$

This is knowns as the **Reynolds decomposition**. The 'mean' component can be a time-average, a spatial average, or an ensemble average, depending on the detailed characteristics of the flow. Note that this is reminiscent of how we decomposed the microscopic velocities of the fluid particles in a 'mean' velocity (describing the fluid elements) and a 'random, microscopic' velocity ($\vec{v} = \vec{u} + \vec{w}$).

Substituting this into the Navier-Stokes equation, and taking the average of that, we obtain what is known as the Reynolds Averaged Navier-Stokes equation, or RANS for short:

$$\frac{\partial \bar{u}_i}{\partial t} + \bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j} = \frac{1}{\rho} \frac{\partial}{\partial x_j} \left[\overline{\sigma}_{ij} - \rho \overline{u'_i u'_j} \right]$$

where, for simplicity, we have ignored gravity (the $\nabla\Phi$-term). This equation looks identical to the Navier-Stokes equation (in absence of gravity), except for the $-\rho\overline{u'_i u'_j}$ term, which is what we call the Reynolds stress tensor:

$$R_{ij} = -\rho\overline{u_i' u_j'}$$

Note that $\overline{u_i' u_j'}$ means the same averaging (time, space or ensemble) as above, but now for the product of u_i' and u_j'. Note that $\bar{u_i'} = 0$, by construction. However, the expectation value for the product of u_i' and u_j' is generally not. As is evident from the equation, the **Reynolds stresses** (which reflect momentum diffusion due to turbulence) act in exactly the same way as the **viscous stresses**. However, they are only present when the flow is turbulent. RANS-modelling is an approximate method used often in engineering to account for turbulence without having to simulate it in detail.

Note also that the Reynolds stress tensor is related to the **two-point correlation tensor**

$$\xi_{ij}(\vec{r}) \equiv \overline{u_i'(\vec{x}, t)\, u_j'(\vec{x} + \vec{r}, t)}$$

in the sense that $R_{ij} = \xi_{ij}(0)$. At large separations, $\vec{r}$, the fluctuating velocities will be uncorrelated so that $\lim_{r\to\infty} \xi_{ij} = 0$. But on smaller scales the fluctuating velocities will be correlated, and there will be a 'characteristic' scale associated with these correlations, called the **correlation length**.

Turbulence: Turbulence is still considered as one of the last "unsolved problems of classical physics" [Richard Feynman]. What we technically mean by this is that we do not yet know how to calculate $\xi_{ij}(\vec{r})$ (and higher order correlation functions, like the three-point, four-point, etc) in a particular situation from a fundamental theory. Salmon (1998) nicely sums up the challenge of defining **turbulence**:

> Every aspect of turbulence is controversial. Even the definition of fluid turbulence is a subject of disagreement. However, nearly everyone would agree with some elements of the following description:
>
> - Turbulence requires the presence of vorticity; irrotational flow is smooth and steady to the extent that the boundary conditions permit.
> - Turbulent flow has a complex structure, involving a broad range of space and time scales.
> - Turbulent flow fields exhibit a high degree of apparent randomness and disorder. However, close inspection often reveals the presence of embedded cohererent flow structures

- Turbulent flows have a high rate of viscous energy dissipation.
- Advected tracers are rapidly mixed by turbulent flows.

However, one further property of turbulence seems to be more fundamental than all of these because it largely explains why turbulence demands a statistical treatment...turbulence is chaotic.

The following is a brief, qualitative description of turbulence:

Turbulence kicks in at sufficiently high Reynolds number (typically $\mathcal{R} > 10^3 - 10^4$). Turbulent flow is characterized by irregular and seemingly random motion. Large vortices (called **eddies**) are created. These contain a large amount of kinetic energy. Due to **vortex stretching** these eddies are stretched thin until they 'break up' in smaller eddies. This results in a **cascade** in which the turbulent energy is transported from large scales to small scales. This cascade is largely **inviscid**, conserving the total turbulent energy. However, once the length scale of the eddies becomes comparable to the mean free path of the particles, the energy is dissipated; the kinetic energy associated with the eddies is transformed into internal energy. The scale at which this happens is called the **Kolmogorov length scale**. The length scales between the scale of turbulence 'injection' and the Kolmogorov length scale at which it is dissipated is called the **inertial range**. Over this inertial range turbulence is believed/observed to be scale invariant. The ratio between the injection scale, L, and the dissipation scale, l, is proportional to the **Reynolds number** according to $L/l \propto \mathcal{R}^{3/4}$. Hence, two turbulent flows that look similar on large scales (comparable L), will dissipate their energies on different scales, l, if their Reynolds numbers are different.

Molecular clouds: an example of turbulence in astrophysics are molecular clouds. These are gas clouds of masses $10^5 - 10^6 M_\odot$, densities $n_H \sim 100 - 500\,\mathrm{cm}^{-3}$, and temperatures $T \sim 10$K. They consist mainly of molecular hydrogen and are the main sites of **star formation**. Observations show that their velocity linewidths are $\sim 6-10$km/s, which is much higher than their sound speed ($c_\mathrm{s} \sim 0.2$km/s). Hence, they are supported against (gravitational) collapse by **supersonic turbulence**. On small scales, however, the turbulent motions compress the gas to high enough densities that stars can form. A numerical simulation of a molecular cloud with supersonic turbulence is available here:
`http://www.youtube.com/watch?v=3z9ZKAkbMhY`

CHAPTER 17

Sound Waves

If a (compressible) fluid in equilibrium is perturbed, and the perturbation is sufficiently small, the perturbation will propagate through the fluid as a **sound wave** (aka acoustic wave), which is a mechanical, longitudinal wave (i.e, a displacement in the same direction as that of propagation). Note that sound waves are perpetuated by the two-body collisions among the particles; this is very different from some of the waves that we encounter in a collisionless plasma (e.g., Langmuir waves, Alvèn waves) or a collisionless galaxy (e.g., spiral waves). These are perpetuated by **collective effects**, as will be discussed in Parts IV and V of these lecture notes.

If the perturbation is small, we may assume that the velocity gradients are so small that viscous effects are negligble (i.e., we can set $\nu = 0$). In addition, we assume that the time scale for conductive heat transport is large, so that energy exchange due to conduction can also safely be ignored. In the absence of these dissipative processes, the wave-induced changes in gas properties are **adiabatic**.

Before proceeding, let us examine the **Reynold's number** of a (propagating) sound wave. Using that $\mathcal{R} = U\,L/\nu$, and setting $U = c_{\rm s}$ (the typical velocity involved is the sound speed, to be defined below), $L = \lambda$ (the characteristic scale of the flow is the wavelength of the acoustic wave), we have that $\mathcal{R} = \lambda\, c_{\rm s}/\nu$. Using the expressions for the viscosity $\mu = \nu\rho$ from the constitutive relations in Chapter 11, we see that $\nu \propto \lambda_{\rm mfp}\, c_{\rm s}$. Hence, we have that

$$\mathcal{R} \equiv \frac{U\,L}{\nu} \propto \frac{\lambda}{\lambda_{\rm mfp}}$$

Thus, as long as the wave-length of the acoustic wave is much larger than the mean-free path of the fluid particles, we have that the Reynolds number is large, and thus that viscosity and conduction can be ignored.

Let $(\rho_0, P_0, \vec{u}_0)$ be a **uniform, equilibrium solution** of the Euler fluid equations (i.e., ignore viscosity). Also, in what follows we will ignore gravity (i.e., $\nabla\Phi = 0$).

Uniformity implies that $\nabla\rho_0 = \nabla P_0 = \nabla\vec{u}_0 = 0$. In addition, since the only allowed motion is uniform motion of the entire system, we can always use a Galilean coordinate transformation so that $\vec{u}_0 = 0$, which is what we adopt in what follows.

Substitution into the continuity and momentum equations, one obtains that $\partial\rho_0/\partial t = \partial\vec{u}_0/\partial t = 0$, indicative of an **equilibrium** solution as claimed.

Perturbation Analysis: Consider a small perturbation away from the above equilibrium solution:

$$\begin{aligned} \rho_0 &\rightarrow \rho_0 + \rho_1 \\ P_0 &\rightarrow P_0 + P_1 \\ \vec{u}_0 &\rightarrow \vec{u}_0 + \vec{u}_1 = \vec{u}_1 \end{aligned}$$

where $|\rho_1/\rho_0| \ll 1$, $|P_1/P_0| \ll 1$ and $\vec{u}_1$ is small (compared to the sound speed, to be derived below).

Substitution in the **continuity** and **momentum** equations yields

$$\begin{aligned} \frac{\partial(\rho_0+\rho_1)}{\partial t} + \nabla\cdot(\rho_0+\rho_1)\vec{u}_1 &= 0 \\ \frac{\partial\vec{u}_1}{\partial t} + (\vec{u}_1\cdot\nabla)\vec{u}_1 &= -\frac{\nabla(P_0+P_1)}{(\rho_0+\rho_1)} \end{aligned}$$

which, using that $\nabla\rho_0 = \nabla P_0 = \nabla\vec{u}_0 = 0$ reduces to

$$\begin{aligned} \frac{\partial\rho_1}{\partial t} + \rho_0\nabla\cdot\vec{u}_1 + \nabla\cdot(\rho_1\vec{u}_1) &= 0 \\ \frac{\partial\vec{u}_1}{\partial t} + \frac{\rho_1}{\rho_0}\frac{\partial\vec{u}_1}{\partial t} + (\vec{u}_1\cdot\nabla)\vec{u}_1 + \frac{\rho_1}{\rho_0}(\vec{u}_1\cdot\nabla)\vec{u}_1 &= -\frac{\nabla P_1}{\rho_0} \end{aligned}$$

The latter follows from first multiplying the momentum equations with $(\rho_0+\rho_1)/\rho_0$. Note that we don't need to consider the **energy equation**; this is because (i) we have assumed that conduction is negligble, and (ii) the disturbance is adiabatic (meaning $\mathrm{d}Q = 0$, and there is thus no heating or cooling).

Next we **linearize** these equations, which means we use that the perturbed values are all small such that terms that contain products of two or more of these quantities

are always negligible compared to those that contain only one such quantity. Hence, the above equations reduce to

$$\begin{aligned} \frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot \vec{u}_1 &= 0 \\ \frac{\partial \vec{u}_1}{\partial t} + \frac{\nabla P_1}{\rho_0} &= 0 \end{aligned}$$

These equations describe the evolution of perturbations in an **inviscid** and **uniform** fluid. As always, these equations need an additional equation for closure. As mentioned above, we don't need the **energy equation**: instead, we can use that the flow is adiabatic, which implies that $P \propto \rho^\gamma$.

Using Taylor series expansion, we then have that

$$P(\rho_0 + \rho_1) = P(\rho_0) + \left(\frac{\partial P}{\partial \rho}\right)_0 \rho_1 + \mathcal{O}(\rho_1^2)$$

where we have used $(\partial P/\partial \rho)_0$ as shorthand for the partial derivative of $P(\rho)$ at $\rho = \rho_0$. And since the flow is isentropic, we have that the partial derivative is for constant entropy. Using that $P(\rho_0) = P_0$ and $P(\rho_0 + \rho_1) = P_0 + P_1$, we find that, when linearized,

$$P_1 = \left(\frac{\partial P}{\partial \rho}\right)_0 \rho_1$$

Note that $P_1 \neq P(\rho_1)$; rather P_1 is the perturbation in pressure associated with the perturbation ρ_1 in the density.

Substitution in the fluid equations of our perturbed quantities yields

$$\begin{aligned} \frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot \vec{u}_1 &= 0 \\ \frac{\partial \vec{u}_1}{\partial t} + \left(\frac{\partial P}{\partial \rho}\right)_0 \frac{\nabla \rho_1}{\rho_0} &= 0 \end{aligned}$$

Taking the partial time derivative of the above **continuity** equation, and using that $\partial \rho_0/\partial t = 0$, gives

$$\frac{\partial^2 \rho_1}{\partial t^2} + \rho_0 \nabla \cdot \frac{\partial \vec{u}_1}{\partial t} = 0$$

Substituting the above **momentum equation**, and realizing that $(\partial P/\partial\rho)_0$ is a constant, then yields

$$\boxed{\frac{\partial^2\rho_1}{\partial t^2} - \left(\frac{\partial P}{\partial\rho}\right)_0 \nabla^2\rho_1 = 0}$$

which we recognize as a **wave equation**, whose solution is a **plane wave**:

$$\rho_1 \propto e^{i(\vec{k}\cdot\vec{x}-\omega t)}$$

with $\vec{k}$ the **wavevector**, $k = |\vec{k}| = 2\pi/\lambda$ the **wavenumber**, λ the **wavelength**, $\omega = 2\pi\nu$ the **angular frequency**, and ν the **frequency**.

To gain some insight, consider the 1D case: $\rho_1 \propto e^{i(kx-\omega t)} \propto e^{ik(x-v_{\rm p}t)}$, where we have defined the **phase velocity** $v_{\rm p} \equiv \omega/k$. This is the velocity with which the wave pattern propagates through space. For our perturbation of a compressible fluid, this phase velocity is called the **sound speed**, $c_{\rm s}$. Substituting the solution $\rho_1 \propto e^{i(kx-\omega t)}$ into the wave equation, we see that

$$\boxed{c_{\rm s} = \frac{\omega}{k} = \sqrt{\left(\frac{\partial P}{\partial\rho}\right)_s}}$$

where we have made it explicit that the flow is assumed to be isentropic. Note that the partial derivative is for the unperturbed medium. This sound speed is sometimes called the **adiabatic speed of sound**, to emphasize that it relies on the assumption of an adiabatic perturbation. If the fluid is an ideal gas, then

$$c_{\rm s} = \sqrt{\gamma\,\frac{k_{\rm B}T}{\mu\,m_{\rm p}}}$$

which shows that the adiabatic sound speed of an ideal fluid increases with temperature.

We can repeat the above derivation by relaxing the assumption of isentropic flow, and assuming instead that (more generally) the flow is **polytropic**. In that case, $P \propto \rho^\Gamma$, with Γ the **polytropic index** (Note: a polytropic EoS is an example of a **barotropic** EoS). The only thing that changes is that now the sound speed becomes

$$c_{\rm s} = \sqrt{\frac{\partial P}{\partial\rho}} = \sqrt{\Gamma\,\frac{P}{\rho}}$$

which shows that the sound speed is larger for a stiffer EoS (i.e., a larger value of Γ).

Note also that, for our barotropic fluid, the sound speed is independent of ω. This implies that all waves move equally fast; the shape of a wave packet is preserved as it moves. We say that an ideal (inviscid) fluid with a barotropic EoS is a **non-dispersive medium**.

To gain further insight, let us look once more at the (1D) solution for our perturbation:

$$\rho_1 \propto e^{i(kx-\omega t)} \propto e^{ikx}\, e^{-i\omega t}$$

Recalling **Euler's formula** ($e^{i\theta} = \cos\theta + i\sin\theta$), we see that:

- The e^{ikx} part describes a periodic, spatial oscillation with wavelength $\lambda = 2\pi/k$.
- The $e^{-i\omega t}$ part describes the time evolution:
  - If ω is **real**, then the solution describes a **sound wave** which propagates through space with a sound speed $c_{\rm s}$.
  - If ω is **imaginary** then the perturation is either exponentially growing ('unstable') or decaying ('damped') with time.

We will return to this in Chapter 19, when we discuss the **Jeans stability criterion**.

As discussed above acoustic waves result from disturbances in a compressible fluid. These disturbances may arise from objects being moved through the fluid. However, sound waves can also be sourced by fluid motions themselves. A familiar example is the noise from jet-engines; the noise emenates from the turbulent wake created by engines. In astrophysics, turbulence will also typically create sound waves. In general these sound waves will not have an important impact on the physics. A potential exception is the heating of the ICM by sound waves created by turbulent wakes created by AGN feedback. Appendix L discusses this topic, deriving an **inhomogeneous wave equation**, known as the **Lighthill equation** that describes how fluid motion can source sound waves. This equation can also be used to include the effects of **viscosity** and **conductivity**. As we stated at the beginning of this chapter, these transport mechanisms can be ignored as long as the wavelength of the sound wave

is much larger than the mean-free path of the fluid particles. However, in the long run viscosity and conductivity will cause some level of momentum dissipation and energy diffusion, which will cause the sound waves to die out.

CHAPTER 18

Shocks

When discussing sound waves in the previous chapter, we considered small (linear) perturbations. In this Chapter we consider the case in which the perturbations are large (non-linear). Typically, a large disturbance results in an abrupt **discontinuity** in the fluid, called a **shock**. Note: not all discontinuities are shocks, but all shocks are discontinuities.

Consider a **polytropic** EoS:

$$P = P_0 \left(\frac{\rho}{\rho_0}\right)^{\Gamma}$$

The sound speed is given by

$$c_{\rm s} = \left(\frac{\partial P}{\partial \rho}\right)^{1/2} = \sqrt{\Gamma \frac{P}{\rho}} = c_{\rm s,0} \left(\frac{\rho}{\rho_0}\right)^{(\Gamma-1)/2}$$

If $\Gamma = 1$, i.e., the EoS is **isothermal**, then the sound speed is a constant, independent of density or pressure. However, if $\Gamma \neq 1$, then the sound speed varies with the local density. An important example, often encountered in (astro)physics is the **adiabatic** EoS, for which $\Gamma = \gamma$ ($\gamma = 5/3$ for a mono-atomic gas). In that case we have that $c_{\rm s}$ increases with density (and pressure, and temperature).

In our discussion of sound waves (Chapter 17), we used perturbation theory, in which we neglected the $\vec{u}_1 \cdot \nabla \vec{u}_1$ term. However, when the perturbations are not small, this term is no longer negligble, and causes non-linearities to develop. The most important of those, is the fact that the sound speed itself varies with density (as we have seen above). This implies that the wave-form of the acoustic wave changes with time; the wave-crest is moving faster than the wave-trough, causing an overall steepening of the wave-form. This steepening continues until the wave-crest tries to overtake the wave-trough, which is not allowed, giving rise to a shock front.

Mach Number: if v is the flow speed of the fluid, and $c_{\rm s}$ is the sound speed, then

the Mach number of the flow is defined as

$$\boxed{\mathcal{M} = \frac{v}{c_{\rm s}}}$$

Note: *simply accelerating a flow to supersonic speeds does* **not** *necessarily generate a shock*. Shocks only arise when an obstruction in the flow causes a deceleration of fluid moving at supersonic speeds. The reason is that disturbances cannot propagate upstream, so that the flow cannot 'adjust itself' to the obstacle because there is no way of propagating a signal (which always goes at the sound speed) in the upstream direction. Consequently, the flow remains undisturbed until it hits the obstacle, resulting in a discontinuous change in flow properties; a shock.

Structure of a Shock: Fig. 21 shows the structure of a planar shock. The shock has a finite, non-zero width (typically a few mean-free paths of the fluid particles), and separates the 'up-stream', pre-shocked gas, from the 'down-stream', shocked gas.

For reasons that will become clear in what follows, it is useful to split the downstream region in two sub-regions; one in which the fluid is out of thermal equilibrium, with net cooling $\mathcal{L} > 0$, and, further away from the shock, a region where the downstream gas is (once again) in thermal equilibrium (i.e., $\mathcal{L} = 0$). If the transition between these two sub-regions falls well outside the shock (i.e., if $x_3 \gg x_2$) the shock is said to be **adiabatic**. In that case, we can derive a relation between the upstream (pre-shocked) properties (ρ_1, P_1, T_1, u_1) and the downstream (post-shocked) properties (ρ_2, P_2, T_2, u_2); these relations are called the **Rankine-Hugoniot jump conditions**. Linking the properties in region three (ρ_3, P_3, T_3, u_3) to those in the pre-shocked gas is in general not possible, except in the case where $T_3 = T_1$. In this case one may consider the shock to be **isothermal**.

Rankine-Hugoniot jump conditions: We now derive the relations between the up- and down-stream quantities, under the assumption that the shock is adiabatic. Consider a rectangular volume V that encloses part of the shock; it has a thickness $\mathrm{d}x > (x_2 - x_1)$ and is centered in the x-direction on the middle of shock. At fixed x the volume is bounded by an area A. If we ignore variations in ρ and $\vec{u}$ in the y-

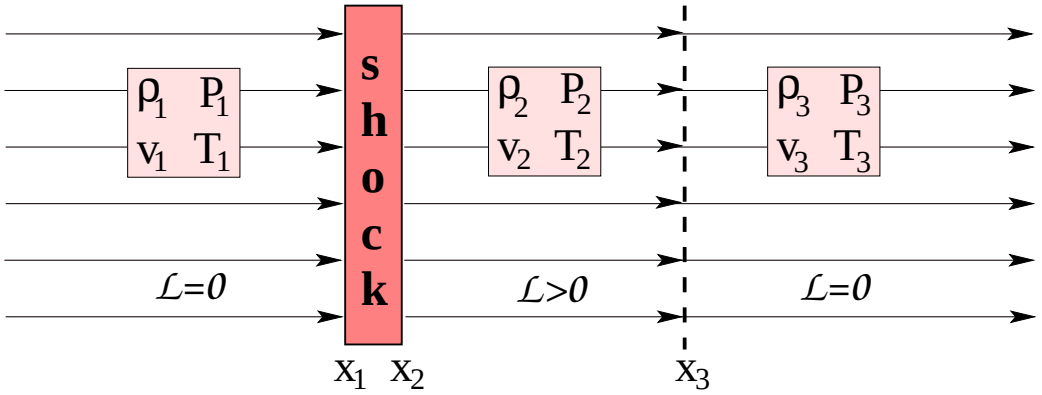


Figure 21: *Structure of a planar shock.*

and z-directions, the **continuity equation** becomes

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho\, u_x) = 0$$

If we integrate this equation over our volume V we obtain

$$\begin{aligned} & \int\int\int \frac{\partial \rho}{\partial t}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z + \int\int\int \frac{\partial}{\partial x}(\rho u_x)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = 0 \\ \Leftrightarrow \quad & \frac{\partial}{\partial t}\int \rho\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z + A\int \frac{\partial}{\partial x}(\rho u_x)\,\mathrm{d}x = 0 \\ \Leftrightarrow \quad & \frac{\partial M}{\partial t} + A\int \mathrm{d}(\rho u_x) = 0 \end{aligned}$$

Since there is no mass accumulation in the shock, and mass does not dissapear in the shock, we have that

$$\rho u_x|_{+\mathrm{d}x/2} = \rho u_x|_{-\mathrm{d}x/2}$$

In terms of the upstream (index 1) and downstream (index 2) quantities:

$$\boxed{\rho_1\, u_1 = \rho_2\, u_2}$$

This equation describes **mass conservation** across a shock.

The momentum equation in the x-direction, ignoring viscosity, is given by

$$\frac{\partial}{\partial t}(\rho\, u_x) = -\frac{\partial}{\partial x}(\rho\, u_x\, u_x + P) - \rho\frac{\partial \Phi}{\partial x}$$

Integrating this equation over V and ignoring any gradient in Φ across the shock, we obtain

$$\boxed{\rho_1\, u_1^2 + P_1 = \rho_2\, u_2^2 + P_2}$$

This equation describes how the shock **converts ram pressure into thermal pressure**.

Finally, applying the same to the **energy equation** under the assumption that the shock is adiabatic (i.e., $\mathrm{d}Q/\mathrm{d}t = 0$), one finds that $(E + P)u$ has to be the same on both sides of the shock, i.e.,

$$\left[\frac{1}{2}u^2 + \Phi + \varepsilon + \frac{P}{\rho}\right]\, \rho\, u = \text{constant}$$

We have already seen that $\rho\, u$ is constant. Hence, if we once more ignore gradients in Φ across the shock, we obtain that

$$\boxed{\frac{1}{2}u_1^2 + \varepsilon_1 + P_1/\rho_1 = \frac{1}{2}u_2^2 + \varepsilon_2 + P_2/\rho_2}$$

This equation describes how the shock **converts kinetic energy into enthalpy**. Qualitatively, a shock converts an ordered flow upstream into a disordered (hot) flow downstream.

The three equations in the rectangular boxes are known as the **Rankine-Hugoniot (RH) jump conditions for an adiabatic shock**. Using straightforward but tedious algebra, these RH jump conditions can be written in a more useful form using the **Mach number** $\mathcal{M}_1$ of the upstream gas:

$$\begin{aligned}
\frac{\rho_2}{\rho_1} &= \frac{u_1}{u_2} = \left[\frac{1}{\mathcal{M}_1^2} + \frac{\gamma-1}{\gamma+1}\left(1 - \frac{1}{\mathcal{M}_1^2}\right)\right]^{-1} \\
\frac{P_2}{P_1} &= \frac{2\gamma}{\gamma+1}\,\mathcal{M}_1^2 - \frac{\gamma-1}{\gamma+1} \\
\frac{T_2}{T_1} &= \frac{P_2\,\rho_2}{P_1\,\rho_1} = \frac{\gamma-1}{\gamma+1}\left[\frac{2}{\gamma+1}\left(\gamma\mathcal{M}_1^2 - \frac{1}{\mathcal{M}_1^2}\right) + \frac{4\gamma}{\gamma-1} - \frac{\gamma-1}{\gamma+1}\right]
\end{aligned}$$

Here we have used that for an **ideal gas**

$$P = (\gamma - 1)\,\rho\,\varepsilon = \frac{k_{\rm B}\,T}{\mu\,m_{\rm p}}\,\rho$$

Given that $\mathcal{M}_1 > 1$, we see that $\rho_2 > \rho_1$ (shocks **compress**), $u_2 < u_1$ (shocks **decelerate**), $P_2 > P_1$ (shocks **increase pressure**), and $T_2 > T_1$ (shocks **heat**). The latter may seem surprising, given that the shock is considered to be **adiabatic**: although the process has been adiabatic, in that $\mathrm{d}Q/\mathrm{d}t = 0$, the gas **has** changed its adiabat; its entropy has increased as a consequence of the shock converting kinetic energy into thermal, internal energy. In general, in the presence of **viscosity**, a change that is adiabatic does not imply that the states before and after are simply linked by the relation $P = K\,\rho^{\gamma}$, with K some constant. Shocks are always viscous, which causes K to change across the shock, such that the entropy increases; it is this aspect of the shock that causes **irreversibility**, thus defining an "arrow of time".

Back to the RH jump conditions: in the limit $\mathcal{M}_1 \gg 1$ we have that

$$\rho_2 = \frac{\gamma + 1}{\gamma - 1}\,\rho_1 = 4\,\rho_1$$

where we have used that $\gamma = 5/3$ for a monoatomic gas. Thus, with an **adiabatic shock** you can achieve a maximum compression in density of a factor four! Physically, the reason why there is a maximal compression is that the pressure and temperature of the downstream fluid diverge as $\mathcal{M}_1^2$. This huge increase in downstream pressure inhibits the amount of compression of the downstream gas. However, this is only true under the assumption that the shock is **adiabatic**. The downstream, post-shocked gas is out of thermal equilibrium, and in general will be cooling (i.e., $\mathcal{L} > 0$). At a certain distance past the shock (i.e., when $x = x_3$ in Fig. 21), the fluid will re-establish **thermal equilibrium** (i.e., $\mathcal{L} = 0$). In some special cases, one can obtain the properties of the fluid in the new equilibrium state; one such case is the example of an **isothermal shock**, for which the downstream gas has the same temperature as the upstream gas (i.e., $T_3 = T_1$).

In the case of an **isothermal shock**, the first two **Rankine-Hugoniot jump con-**

ditions are still valid, i.e.,

$$\begin{aligned} \rho_1\, u_1 &= \rho_3\, u_3 \\ \rho_1\, u_1^2 + P_1 &= \rho_3\, u_3^2 + P_3 \end{aligned}$$

However, the third condition, which derives from the energy equation, is no longer valid. After all, in deriving that one we had assumed that the shock was adiabatic. In the case of an isothermal shock we have to replace the third RH jump condition with $T_1 = T_3$. The latter implies that $c_{\rm s}^2 = P_3/\rho_3 = P_1/\rho_1$, and allows us to rewrite the second RH condition as

$$\begin{aligned} & \rho_1(u_1^2 + c_{\rm s}^2) = \rho_3(u_3^2 + c_{\rm s}^2) \\ \Leftrightarrow \quad & u_1^2 - \tfrac{\rho_3}{\rho_1} u_3^2 = \tfrac{\rho_3}{\rho_1} c_{\rm s}^2 - c_{\rm s}^2 \\ \Leftrightarrow \quad & u_1^2 - u_1 u_3 = (\tfrac{u_1}{u_3} - 1)\, c_{\rm s}^2 \\ \Leftrightarrow \quad & u_1 u_3 (u_1 - u_3) = (u_1 - u_3)\, c_{\rm s}^2 \\ \Leftrightarrow \quad & c_{\rm s}^2 = u_1 u_3 \end{aligned}$$

Here the second step follows from using the first RH jump condition. If we now substitute this result back into the first RH jump condition we obtain that

$$\frac{\rho_3}{\rho_1} = \frac{u_1}{u_3} = \left(\frac{u_1}{c_{\rm s}}\right)^2 = \mathcal{M}_1^2$$

Hence, in the case of **isothermal shock** (or an adiabatic shock, but sufficiently far behind the shock in the downstream fluid), we have that there is no restriction to how much compression the shock can achieve; depending on the Mach number of the shock, the compression can be huge.

Figure 22: *An actual example of a supernova blastwave. The red colors show the optical light emitted by the supernova ejecta, while the green colors indicate X-ray emission coming from the hot bubble of gas that has been shock-heated when the blast-wave ran over it.*

Supernova Blastwave: An important example of a shock in astrophysics are supernova blastwaves. When a supernova explodes, it blasts a shell of matter (the 'ejecta') at high (highly supersonic) speed into the surrounding medium. The kinetic energy of this shell material is roughly $E_{\rm SN} = 10^{51}$ erg. This is roughly 100 times larger than the amount of energy emitted in radiation by the supernova explosion (which is what we 'see'). For comparison, the entire Milky Way has a luminosity of $\sim 10^{10}\,{\rm L}_\odot \simeq 4 \times 10^{43}\,{\rm erg} s^{-1}$, which amounts to an energy emitted by stars over an entire year that is of the order of 1.5×10^{51} erg. Hence, the kinetic energy released by a single SN is larger than the energy radiated by stars, by the entire galaxy, in an entire year!
The mass of the ejecta is of the order of 1 Solar mass, which implies (using that $E_{\rm SN} = \frac{1}{2} M_{\rm ej} v_{\rm ej}^2$), that the ejecta have a velocity of $\sim 10,000\,{\rm km\,s^{-1}}$!! Initially, this shell material has a mass that is much larger than the mass of the surroundings swept up by the shock, and to lowest order the shell undergoes free expansion. This phase is therefore called the **free-expansion phase**. As the shock moves out, it engulves more and more interstellar material, which is heated (and compressed) by the shock. Hence, the interior of the shell (=shock) is a super-hot bubble of over-pressurized

gas, which 'pushes' the shock outwards. As more and more material is swept-up, and accelerated outwards, the mass of the shell increases, which causes the velocity of the shell to decelerate. At the early stages, the cooling of the hot bubble is negligble, and the blastwave is said to be in the **adiabatic phase**, also known as the **Sedov-Taylor phase**. At some point, though, the hot bubble starts to cool, radiating away the kinetic energy of the supernova, and lowering the interior pressure up to the point that it no longer pushes the shell outwards. This is called the **radiative phase**. From this point on, the shell expands purely by its inertia, being slowed down by the work it does against the surrounding material. This phase is called the **snow-plow phase**. Ultimately, the velocity of the shell becomes comparable to the sound speed of the surrounding material, after which it continues to move outward as a sound wave, slowly dissipating into the surroundings.

During the adiabatic phase, we can use a simple dimensional analysis to solve for the evolution of the shock radius, $r_{\rm sh}$, with time. Since the only physical parameters that can determine $r_{\rm sh}$ in this adiabatic phase are time, t, the initial energy of the SN explosion, ε_0, and the density of the surrounding medium, ρ_0, we have that

$$r_{\rm sh} = f(t, \varepsilon_0, \rho_0) = A t^{\eta}\, \varepsilon_0^{\alpha}\, \rho_0^{\beta}$$

It is easy to check that there is only one set of values for η, α and β for which the product on the right has the dimensions of length (which is the dimension of $r_{\rm sh}$. This solution has $\eta = 2/5$, $\alpha = 1/5$ and $\beta = -1/5$, such that

$$r_{\rm sh} = A \left(\frac{\varepsilon}{\rho_0}\right)^{1/5} t^{2/5}$$

and thus

$$v_{\rm sh} = \frac{{\rm d}r_{\rm sh}}{{\rm d}t} = \frac{2A}{5} \left(\frac{\varepsilon}{\rho_0}\right)^{1/5} t^{-3/5}$$

which shows that indeed the shock decelerates as it moves outwards.

CHAPTER 19

Fluid Instabilities

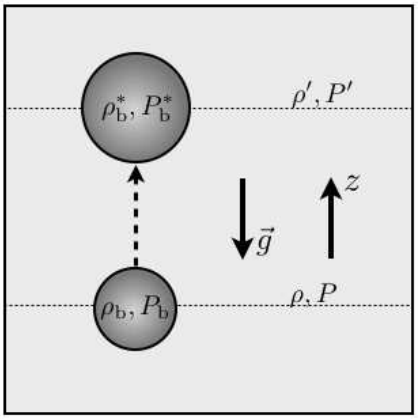
In this Chapter we discuss the following instabilities:

- convective instability (Schwarzschild criterion)
- interface instabilities (Rayleigh-Taylor & Kelvin-Helmholtz)
- gravitational instability (Jeans criterion)
- thermal instability (Field criterion)

Convective Instability: In astrophysics we often need to consider fluids heated from "below" (e.g., stars, Earth's atmosphere where Sun heats surface, etc.) This results in a temperature gradient: hot at the base, colder further "up". Since warmer fluids are more buoyant ('lighter'), they like to be further up than colder ('heavier') fluids. The question we need to address is under what conditions this adverse temperature gradient becomes unstable, developing "overturning" motions known as thermal **convection**.

Consider a blob with density $\rho_{\rm b}$ and pressure $P_{\rm b}$ embedded in an ambient medium of density ρ and pressure P. Suppose the blob is displaced by a small distance δz upward. After the displacement the blob will have conditions $(\rho_{\rm b}^*, P_{\rm b}^*)$ and its new ambient medium is characterized by (ρ', P'), where

$$\rho' = \rho + \frac{{\rm d}\rho}{{\rm d}z}\,\delta z \qquad\qquad P' = P + \frac{{\rm d}P}{{\rm d}z}\,\delta z$$



Initially the blob is assumed to be in **mechanical** and **thermal equilibrium** with its ambient medium, so that $\rho_{\rm b} = \rho$ and $P_{\rm b} = P$. After the displacement the blob needs to re-establish a new mechanical and thermal equilibrium. In general, the time scale on which it re-establishes mechanical (pressure) equilibrium is the sound crossing time, $\tau_{\rm s}$, while re-establishing thermal equilibrium

proceeds much slower, on the conduction time, $\tau_{\rm c}$. Given that $\tau_{\rm s} \ll \tau_{\rm c}$ we can assume that $P_{\rm b}^* = P'$, and treat the displacement as **adiabatic**. The latter implies that the process can be described by an adiabatic EoS: $P \propto \rho^\gamma$. Hence, we have that

$$\rho_{\rm b}^* = \rho_{\rm b} \left(\frac{P_{\rm b}^*}{P_{\rm b}}\right)^{1/\gamma} = \rho_{\rm b} \left(\frac{P'}{P}\right)^{1/\gamma} = \rho_{\rm b} \left[1 + \frac{1}{P}\frac{{\rm d}P}{{\rm d}z}\,\delta z\right]^{1/\gamma}$$

In the limit of small displacements δz, we can use Taylor series expansion to show that, to first order,

$$\rho_{\rm b}^* = \rho + \frac{\rho}{\gamma\, P}\,\frac{{\rm d}P}{{\rm d}z}\,\delta z$$

where we have used that initially $\rho_{\rm b} = \rho$, and that the Taylor series expansion, $f(x) \simeq f(0)+f'(0)x+\frac{1}{2}f''(0)x^2+...$, of $f(x) = [1+x]^{1/\gamma}$ is given by $f(x) \simeq 1+\frac{1}{\gamma}x+....$ Suppose we have a stratified medium in which ${\rm d}\rho/{\rm d}z < 0$ and ${\rm d}P/{\rm d}z < 0$. In that case, if $\rho_{\rm b}^* > \rho'$ the blob will be heavier than its surrounding and it will sink back to its original position; the system is stable to convection. If, on the other hand, $\rho_{\rm b}^* < \rho'$ then the displacement has made the blob more buoyant, resulting in **instability**. Hence, using that $\rho' = \rho + ({\rm d}\rho/{\rm d}z)\,\delta z$ we see that stability requires that

$$\frac{{\rm d}\rho}{{\rm d}z} < \frac{\rho}{\gamma\, P}\,\frac{{\rm d}P}{{\rm d}z}$$

This is called the **Schwarzschild criterion for convective stability**.

It is often convenient to rewrite this criterion in a form that contains the temperature. Using that

$$\rho = \rho(P,T) = \frac{\mu\, m_{\rm p}}{k_{\rm B}T}P$$

it is straightforward to show that

$$\frac{{\rm d}\rho}{{\rm d}z} = \frac{\rho}{P}\,\frac{{\rm d}P}{{\rm d}z} - \frac{\rho}{T}\,\frac{{\rm d}T}{{\rm d}z}$$

Substitution in $\rho' = \rho + ({\rm d}\rho/{\rm d}z)\,\delta z$ then yields that

$$\rho_{\rm b}^* - \rho' = \left[-(1-\frac{1}{\gamma})\frac{\rho}{P}\frac{{\rm d}P}{{\rm d}z} + \frac{\rho}{T}\frac{{\rm d}T}{{\rm d}z}\right]\,\delta z$$

Since stability requires that $\rho_{\rm b}^* - \rho' > 0$, and using that $\delta z > 0$, ${\rm d}P/{\rm d}z < 0$ and ${\rm d}T/{\rm d}z < 0$ we can rewrite the above Schwarzschild criterion for stability as

$$\left|\frac{{\rm d}T}{{\rm d}z}\right| < \left(1-\frac{1}{\gamma}\right)\,\frac{T}{P}\,\left|\frac{{\rm d}P}{{\rm d}z}\right|$$

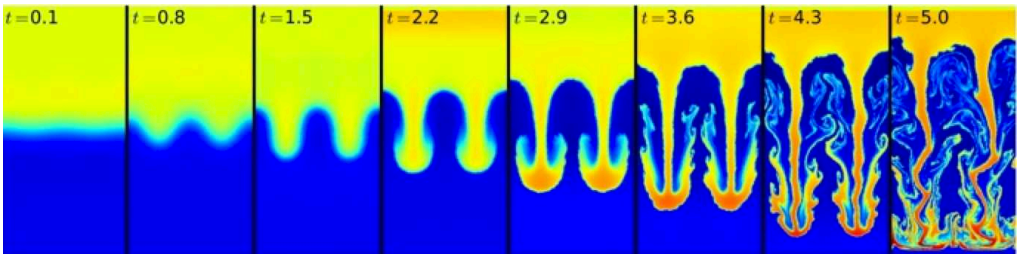


Figure 23: *Example of Rayleigh-Taylor instability in a hydro-dynamical simulation.*

This shows that if the temperature gradient becomes too large the system becomes convectively unstable: blobs will rise up until they start to loose their thermal energy to the ambient medium, resulting in convective energy transport that tries to "overturn" the hot (high entropy) and cold (low entropy) material. In fact, without any proof we mention that in terms of the **specific entropy**, s, one can also write the Schwarzschild criterion for convective stability as $\mathrm{d}s/\mathrm{d}z > 0$.

To summarize, the **Schwarzschild criterion for convective stability** is given by either of the following three expressions:

$$\left|\frac{\mathrm{d}T}{\mathrm{d}z}\right| < \left(1 - \frac{1}{\gamma}\right) \frac{T}{P} \left|\frac{\mathrm{d}P}{\mathrm{d}z}\right|$$

$$\frac{\mathrm{d}\rho}{\mathrm{d}z} < \frac{\rho}{\gamma\, P} \frac{\mathrm{d}P}{\mathrm{d}z}$$

$$\frac{\mathrm{d}s}{\mathrm{d}z} > 0$$

Rayleigh-Taylor Instability: The Rayleigh-Taylor (RT) instability is an instability of an interface between two fluids of different densities that occurs when one of the fluids is accelerated into the other. Examples include supernova explosions in which expanding core gas is accelerated into denser shell gas and the common terrestrial example of a denser fluid such as water suspended above a lighter fluid such as oil in the Earth's gravitational field.

It is easy to see where the RT instability comes from. Consider a fluid of density ρ_2 sitting on top of a fluid of density $\rho_1 < \rho_2$ in a gravitational field that is pointing in the downward direction. Consider a small perturbation in which the initially horizontal interface takes on a small amplitude, sinusoidal deformation. Since this implies moving a certain volume of denser material down, and an equally large volume of the lighter material up, it is immediately clear that the potential energy of this 'perturbed' configuration is lower than that of the initial state, and therefore energetically favorable. Simply put, the initial configuration is unstable to small deformations of the interface.

Stability analysis (i.e., perturbation analysis of the fluid equations) shows that the **dispersion relation** corresponding to the RT instability is given by

$$\omega = \pm i\, k \sqrt{\frac{g}{k}\, \frac{\rho_2 - \rho_1}{\rho_2 + \rho_1}}$$

where g is the gravitational acceleration, and the factor $(\rho_2 - \rho_1)/(\rho_2 + \rho_1)$ is called the **Atwood number**. Since the wavenumber of the perturbation $k > 0$ we see that ω is imaginary, which implies that the perturbations will grow exponentially (i.e., the system is unstable). If $\rho_1 > \rho_2$ though, ω is real, and the system is stable (perturbations to the interface propagate as waves).

Kelvin-Helmholtz Instability: the Kelvin-Helmholtz (KH) instability is an interface instability that arises when two fluids with different densities have a velocity difference across their interface. Similar to the RT instability, the KH instability manifests itself as a small wavy pattern in the interface which develops into turbulence and which causes **mixing**. Examples where KH instability plays a role are wind blowing over water, (astrophysical) jets, the cloud bands on Jupiter (in particular the famous red spot), and clouds of denser gas falling through the hot, low density intra-cluster medium (ICM).

Stability analysis (i.e., perturbation analysis of the fluid equations) shows that the the **dispersion relation** corresponding to the KH instability is given by

$$\frac{\omega}{k} = \frac{(\rho_1 u_1 + \rho_2 u_2) \pm i\,(u_1 - u_2)\,(\rho_1 \rho_2)^{1/2}}{\rho_1 + \rho_2}$$

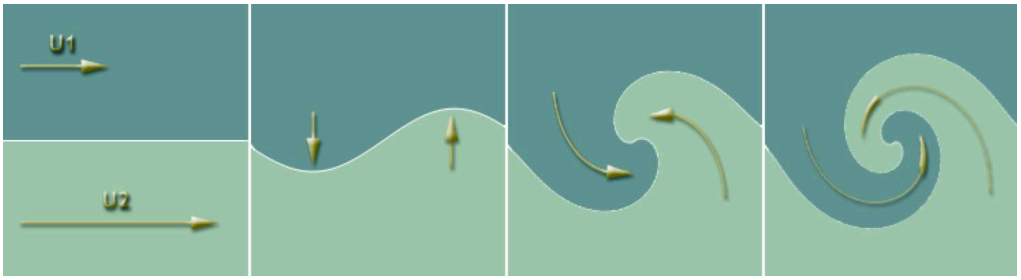


Figure 24: *Illustration of onset of Kelvin-Helmholtz instability*

Note that this dispersion relation has both real and imaginary parts, given by

$$\frac{\omega_{\rm R}}{k} = \frac{(\rho_1 u_1 + \rho_2 u_2)}{\rho_1 + \rho_2}$$

and

$$\frac{\omega_{\rm I}}{k} = (u_1 - u_2)\,\frac{(\rho_1 \rho_2)^{1/2}}{\rho_1 + \rho_2}$$

Since the imaginary part is non-zero, except for $u_1 = u_2$, we we have that, in principle, *any* velocity difference across an interface is KH unstable. In practice, **surface tension** can stabilize the short wavelength modes so that typically **KH instability** kicks in above some velocity treshold.

As an example, consider a cold cloud of radius $R_{\rm c}$ falling into a cluster of galaxies. The latter contains a hot intra-cluster medium (ICM), and as the cloud moves through this hot ICM, KH instabilities can develop on its surface. If the cloud started out at a large distance from the cluster with zero velocity, than at infall it has a velocity $v \sim v_{\rm esc} \sim c_{\rm s,h}$, where the latter is the sound speed of the hot ICM, assumed to be in hydrostatic equilibrium. Defining the cloud's overdensity $\delta = \rho_{\rm c}/\rho_{\rm h} - 1$, we can write the (imaginary part of the) dispersion relation as

$$\omega = \frac{\rho_{\rm h}\,(\rho_{\rm c}/\rho_{\rm h})^{1/2}}{\rho_{\rm h}[1 + (\rho_{\rm c}/\rho_{\rm h})]}\,c_{\rm s,h}\,k = \frac{(\delta+1)^{1/2}}{\delta+2}\,c_{\rm s,h}\,k$$

The mode that will destroy the cloud has $k \sim 1/R_{\rm c}$, so that the time-scale for cloud destruction is

$$\tau_{\rm KH} \simeq \frac{1}{\omega} \simeq \frac{R_{\rm c}}{c_{\rm s,h}}\frac{\delta+2}{(\delta+1)^{1/2}}$$

Assuming pressure equilibrium between cloud and ICM, and adopting the EoS of an ideal gas, implies that $\rho_{\rm h}\, T_{\rm h} = \rho_{\rm c}\, T_{\rm c}$, so that

$$\frac{c_{\rm s,h}}{c_{\rm s,c}} = \frac{T_{\rm h}^{1/2}}{T_{\rm c}^{1/2}} = \frac{\rho_{\rm c}^{1/2}}{\rho_{\rm h}^{1/2}} = (\delta+1)^{1/2}$$

Hence, one finds that the **Kelvin-Helmholtz time** for cloud destruction is

$$\tau_{\rm KH} \simeq \frac{1}{\omega} \simeq \frac{R_{\rm c}}{c_{\rm s,c}}\,\frac{\delta+2}{\delta+1}$$

Note that $\tau_{\rm KH} \sim \zeta(R_{\rm c}/c_{\rm s,c}) = \zeta\tau_{\rm s}$, with $\zeta = 1(2)$ for $\delta \gg 1 (\ll 1)$. Hence, the **Kelvin-Helmholtz instability** will typically destroy clouds falling into a hot "atmosphere" on a time scale between one and two **sound crossing times**, $\tau_{\rm s}$, of the cloud. Note, though, that magnetic fields and/or radiative cooling at the interface may stabilize the clouds.

Gravitational Instability: In our discussion of sound waves we used perturbation analysis to derive a dispersion relation $\omega^2 = k^2\, c_{\rm s}^2$. In deriving that equation we ignored gravity by setting $\nabla\Phi = 0$ (see Chapter 17). If you do not ignore gravity, then you add one more perturbed quantity; $\Phi = \Phi_0 + \Phi_1$ and one more equation, namely the **Poisson equation** $\nabla^2\Phi = 4\pi G\rho$.
It is not difficult to show that this results in a modified **dispersion relation**:

$$\boxed{\omega^2 = k^2\, c_{\rm s}^2 - 4\pi G\rho_0 = c_{\rm s}^2\,\left(k^2 - k_{\rm J}^2\right)}$$

where we have introduced the **Jeans wavenumber**

$$k_{\rm J} = \frac{\sqrt{4\pi G\rho_0}}{c_{\rm s}}$$

to which we can also associate a **Jeans length**

$$\boxed{\lambda_{\rm J} \equiv \frac{2\pi}{k_{\rm J}} = \sqrt{\frac{\pi}{G\rho_0}}\, c_{\rm s}}$$

and a **Jeans mass**

$$M_{\rm J} = \frac{4}{3}\pi\rho_0 \left(\frac{\lambda_{\rm J}}{2}\right)^3 = \frac{\pi}{6}\,\rho_0\,\lambda_{\rm J}^3$$

From the dispersion relation one immediately sees that the system is **unstable** (i.e., ω is imaginary) if $k < k_{\rm J}$ (or, equivalently, $\lambda > \lambda_{\rm J}$ or $M > M_{\rm J}$). This is called the **Jeans criterion for gravitational instability**. It expresses when pressure forces (which try to disperse matter) are no longer able to overcome gravity (which tries to make matter collapse), resulting in exponential gravitational collapse on a time scale

$$\boxed{\tau_{\rm ff} = \sqrt{\frac{3\,\pi}{32\,G\,\rho}}}$$

known as the **free-fall time** for gravitational collapse.

The Jeans stability criterion is of utmost importance in astrophysics. It is used to describes the formation of galaxies and large scale structure in an expanding space-time (in this case the growth-rate is not exponential, but only power-law), to describe the formation of stars in molecular clouds within galaxies, and it may even play an important role in the formation of planets in protoplanetary disks.

In deriving the Jeans Stability criterion you will encounter a somewhat puzzling issue. Consider the **Poisson equation** for the unperturbed medium (which has density ρ_0 and gravitational potential Φ_0):

$$\nabla^2\Phi_0 = 4\pi G\rho_0$$

Since the initial, unperturbed medium is supposed to be homogeneous there can be no gravitational force; hence $\nabla\Phi_0 = 0$ everywhere. The above Poisson equation then implies that $\rho_0 = 0$. In other words, an unperturbed, homogeneous density field of non-zero density does not seem to exist. Sir James Jeans 'ignored' this 'nuisance' in his derivation, which has since become known as the **Jeans swindle**. The problem arises because Newtonian physics is not equipped to deal with systems of infinite extent (a requirement for a perfectly homogeneous density distribution). See Kiessling (1999; arXiv:9910247) for a detailed discussion, including an elegant demonstration that the Jeans swindle is actually vindicated!

Thermal Instability: Let $\mathcal{L} = \mathcal{L}(\rho, T) = \mathcal{C} - \mathcal{H}$ be the net cooling rate. If $\mathcal{L} = 0$ the system is said to be in **thermal equilibrium** (TE), while $\mathcal{L} > 0$ and $\mathcal{L} < 0$ correspond to cooling and heating, respectively.

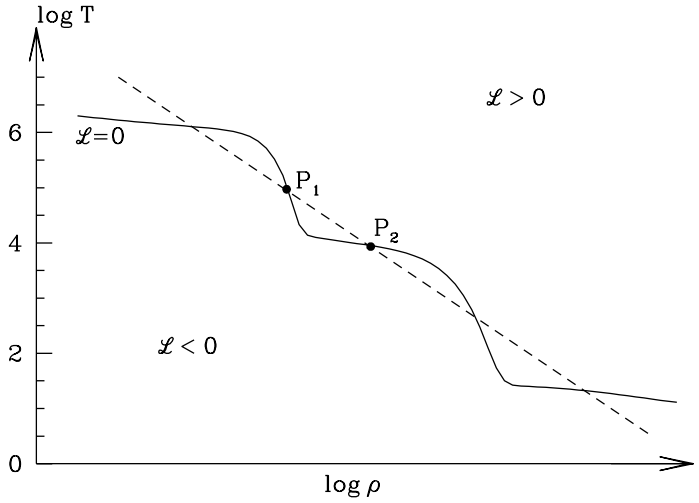


Figure 25: *The locus of thermal equilibrium ($\mathcal{L} = 0$) in the (ρ, T) plane, illustrating the principle of thermal instability. The dashed line indicates a line of constant pressure.*

The condition $\mathcal{L}(\rho, T) = 0$ corresponds to a curve in the (ρ, T)-plane with a shape similar to that shown in Fig. 25. It has flat parts at $T \sim 10^6$K, at $T \sim 10^4$K, at $T \sim 10 - 100$K. This can be understood from simple atomic physics (see for example § 8.5.1 of Mo, van den Bosch & White, 2010). Above the TE curve we have that $\mathcal{L} > 0$ (net cooling), while below it $\mathcal{L} < 0$ (net heating). The dotted curve indicates a line of constant pressure ($T \propto \rho^{-1}$). Consider a blob in thermal and mechanical (pressure) equilibrium with its ambient medium, and with a pressure indicated by the dashed line. There are five possible solutions for the density and temperature of the blob, two of which are indicated by P_1 and P_2; here confusingly the P refers to 'point' rather than 'pressure'. Suppose I have a blob located at point P_2. If I heat the blob, displacing it from TE along the constant pressure curve (i.e., the blob is assumed small enough that the sound crossing time, on which the blob re-established mechanical equilibrium, is short). The blob now finds itself in the region where $\mathcal{L} > 0$ (i.e, net cooling), so that it will cool back to its original location on the TE-curve; the blob is **stable**. For similar reasons, it is easy to see that a blob located at point P_1 is **unstable**. This instability is called **thermal instability**, and it explains why the ISM is a **three-phase** medium, with gas of three different temperatures ($T \sim 10^6\,\mathrm{K}, 10^4\,\mathrm{K}$, and $\sim 10 - 100\,\mathrm{K}$) coexisting in pressure equilibrium. Gas at any other temperature but in pressure equilibrium is thermally unstable.

It is easy to see that the requirement for **thermal instability** translates into

$$\left(\frac{\partial \mathcal{L}}{\partial T}\right)_P < 0$$

which is known as the **Field criterion for thermal instability** (after astrophysicist George B. Field).

Fragmentation and Shattering: Consider the Jeans criterion, expressing a balance between gravity and pressure. Using that the **Jeans mass** $M_{\rm J} \propto \rho\,\lambda_{\rm J}^3$ and that $\lambda_{\rm J} \propto \rho^{-1/2}\,c_{\rm s}$, we see that

$$M_{\rm J} \propto \rho^{-1/2}\,T^{3/2}$$

where we have used that $c_{\rm s} \propto T^{1/2}$. Now consider a **polytropic equation of state**, which has $P \propto \rho^{\Gamma}$, with Γ the **polytropic index**. Assuming an ideal gas, such that

$$P = \frac{k_{\rm B}\,T}{\mu\,m_{\rm p}}\rho$$

we thus see that a polytropic ideal gas must have that $T \propto \rho^{\Gamma-1}$. Substituting that in the expression for the Jeans mass, we obtain that

$$M_{\rm J} \propto \rho^{\frac{3}{2}\Gamma-2} = \rho^{\frac{3}{2}(\Gamma-\frac{4}{3})}$$

Thus, we see that for $\Gamma > 4/3$ the Jeans mass will increase with increasing density, while the opposite is true for $\Gamma < 4/3$. Now consider a system that is (initially) larger than the Jeans mass. Since pressure can no longer support it against its own gravity, the system will start to collapse, which increases the density. If $\Gamma < 4/3$, the Jeans mass will becomes smaller as a consequence of the collapse, and now small subregions of the system will find themselves having a mass larger than the Jeans mass $\Rightarrow$ the system will start to **fragment**.

If the collapse is adiabatic (i.e., we can ignore cooling), then $\Gamma = \gamma = 5/3 > 4/3$ and there will be no fragmentation. However, if cooling is very efficient, such that while the cloud collapses it maintains the same temperature, the EoS is now isothermal, which implies that $\Gamma = 1 < 4/3$: the cloud will fragment into smaller collapsing clouds. Fragmentation is believed to underly the formation of star clusters.

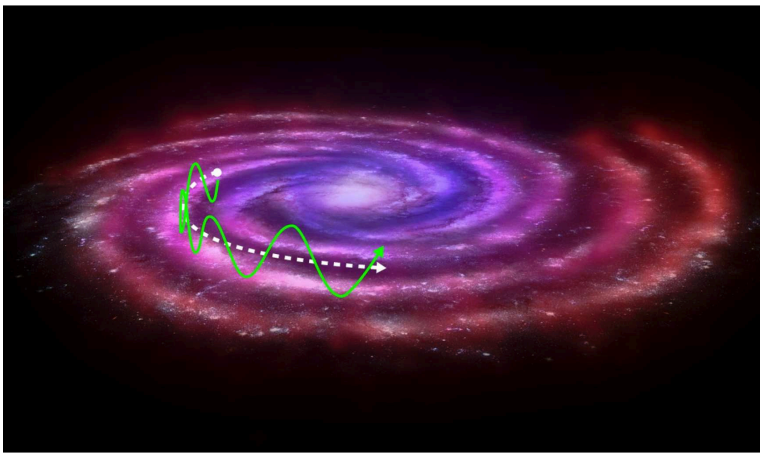
A very similar process operates related to the **thermal instability**. In the discussion of the **Field criterion** we had made the assumption *"the blob is assumed small enough that the sound crossing time, on which the blob re-established mechanical equilibrium, is short"*. Here 'short' means compared to the cooling time of the cloud. Let's define the **cooling length** $l_{\rm cool} \equiv c_{\rm s}\tau_{\rm cool}$, where $c_{\rm s}$ is the cloud's sound speed and

$\tau_{\rm cool}$ is the cooling time (the time scale on which it radiates away most of its internal energy). The above assumption thus implies that the size of the cloud, $l_{\rm cloud} \ll l_{\rm cool}$. As a consequence, whenever the cloud cools somewhat, it can immediately re-establish pressure equilibrium with its surrounding (i.e., the sound crossing time, $\tau_{\rm s} = l_{\rm cloud}/c_{\rm s}$ is much smaller than the cooling time $\tau_{\rm cool} = l_{\rm cool}/c_{\rm s}$).

Now consider a case in which $l_{\rm cloud} \gg l_{\rm cool}$ (i.e., $\tau_{\rm cool} \ll \tau_{\rm s}$). As the cloud cools, it cannot maintain pressure equilibrium with its surroundings; it takes too long for mechanical equilibrium to be established over the entire cloud. What happens is that smaller subregions, of order the size $l_{\rm cool}$, will fragment. The smaller fragments will be able to maintain pressure equilibrium with their surroundings. But as the small cloudlets cool further, the cooling length $l_{\rm cool}$ shrinks. To see this, realize that when T drops this lowers the sound speed and decreases the cooling time; after all, we are in the regime of thermal instability, so $(\partial\mathcal{L}/\partial T)_P < 0$. As a consequence, $l_{\rm cool} = c_{\rm s}\tau_{\rm cool}$ drops as well. So the small cloudlet soon finds itself larger than the cooling length, and it in turn will fragment. This process of **shattering** continues until the cooling time becomes sufficiently long and the cloudlets are no longer thermally unstable.

This process of **shattering** is believed to play an important role in the inter-galactic medium (IGM) in between galaxies, and the circum-galactic medium (CGM) in the halos of galaxies.

Part IV: Collisionless Dynamics



The following chapters give an elementary introduction into the rich topic of collisionless dynamics. The main goal is to highlight how the lack of collisions among the constituent particles give rise to a dynamics that differs remarkably from collisional fluids. We also briefly discuss the theory of orbits, which are the building blocks of collisionless systems, the Virial theorem, and the gravothermal catastrophe, which is a consequence of the negative heat capacity of a gravitational system. Finally, we briefly discuss interactions ('collisions') among collisionless systems.

Collisionless Dynamics is a rich topic, and one could easily devote an entire course to it (for example the Yale Graduate Course 'ASTR 518; Galactic Dynamics'). The following chapters therefore only scratch the surface of this rich topic. Readers who want to get more indepth information are referred to the following excellent textbooks

- *Galactic Dynamics* by J. Binney & S. Tremaine
- *Galactic Nuclei* by D. Merritt
- *Galaxy Formation and Evolution* by H.J. Mo, F. van den Bosch & S. White

CHAPTER 20

Jeans Equations and Dynamical Modelling

In this chapter we consider collisionless fluids, such as galaxies and dark matter halos. As discussed in previous chapters, their dynamics is governed by the **Collisionless Boltzmann equation** (CBE)

$$\boxed{\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + v_i \frac{\partial f}{\partial x_i} - \frac{\partial \Phi}{\partial x_i}\frac{\partial f}{\partial v_i} = 0}$$

By taking the velocity moment of the CBE (see Chapter 7), we obtain the **Jeans equations**

$$\boxed{\frac{\partial u_i}{\partial t} + u_i \frac{\partial u_j}{\partial x_i} = \frac{1}{\rho}\frac{\partial \hat{\sigma}_{ij}}{\partial x_i} - \frac{\partial \Phi}{\partial x_i}}$$

which are the equivalent of the Navier-Stokes equations (or Euler equations), but for a collisionless fluid. The quantity $\hat{\sigma}_{ij}$ in the above expression is the **stress tensor**, defined as

$$\hat{\sigma}_{ij} = -\rho \left\langle w_i w_j \right\rangle = \rho \langle v_i \rangle \left\langle v_j \right\rangle - \rho \langle v_i v_j \rangle$$

In this chapter, we write a hat on top of the stress tensor, in order to distinguish it from the **velocity dispersion tensor** given by

$$\boxed{\sigma_{ij}^2 = \left\langle v_i v_j \right\rangle - \left\langle v_i \right\rangle \left\langle v_j \right\rangle = -\frac{\hat{\sigma}_{ij}}{\rho}}$$

This notation may cause some confusion, but it is adapted here in order to be consistent with the notation in standard textbooks on galactic dynamics. For the same reason, in what follows we will write $\langle v_i \rangle$ instead of u_i.

As we have discussed in detail in chapter 11, for a collisional fluid the stress tensor is given by

$$\hat{\sigma}_{ij} = -\rho \sigma_{ij}^2 = -P\delta_{ij} + \tau_{ij}$$

and therefore completely specified by two scalar quantities; the pressure P and the shear viscosity μ (as always, we ignore bulk viscosity). Both P and μ are related to ρ and T via **constitutive equations**, which allow for closure in the equations.

In the case of a collisionless fluid, though, no constistutive relations exist, and the (symmetric) velocity dispersion tensor has 6 unknowns. As a consequence, the Jeans equations do not form a closed set. Adding higher-order moment equations of the CBE will yield more equations, but this also adds new, higher-order unknowns such as $\langle v_i v_j v_k \rangle$, etc. As a consequence, the set of CBE moment equations never closes!

Note that σ_{ij}^2 is a **local** quantity; $\sigma_{ij}^2 = \sigma_{ij}^2(\vec{x})$. At each point $\vec{x}$ it defines the **velocity ellipsoid**; an ellipsoid whose principal axes are defined by the orthogonal eigenvectors of σ_{ij}^2 with lengths that are proportional to the square roots of the respective eigenvalues.

Since these eigenvalues are typically not the same, a collisionless fluid experiences **anisotropic** pressure-like forces. In order to be able to close the set of Jeans equations, it is common to make certain assumptions about the symmetry of the fluid. For example, a common assumption is that the fluid is **isotropic**, such that the (local) velocity dispersion tensor is specified by a single quantity; the local velocity dispersion σ^2. Note, though, that if with this approach, a solution is found, the solution may not correspond to a physical distribution function (DF) (i.e., in order to be physical, $f \geq 0$ everywhere). Thus, although any real DF obeys the Jeans equations, not every solution to the Jeans equations corresponds to a physical DF!!!

As a worked out example, we now derive the Jeans equations under **cylindrical symmetry**. We therefore write the Jeans equations in the cylindrical coordinate system (R, ϕ, z). The first step is to write the CBE in cylindrical coordinates.

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \dot{R}\frac{\partial f}{\partial R} + \dot{\phi}\frac{\partial f}{\partial \phi} + \dot{z}\frac{\partial f}{\partial z} + \dot{v}_R\frac{\partial f}{\partial v_R} + \dot{v}_\phi\frac{\partial f}{\partial v_\phi} + \dot{v}_z\frac{\partial f}{\partial v_z}$$

Recall from vector calculus (see Appendices A and D) that

$$\vec{v} = \dot{R}\vec{e}_R + R\dot{\phi}\vec{e}_\phi + \dot{z}\vec{e}_z = v_R\vec{e}_R + v_\phi\vec{e}_\phi + v_z\vec{e}_z$$

from which we obtain the acceleration vector

$$\vec{a} = \frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = \ddot{R}\vec{e}_R + \dot{R}\dot{\vec{e}}_R + \dot{R}\dot{\phi}\vec{e}_\phi + R\ddot{\phi}\vec{e}_\phi + R\dot{\phi}\dot{\vec{e}}_\phi + \ddot{z}\vec{e}_z + \dot{z}\dot{\vec{e}}_z$$

Using that $\dot{\vec{e}}_R = \dot{\phi}\vec{e}_\phi$, $\dot{\vec{e}}_\phi = -\dot{\phi}\vec{e}_R$, and $\dot{\vec{e}}_z = 0$ we have that

$$\vec{a} = \left[\ddot{R} - R\dot{\phi}^2\right]\vec{e}_R + \left[2\dot{R}\dot{\phi} + R\ddot{\phi}\right]\vec{e}_\phi + \ddot{z}\vec{e}_z$$

Next we use that

$$\begin{aligned} v_R = \dot{R} &\Rightarrow \dot{v}_R = \ddot{R} \\ v_\phi = R\dot{\phi} &\Rightarrow \dot{v}_\phi = \dot{R}\dot{\phi} + R\ddot{\phi} \\ v_z = \dot{z} &\Rightarrow \dot{v}_z = \ddot{z} \end{aligned}$$

to write the acceleration vector as

$$\vec{a} = \left[\dot{v}_R - \frac{v_\phi^2}{R}\right]\vec{e}_R + \left[\frac{v_R v_\phi}{R} + \dot{v}_\phi\right]\vec{e}_\phi + \dot{v}_z\,\vec{e}_z$$

Newton's equation of motion in vector form reads

$$\vec{a} = -\nabla\Phi = \frac{\partial\Phi}{\partial R}\,\vec{e}_R + \frac{1}{R}\frac{\partial\Phi}{\partial\phi}\,\vec{e}_\phi + \frac{\partial\Phi}{\partial z}\,\vec{e}_z$$

Combining this with the above we see that

$$\begin{aligned} \dot{v}_R &= -\frac{\partial\Phi}{\partial R} + \frac{v_\phi^2}{R} \\ \dot{v}_\phi &= -\frac{1}{R}\frac{\partial\Phi}{\partial R} + \frac{v_R v_\phi}{R} \\ \dot{v}_z &= -\frac{\partial\Phi}{\partial z} \end{aligned}$$

which allows us to the write the **CBE in cylindrical coordinates** as

$$\boxed{\frac{\partial f}{\partial t} + v_R\frac{\partial f}{\partial R} + \frac{v_\phi}{R}\frac{\partial f}{\partial\phi} + v_z\frac{\partial f}{\partial z} + \left[\frac{v_\phi^2}{R} - \frac{\partial\Phi}{\partial R}\right]\frac{\partial f}{\partial v_R} - \frac{1}{R}\left[v_R v_\phi + \frac{\partial\Phi}{\partial\phi}\right]\frac{\partial f}{\partial v_\phi} - \frac{\partial\Phi}{\partial z}\frac{\partial f}{\partial v_z} = 0}$$

The **Jeans equations** follow from multiplication with v_R, v_ϕ, and v_z and integrating over velocity space. Note that the **cylindrical symmetry** requires that all derivatives with respect to ϕ vanish. The remaining terms are:

$$
\begin{aligned}
\int v_R \frac{\partial f}{\partial t} \mathrm{d}^3\vec{v} &= \frac{\partial}{\partial t}\int v_R f \mathrm{d}^3\vec{v} = \frac{\partial(\rho\langle v_R\rangle)}{\partial t} \\
\int v_R^2 \frac{\partial f}{\partial R} \mathrm{d}^3\vec{v} &= \frac{\partial}{\partial R}\int v_R^2 f \mathrm{d}^3\vec{v} = \frac{\partial(\rho\langle v_R^2\rangle)}{\partial R} \\
\int v_R v_z \frac{\partial f}{\partial z} \mathrm{d}^3\vec{v} &= \frac{\partial}{\partial z}\int v_R v_z f \mathrm{d}^3\vec{v} = \frac{\partial(\rho\langle v_R v_z\rangle)}{\partial z} \\
\int \frac{v_R v_\phi^2}{R} \frac{\partial f}{\partial v_R} \mathrm{d}^3\vec{v} &= \frac{1}{R}\left[\int \frac{\partial(v_R v_\phi^2 f)}{\partial v_R}\mathrm{d}^3\vec{v} - \int \frac{\partial(v_R v_\phi^2)}{\partial v_R} f \mathrm{d}^3\vec{v}\right] = -\rho\frac{\langle v_\phi^2\rangle}{R} \\
\int v_R \frac{\partial\Phi}{\partial R}\frac{\partial f}{\partial v_R} \mathrm{d}^3\vec{v} &= \frac{\partial\Phi}{\partial R}\left[\int \frac{\partial(v_R f)}{\partial v_R}\mathrm{d}^3\vec{v} - \int \frac{\partial v_R}{\partial v_R} f \mathrm{d}^3\vec{v}\right] = -\rho\frac{\partial\Phi}{\partial R} \\
\int \frac{v_R^2 v_\phi}{R} \frac{\partial f}{\partial v_\phi} \mathrm{d}^3\vec{v} &= \frac{1}{R}\left[\int \frac{\partial(v_R^2 v_\phi f)}{\partial v_\phi}\mathrm{d}^3\vec{v} - \int \frac{\partial(v_R^2 v_\phi)}{\partial v_\phi} f \mathrm{d}^3\vec{v}\right] = -\rho\frac{\langle v_R^2\rangle}{R} \\
\int v_R \frac{\partial\Phi}{\partial z}\frac{\partial f}{\partial v_z} \mathrm{d}^3\vec{v} &= \frac{\partial\Phi}{\partial z}\left[\int \frac{\partial(v_R f)}{\partial v_z}\mathrm{d}^3\vec{v} - \int \frac{\partial v_z}{\partial v_R} f \mathrm{d}^3\vec{v}\right] = 0
\end{aligned}
$$

Working out the similar terms for the other Jeans equations we finally obtain the **Jeans Equations in Cylindrical Coordinates**:

$$
\frac{\partial(\rho\langle v_R\rangle)}{\partial t} + \frac{\partial(\rho\langle v_R^2\rangle)}{\partial R} + \frac{\partial(\rho\langle v_R v_z\rangle)}{\partial z} + \rho\left[\frac{\langle v_R^2\rangle - \langle v_\phi^2\rangle}{R} + \frac{\partial\Phi}{\partial R}\right] = 0
$$

$$
\frac{\partial(\rho\langle v_\phi\rangle)}{\partial t} + \frac{\partial(\rho\langle v_R v_\phi\rangle)}{\partial R} + \frac{\partial(\rho\langle v_\phi v_z\rangle)}{\partial z} + 2\rho\frac{\langle v_R v_\phi\rangle}{R} = 0
$$

$$
\frac{\partial(\rho\langle v_z\rangle)}{\partial t} + \frac{\partial(\rho\langle v_R v_z\rangle)}{\partial R} + \frac{\partial(\rho\langle v_z^2\rangle)}{\partial z} + \rho\left[\frac{\langle v_R v_z\rangle}{R} + \frac{\partial\Phi}{\partial z}\right] = 0
$$

These are 3 equations with 9 unknowns, which can only be solved if we make additional assumptions. In particular, one often makes the following assumptions:

1 System is static $\Rightarrow$ the $\frac{\partial}{\partial t}$-terms are zero and $\langle v_R\rangle = \langle v_z\rangle = 0$.

2 Velocity dispersion tensor is diagonal $\Rightarrow \langle v_i v_j\rangle = 0$ (if $i \neq j$).

3 Meridional isotropy $\Rightarrow \langle v_R^2\rangle = \langle v_z^2\rangle = \sigma_R^2 = \sigma_z^2 \equiv \sigma^2$.

Under these assumptions we have 3 unknowns left: $\langle v_\phi \rangle$, $\langle v_\phi^2 \rangle$, and σ^2, and the Jeans equations reduce to

$$\boxed{\begin{aligned} &\frac{\partial(\rho\sigma^2)}{\partial R} + \rho\left[\frac{\sigma^2 - \langle v_\phi^2\rangle}{R} + \frac{\partial\Phi}{\partial R}\right] = 0 \\ &\frac{\partial(\rho\sigma^2)}{\partial z} + \rho\frac{\partial\Phi}{\partial z} = 0 \end{aligned}}$$

Since we now only have two equations left, the system is still not closed. If from the surface brightness we can estimate the mass density, $\rho(R,z)$, and hence (using the Poisson equation) the potential $\Phi(R,z)$, we can solve the second of these Jeans equations for the **meridional velocity dispersion**:

$$\sigma^2(R,z) = \frac{1}{\rho}\int_z^\infty \rho\frac{\partial\Phi}{\partial z}\mathrm{d}z$$

and the first Jeans equation then gives the **mean square azimuthal velocity** $\langle v_\phi^2 \rangle = \langle v_\phi \rangle^2 + \sigma_\phi^2$:

$$\langle v_\phi^2\rangle(R,z) = \sigma^2(R,z) + R\frac{\partial\Phi}{\partial R} + \frac{R}{\rho}\frac{\partial(\rho\sigma^2)}{\partial R}$$

Thus, although $\langle v_\phi^2 \rangle$ is uniquely specified by the Jeans equations, we don't know how it splits in the actual **azimuthal streaming**, $\langle v_\phi \rangle$, and the **azimuthal dispersion**, σ_ϕ^2. Additional assumptions are needed for this.

A similar analysis, but for a spherically symmetric system, using the spherical coordinate system (r, θ, ϕ), gives the following **Jeans equations in Spherical Symmetry**

$$\boxed{\begin{aligned}
&\frac{\partial(\rho\langle v_r\rangle)}{\partial t} + \frac{\partial(\rho\langle v_r^2\rangle)}{\partial r} + \frac{\rho}{r}\left[2\langle v_r^2\rangle - \langle v_\theta^2\rangle - \langle v_\phi^2\rangle\right] + \rho\frac{\partial\Phi}{\partial r} = 0 \\
&\frac{\partial(\rho\langle v_\theta\rangle)}{\partial t} + \frac{\partial(\rho\langle v_r v_\theta\rangle)}{\partial r} + \frac{\rho}{r}\left[3\langle v_r v_\theta\rangle + \left(\langle v_\theta^2\rangle - \langle v_\phi^2\rangle\right)\cot\theta\right] = 0 \\
&\frac{\partial(\rho\langle v_\phi\rangle)}{\partial t} + \frac{\partial(\rho\langle v_r v_\phi\rangle)}{\partial r} + \frac{\rho}{r}\left[3\langle v_r v_\phi\rangle + 2\langle v_\theta v_\phi\rangle\cot\theta\right] = 0
\end{aligned}}$$

If we now make the additional assumptions that the system is **static** and that also the *kinematic* properties of the system are spherically symmetric, then there can be *no streaming motions* and all mixed second-order moments vanish. Consequently, the **velocity dispersion tensor** is diagonal with $\sigma_\theta^2 = \sigma_\phi^2$. Under these assumptions only one of the three Jeans equations remains:

$$\frac{\partial(\rho\sigma_r^2)}{\partial r} + \frac{2\rho}{r}\left[\sigma_r^2 - \sigma_\theta^2\right] + \rho\frac{\partial\Phi}{\partial r} = 0$$

Notice that this single equation still constains two unknown, $\sigma_r^2(r)$ and $\sigma_\theta^2(r)$ (if we assume that the density and potential are known), and can thus not be solved.
It is useful to define the **anisotropy parameter**

$$\beta(r) \equiv 1 - \frac{\sigma_\theta^2(r) + \sigma_\phi^2(r)}{2\sigma_r^2(r)} = 1 - \frac{\sigma_\theta^2(r)}{\sigma_r^2(r)}$$

where the second equality only holds under the assumption that the kinematics are spherically symmetric.

With β thus defined the (spherical) Jeans equation can be written as

$$\boxed{\frac{1}{\rho}\frac{\partial(\rho\langle v_r^2\rangle)}{\partial r} + 2\frac{\beta\langle v_r^2\rangle}{r} = -\frac{\mathrm{d}\Phi}{\mathrm{d}r}}$$

If we now use that $\mathrm{d}\Phi/\mathrm{d}r = GM(r)/r$, we can write the following expression for the

enclosed (dynamical) mass:

$$\boxed{M(r) = -\frac{r\langle v_r^2\rangle}{G}\left[\frac{\mathrm{d}\ln\rho}{\mathrm{d}\ln r} + \frac{\mathrm{d}\ln\langle v_r^2\rangle}{\mathrm{d}\ln r} + 2\beta\right]}$$

Hence, if we know $\rho(r)$, $\langle v_r^2\rangle(r)$, and $\beta(r)$, we can use the **spherical Jeans equation** to infer the mass profile $M(r)$.

Consider an external, spherical galaxy. Observationally, we can measure the projected **surface brightness** profile, $\Sigma(R)$, which is related to the 3D **luminosity density** $\nu(r) = \rho(r)/\Upsilon(r)$

$$\Sigma(R) = 2\int_R^\infty \frac{\nu\, r\, \mathrm{d}r}{\sqrt{r^2 - R^2}}$$

with $\Upsilon(r)$ the **mass-to-light ratio**. Similarly, the **line-of-sight velocity dispersion**, $\sigma_\mathrm{p}^2(R)$, which can be inferred from spectroscopy, is related to both $\langle v_r^2\rangle(r)$ and $\beta(r)$ according to (see Figure 26)

$$\begin{aligned}
\Sigma(R)\sigma_p^2(R) &= 2\int_R^\infty \langle (v_r\cos\alpha - v_\theta\sin\alpha)^2\rangle \frac{\nu\, r\, \mathrm{d}r}{\sqrt{r^2 - R^2}} \\
&= 2\int_R^\infty \left(\langle v_r^2\rangle\cos^2\alpha + \langle v_\theta^2\rangle\sin^2\alpha\right)\frac{\nu\, r\, \mathrm{d}r}{\sqrt{r^2 - R^2}} \\
&= 2\int_R^\infty \left(1 - \beta\frac{R^2}{r^2}\right)\frac{\nu\,\langle v_r^2\rangle\, r\, \mathrm{d}r}{\sqrt{r^2 - R^2}}
\end{aligned}$$

The 3D luminosity density is trivially obtained from the observed $\Sigma(R)$ using the **Abel transform**

$$\nu(r) = -\frac{1}{\pi}\int_r^\infty \frac{\mathrm{d}\Sigma}{\mathrm{d}R}\frac{\mathrm{d}R}{\sqrt{R^2 - r^2}}$$

In practice, one often uses forward modeling instead: assume a functional form for $\nu(r)$, project and fit to the observed surface brightness profiles $\Sigma(R)$ to constrain the free parameters of the function that describes $\nu(r)$.

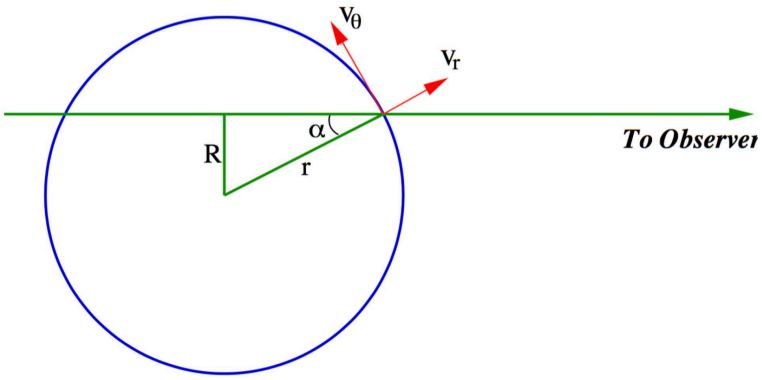


Figure 26: *Geometry related to projection*

In general, we have three unknowns: $M(r)$ [or equivalently $\rho(r)$ or $\Upsilon(r)$], $\langle v_r^2 \rangle(r)$ and $\beta(r)$. With our two observables $\Sigma(R)$ and $\sigma_{\rm p}^2(R)$, these can only be determined if we make additional assumptions.

EXAMPLE 1: Assume isotropy: $\beta(r) = 0$. In this case we can use the **Abel transform** to obtain

$$\nu(r)\langle v_r^2 \rangle(r) = -\frac{1}{\pi} \int_r^\infty \frac{{\rm d}(\Sigma \sigma_p^2)}{{\rm d}R} \frac{{\rm d}R}{\sqrt{R^2 - r^2}}$$

and the enclosed mass follows from the Jeans equation

$$M(r) = -\frac{r\langle v_r^2 \rangle}{G} \left[\frac{{\rm d}\ln \nu}{{\rm d}\ln r} + \frac{{\rm d}\ln \langle v_r^2 \rangle}{{\rm d}\ln r} \right]$$

Note that the first term uses the *luminosity* density $\nu(r)$ rather than the *mass* density $\rho(r)$. This is because $\sigma_{\rm p}^2$ is weighted by light rather than mass.
The mass-to-light ratio now follows from

$$\Upsilon(r) = \frac{M(r)}{4\pi \int_0^r \nu(r)\, r^2\, {\rm d}r}$$

which can be used to constrain the mass of a potential dark matter halo or central supermassive black hole (but always under assumption that the system is isotropic).

EXAMPLE 2: Assume a constant mass-to-light ratio: $\Upsilon(r) = \Upsilon_0$. In this case the luminosity density $\nu(r)$ immediately yields the enclosed mass:

$$M(r) = 4\pi\Upsilon_0 \int_0^r \nu(r)\, r^2\, \mathrm{d}r$$

We can now use the **spherical Jeans Equation** to write $\beta(r)$ in terms of $M(r)$, $\nu(r)$ and $\langle v_r^2 \rangle(r)$. Substituting this in the equation for $\Sigma(R)\sigma_p^2(R)$ yields a solution for $\langle v_r^2 \rangle(r)$, and thus for $\beta(r)$. As long as $\beta(r) \leq 1$ the model is said to be **self-consistent** within the context of the Jeans equations.

Almost always, radically different models (based on radically different assumptions) can be constructed, that are all consistent with the data and the Jeans equations. This is often referred to as the **mass-anisotropy degeneracy**. Note, however, that none of these models need to be physical: they can still have $f < 0$.

CHAPTER 21

The Jeans Theorem

Integrals of Motion: An integral of motion is a function $I(\vec{x}, \vec{v})$ of the phase-space coordinates that is constant along all orbits, i.e.,

$$I[\vec{x}(t_1), \vec{v}(t_1)] = I[\vec{x}(t_2), \vec{v}(t_2)]$$

for any t_1 and t_2. The value of the integral of motion can be the same for different orbits. Note that an integral of motion can not depend on time. Orbits can have from zero to five integrals of motion. If the Hamiltonian does not depend on time, then energy is always an integral of motion.

Integrals of motion come in two kinds:

- **Isolating Integrals of Motion:** these reduce the dimensionality of the particle's trajectory in 6-dimensional phase-space by one. Therefore, an orbit with n isolating integrals of motion is restricted to a $6 - n$ dimensional manifold in 6-dimensional phase-space. Energy is always an isolating integral of motion.
- **Non-Isolating Integrals of Motion:** these are integrals of motion that do *not* reduce the dimensionality of the particle's trajectory in phase-space. They are of essentially no practical value for the dynamics of the system.

Orbits: If in a system with n degrees of freedom a particular orbit admits n independent isolating integrals of motion, the orbit is said to be **regular**, and its corresponding trajectory $\Gamma(t)$ is confined to a $2n - n = n$ dimensional manifold in phase-space. Topologically this manifold is called an **invariant torus** (or torus for short), and is uniquely specified by the n isolating integrals. A regular orbit has n fundamental frequencies, ω_i, with which it circulates or librates in its n-dimensional manifold. If two or more of these frequencies are commensurable (i.e., $l\omega_i + m\omega_j = 0$ with l and m integers), then the orbit is a **resonant orbit**, and has a dimensionality that is one lower than that of the non-resonant regular orbits (i.e., $l\omega_i + m\omega_j$ is an extra isolating integral of motion). Orbits with fewer than n isolating integrals of motion are called **irregular** or **stochastic**.

Every spherical potential admits at least four isolating integrals of motion, namely

energy, E, and the three components of the angular momentum vector $\vec{L}$. Orbits in a flattened, axisymmetric potential frequently (but not always) admit three isolating integrals of motion: E, L_z (where the z-axis is the system's symmetry axis), and a **non-classical** third integral I_3 (the integral is called non-classical since there is no analytical expression of I_3 as function of the phase-space variables).

Since an integral of motion, $I(\vec{x}, \vec{v})$ is constant along an orbit, we have that

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{\partial I}{\partial x_i}\frac{\mathrm{d}x_i}{\mathrm{d}t} + \frac{\partial I}{\partial v_i}\frac{\mathrm{d}v_i}{\mathrm{d}t} = \vec{v}\cdot\nabla I - \nabla\Phi\cdot\frac{\partial I}{\partial \vec{v}} = 0$$

Compare this to the **CBE** for a steady-state (static) system:

$$\vec{v}\cdot\nabla f - \nabla\Phi\cdot\frac{\partial f}{\partial \vec{v}} = 0$$

Thus the condition for I to be an **integral of motion** is identical with the condition for I to be a **steady-state solution of the CBE**. This implies the following:

Jeans Theorem: *Any steady-state solution of the CBE depends on the phase-space coordinates only through integrals of motion. Any function of these integrals is a steady-state solution of the CBE.*

Strong Jeans Theorem: *The DF of a steady-state system in which almost all orbits are regular can be written as a function of the independent isolating integrals of motion.*

Hence, the DF of any steady-state spherical system can be expressed as $f = f(E, \vec{L})$. If the system is spherically symmetric in *all* its properties, then $f = f(E, L^2)$, i.e., the DF can only depend on the magnitude of the angular momentum vector, not on its direction.

An even simpler case to consider is the one in which $f = f(E)$: Since $E = \Phi(\vec{r}) + \frac{1}{2}[v_r^2 + v_\theta^2 + v_\phi^2]$ we have that

$$\begin{aligned}
\langle v_r^2 \rangle &= \frac{1}{\rho} \int \mathrm{d}v_r \mathrm{d}v_\theta \mathrm{d}v_\phi \, v_r^2 \, f \left(\Phi + \frac{1}{2}[v_r^2 + v_\theta^2 + v_\phi^2] \right) \\
\langle v_\theta^2 \rangle &= \frac{1}{\rho} \int \mathrm{d}v_r \mathrm{d}v_\theta \mathrm{d}v_\phi \, v_\theta^2 \, f \left(\Phi + \frac{1}{2}[v_r^2 + v_\theta^2 + v_\phi^2] \right) \\
\langle v_\phi^2 \rangle &= \frac{1}{\rho} \int \mathrm{d}v_r \mathrm{d}v_\theta \mathrm{d}v_\phi \, v_\phi^2 \, f \left(\Phi + \frac{1}{2}[v_r^2 + v_\theta^2 + v_\phi^2] \right)
\end{aligned}$$

Since these equations differ only in the labelling of one of the variables of integration, it is immediately evident that $\langle v_r^2 \rangle = \langle v_\theta^2 \rangle = \langle v_\phi^2 \rangle$. Hence, assuming that $f = f(E)$ is identical to assuming that the system is **isotropic** (and thus $\beta(r) = 0$). And since

$$\langle v_i \rangle = \frac{1}{\rho} \int \mathrm{d}v_r \mathrm{d}v_\theta \mathrm{d}v_\phi \, v_i \, f \left(\Phi + \frac{1}{2}[v_r^2 + v_\theta^2 + v_\phi^2] \right)$$

it is also immediately evident that $\langle v_r \rangle = \langle v_\theta \rangle = \langle v_\phi \rangle = 0$. Thus, a system with $f = f(E)$ has no net sense of rotation.

The more general $f(E, L^2)$ models typically are **anisotropic**. Models with $0 < \beta \leq 1$ are radially anisotropic. In the extreme case of $\beta = 1$ all orbits are purely radial and $f = g(E)\,\delta(L)$, with $g(E)$ some function of energy. Tangentially anisotropic models have $\beta < 0$, with $\beta = -\infty$ corresponding to a model in which all orbits are circular. In that case $f = g(E)\,\delta[L - L_{\rm max}(E)]$, where $L_{\rm max}(E)$ is the maximum angular momentum for energy E. Another special case is the one in which $\beta(r) = \beta$ is constant; such models have $f = g(E)\,L^{-2\beta}$.

Next we consider **axisymmetric** systems. If we only consider systems for which most orbits are regular, then the **strong Jeans Theorem** states that, in the most general case, $f = f(E, L_z, I_3)$. For a static, axisymmetric system

$$\langle v_R \rangle = \langle v_z \rangle = 0 \qquad \langle v_R v_\phi \rangle = \langle v_z v_\phi \rangle = 0$$

but note that, in this general case, $\langle v_R v_z \rangle \neq 0$; Hence, in general, in a three-integral model with $f = f(E, L_z, I_3)$ the **velocity ellipsoid** is not aligned with (R, ϕ, z), and the velocity dispersion tensor contains four unknowns: $\langle v_R^2 \rangle$, $\langle v_\phi^2 \rangle$, $\langle v_z^2 \rangle$, and $\langle v_R v_z \rangle$. In this case there are two non-trivial Jeans Equations:

$$\boxed{\begin{aligned}&\frac{\partial(\rho\langle v_R^2\rangle)}{\partial R}+\frac{\partial(\rho\langle v_R v_z\rangle)}{\partial z}+\rho\left[\frac{\langle v_R^2\rangle-\langle v_\phi^2\rangle}{R}+\frac{\partial\Phi}{\partial R}\right]=0\\&\frac{\partial(\rho\langle v_R v_z\rangle)}{\partial R}+\frac{\partial(\rho\langle v_z^2\rangle)}{\partial z}+\rho\left[\frac{\langle v_R v_z\rangle}{R}+\frac{\partial\Phi}{\partial z}\right]=0\end{aligned}}$$

which clearly doesn't suffice to solve for the four unknowns (modelling three-integral axisymmetric systems is best done using the Schwarzschild orbit superposition technique). To make progress with Jeans modeling, one has to make additional assumptions. A typical assumption is that the DF has the **two-integral form** $f = f(E, L_z)$.

In that case, $\langle v_R v_z\rangle = 0$ [velocity ellipsoid now *is* aligned with (R, ϕ, z)] and $\langle v_R^2\rangle = \langle v_z^2\rangle$ (see Binney & Tremaine 2008), so that the **Jeans equations** reduce to

$$\boxed{\begin{aligned}&\frac{\partial(\rho\langle v_R^2\rangle)}{\partial R}+\rho\left[\frac{\langle v_R^2\rangle-\langle v_\phi^2\rangle}{R}+\frac{\partial\Phi}{\partial R}\right]=0\\&\frac{\partial(\rho\langle v_z^2\rangle)}{\partial z}+\rho\frac{\partial\Phi}{\partial z}=0\end{aligned}}$$

which is a closed set for the two unknowns $\langle v_R^2\rangle$ $(= \langle v_z^2\rangle)$ and $\langle v_\phi^2\rangle$. Note, however, that the Jeans equations provide no information about how $\langle v_\phi^2\rangle$ splits in streaming and random motions. In practice one often writes that

$$\langle v_\phi\rangle^2 = k\left[\langle v_\phi^2\rangle - \langle v_R^2\rangle\right]$$

with k a free parameter. When $k = 1$ the azimuthal dispersion is $\sigma_\phi^2 \equiv \langle v_\phi^2\rangle - \langle v_\phi\rangle^2 = \sigma_R^2 = \sigma_z^2$ everywhere. Such models are called **oblate isotropic rotators**.

CHAPTER 22

Virial Theorem & Gravothermal Catastrophe

Consider a gravitational system consisting of N particles (e.g., stars, fluid elements). The **total energy** of the system is $E = K + W$, where

$$\textbf{Total Kinetic Energy:} \quad K = \sum_{i=1}^{N} \tfrac{1}{2}\, m_i\, v_i^2$$

$$\textbf{Total Potential Energy:} \quad W = -\tfrac{1}{2} \sum_{i=1}^{N} \sum_{j \neq i} \frac{G\, m_i m_j}{|\vec{r}_i - \vec{r}_j|}$$

The latter follows from the fact that **gravitational binding energy** between a pair of masses is proportional to the product of their masses, and inversely proportional to their separation. The factor 1/2 corrects for double counting the number of pairs.

Potential Energy in Continuum Limit: To infer an expression for the gravitational potential energy in the continuum limit, it is useful to rewrite the above expression as

$$W = \frac{1}{2} \sum_{i=1}^{N} m_i\, \Phi_i$$

where

$$\Phi_i = -\sum_{j \neq i} \frac{G\, m_j}{r_{ij}}$$

where $r_{ij} = |\vec{r}_i - \vec{r}_j|$. In the continuum limit this simply becomes

$$W = \frac{1}{2} \int \rho(\vec{x})\, \Phi(\vec{x})\, \mathrm{d}^3\vec{x}$$

One can show (see e.g., Binney & Tremaine 2008) that this is equal to the trace of the **Chandrasekhar Potential Energy Tensor**

$$W_{ij} \equiv -\int \rho(\vec{x})\, x_i\, \frac{\partial \Phi}{\partial x_j}\, \mathrm{d}^3\vec{x}$$

In particular,

$$W = \mathrm{Tr}(W_{ij}) = \sum_{i=1}^{3} W_{ii} = -\int \rho(\vec{x})\, \vec{x} \cdot \nabla\Phi \, \mathrm{d}^3\vec{x}$$

which is another, equally valid, expression for the gravitational potential energy in the continuum limit.

Virial Theorem: A stationary, gravitational system obeys

$$\boxed{2K + W = 0}$$

Actually, the correct virial equation is $2K + W + \Sigma = 0$*, where* Σ *is the surface pressure. In many, but certainly not all, applications in astrophysics this term can be ignored. Many textbooks don't even mention the surface pressure term.*

Combining the **virial equation** with the expression for the total energy, $E = K + W$, we see that for a system that obeys the virial theorem

$$\boxed{E = -K = W/2}$$

Example: Consider a cluster consisting of N galaxies. If the cluster is in virial equilibrium then

$$2\sum_{i=1}^{N} \frac{1}{2} m\, v_i^2 - \frac{1}{2}\sum_{i=1}^{N}\sum_{j\neq i} \frac{G\, m_i m_j}{r_{ij}} = 0$$

If we assume, for simplicity, that all galaxies have equal mass then we can rewrite this as

$$N\, m\, \frac{1}{N}\sum_{i=1}^{N} v_i^2 - \frac{G\,(Nm)^2}{2}\, \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j\neq i} \frac{1}{r_{ij}} = 0$$

Using that $M = N\,m$ and $N(N-1) \simeq N^2$ for large N, this yields

$$M = \frac{2\,\langle v^2 \rangle}{G\,\langle 1/r \rangle}$$

with

$$\langle 1/r \rangle = \frac{1}{N(N-1)} \sum_{i=1}^{N} \sum_{j \neq i} \frac{1}{r_{ij}}$$

It is useful to define the **gravitational radius** $r_{\rm g}$ such that

$$W = -\frac{G\,M^2}{r_{\rm g}}$$

Using the relations above, it is clear that $r_{\rm g} = 2/\langle 1/r \rangle$. We can now rewrite the above equation for M in the form

$$M = \frac{r_{\rm g} \langle v^2 \rangle}{G}$$

Hence, one can infer the mass of our cluster of galaxies from its velocity dispersion and its gravitation radius. In general, though, neither of these is observable, and one uses instead

$$M = \alpha \frac{R_{\rm eff} \langle v_{\rm los}^2 \rangle}{G}$$

where $v_{\rm los}$ is the line-of-sight velocity, $R_{\rm eff}$ is some measure for the 'effective' radius of the system in question, and α is a parameter of order unity that depends on the radial distribution of the galaxies. Note that, under the assumption of **isotropy**, $\langle v_{\rm los}^2 \rangle = \langle v^2 \rangle / 3$ and one can also infer the mean reciprocal pair separation from the projected pair separations; in other words under the assumption of isotropy one can infer α, and thus use the above equation to compute the total, gravitational mass of the cluster. This method was applied by **Fritz Zwicky** in 1933, who inferred that the total dynamical mass in the Coma cluster is much larger than the sum of the masses of its galaxies. This was the first observational evidence for **dark matter**, although it took the astronomical community until the late 70's to generally accept this notion.

For a self-gravitating fluid

$$K = \sum_{i=1}^{N} \frac{1}{2} m_i v_i^2 = \frac{1}{2} N m \langle v^2 \rangle = \frac{3}{2} N k_{\rm B} T$$

where the last step follows from the kinetic theory of ideal gases of monoatomic particles. In fact, we can use the above equation for *any* fluid (including a collisionless one), if we interpret T as an **effective** temperature that measures the rms velocity of the constituent particles. If the system is in virial equilibrium, then

$$E = -K = -\frac{3}{2} N k_{\rm B} T$$

which, as we show next, has some important implications...

Heat Capacity: the amount of heat required to increase the temperature by one degree Kelvin (or Celsius). For a self-gravitating fluid this is

$$C \equiv \frac{{\rm d}E}{{\rm d}T} = -\frac{3}{2} N k_{\rm B}$$

which is negative! This implies that **by losing energy, a gravitational system gets hotter**!! This is a very counter-intuitive result, that often leads to confusion and wrong expectations. Below we give three examples of implications of the **negative heat capacity** of gravitating systems,

Example 1: Drag on satellites Consider a satellite orbiting Earth. When it experiences friction against the (outer) atmosphere, it loses energy. This causes the system to become more strongly bound, and the orbital radius to shrink. Consequently, the energy loss results in the gravitational potential energy, W, becoming more negative. In order for the satellite to re-establish virial equilibrium ($2K + W = 0$), its kinetic energy needs to **increase**. Hence, contrary to common intuition, *friction causes the satellite to speed up*, as it moves to a lower orbit (where the circular velocity is higher).

Example 2: Stellar Evolution A star is a gaseous, self-gravitating sphere that radiates energy from its surface at a luminosity L. Unless this energy is replenished (i.e., via some energy production mechanism in the star's interior), the star will react by shrinking (i.e., the energy loss implies an increase in binding energy, and thus a

potential energy that becomes more negative). In order for the star to remain in virial equilibrium its kinetic energy, which is proportional to temperature, has to increase; *the star's energy loss results in an increase of its temperature.*

In the Sun, hydrogen burning produces energy that replenishes the energy loss from the surface. As a consequence, the system is in equilibrium, and will not contract. However, once the Sun has used up all its hydrogren, it will start to contract and heat up, because of the **negative heat capacity**. This continues until the temperature in the core becomes sufficiently high that helium can start to fuse into heavier elements, and the Sun settles in a new equilibrium.

Example 3: Core Collapse *a system with negative heat capacity in contact with a heat bath is thermodynamically unstable.* Consider a self-gravitating fluid of 'temperature' T_1, which is in contact with a heat bath of temperature T_2. Suppose the system is in thermal equilibrium, so that $T_1 = T_2$. If, due to some small disturbance, a small amount of heat is tranferred from the system to the heat bath, the negative heat capacity implies that this results in $T_1 > T_2$. Since heat always flows from hot to cold, more heat will now flow from the system to the heat bath, further increasing the temperature difference, and T_1 will continue to rise without limit. This run-away instability is called the **gravothermal catastrophe**. An example of this instability is the **core collapse of globular clusters:** Suppose the formation of a gravitational system results in the system having a declining velocity dispersion profile, $\sigma^2(r)$ (i.e., σ decreases with increasing radius). This implies that the central region is (dynamically) hotter than the outskirts. **IF** heat can flow from the center to those outskirts, the gravothermal catastrophe kicks in, and σ in the central regions will grow without limits. Since $\sigma^2 = GM(r)/r$, the central mass therefore gets compressed into a smaller and smaller region, while the outer regions expand. This is called **core collapse**. Note that this does NOT lead to the formation of a supermassive black hole, because regions at smaller r always shrink faster than regions at somewhat larger r. In dark matter halos, and elliptical galaxies, the velocity dispersion profile is often declining with radius. However, in those systems the two-body relaxation time is soo long that there is basically no heat flow (which requires two-body interactions). However, globular clusters, which consist of $N \sim 10^4$ stars, and have a crossing time of only $t_{\rm cross} \sim 5 \times 10^6$yr, have a two-body relaxation time of only $\sim 5 \times 10^8$yr. Hence, heat flow in globular clusters is not negligible, and they can (and do) undergo core collapse. The collapse does not proceed indefinitely, because of binaries (see `Galactic Dynamics` by Binney & Tremaine for more details).

CHAPTER 23

Collisions & Encounters of Collisionless Systems

Consider an encounter between two collisionless N-body systems (i.e., dark matter halos or galaxies): a perturber P and a system S. Let q denote a particle of S and let b be the impact parameter, v_∞ the initial speed of the encounter, and R_0 the distance of closest approach (see Fig. 27).

Typically what happens in an encounter is that **orbital energy** (of P wrt S) **is converted into random motion energy** of the constituent particles of P and S (i.e., q gains kinetic energy wrt S).

The **velocity impulse** $\Delta\vec{v}_{\rm q} = \int \vec{g}(t)\,{\rm d}t$ of q due to the gravitational field $\vec{g}(t)$ from P decreases with increasing v_∞ (simply because Δt will be shorter). Consequently, when v_∞ increases, less and less orbital energy is transferred to random motion, and there is a critical velocity, $v_{\rm crit}$, such that

$$\begin{aligned} v_\infty > v_{\rm crit} \quad &\Rightarrow \quad S \text{ and } P \text{ escape from each other} \\ v_\infty < v_{\rm crit} \quad &\Rightarrow \quad S \text{ and } P \text{ merge together} \end{aligned}$$

There are only two cases in which we can calculate the outcome of the encounter analytically:

- **high speed encounter** ($v_\infty \gg v_{\rm crit}$). In this case the encounter is said to be **impulsive** and one can use the impulsive approximation to compute its outcome.
- **large mass ratio** ($M_{\rm P} \ll M_{\rm S}$). In this case one can use the treatment of **dynamical friction** to describe how P loses orbital energy and angular momentum to S.

In all other cases, one basically has to resort to numerical simulations to study the outcome of the encounter. In what follows we present treatments of first the impulse approximation and then dynamical friction.

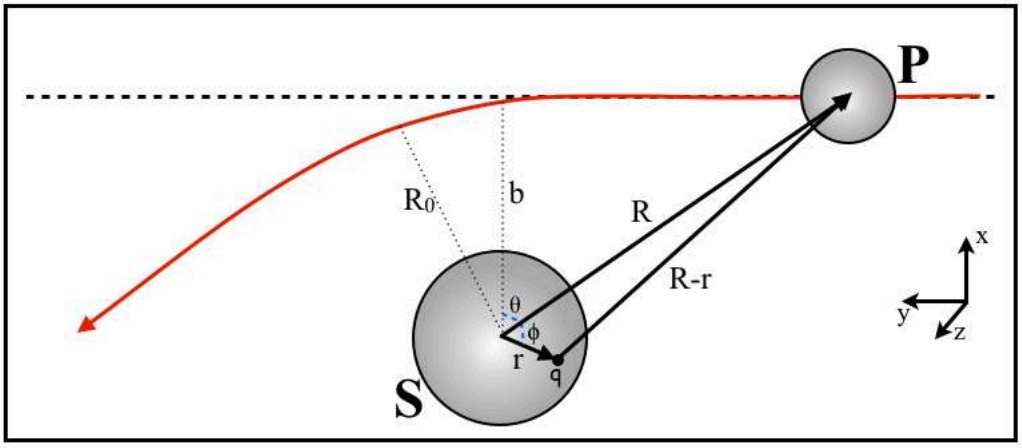


Figure 27: *Schematic illustration of an encounter with impact parameter b between a perturber P and a subject S.*

Impulse Approximation: In the limit where the encounter velocity v_∞ is much larger than the internal velocity dispersion of S, the change in the internal energy of S can be approximated analytically. The reason is that, in a high-speed encounter, the time-scale over which the tidal forces from P act on q is much shorter than the dynamical time of S (or q). Hence, we may consider q to be stationary (fixed wrt S) during the encounter. Consequently, q only experiences a change in its **kinetic** energy, while its potential energy remains unchanged:

$$\Delta E_{\rm q} = \frac{1}{2}(\vec{v}+\Delta\vec{v})^2 - \frac{1}{2}\vec{v}^2 = \vec{v}\cdot\Delta\vec{v} + \frac{1}{2}|\Delta\vec{v}|^2$$

We are interested in $\Delta E_{\rm S} = \sum_q \Delta E_{\rm q}$, where the summation is over all its constituent particles:

$$\Delta E_{\rm S} = \int \Delta E_{\rm q}(\vec{r})\,\rho(r)\,{\rm d}^3\vec{r} \simeq \frac{1}{2}\int |\Delta\vec{v}|^2\,\rho(r){\rm d}^3\vec{r}$$

where we have used that, because of symmetry, the integral

$$\int \vec{v}\cdot\Delta\vec{v}\,\rho(r)\,{\rm d}^3\vec{r} \simeq 0$$

In the large v_∞ limit, we have that the distance of closest approach $R_0 \to b$, and the velocity of P wrt S is $v_{\rm P}(t) \simeq v_\infty\vec{e}_y \equiv v_{\rm P}\vec{e}_y$. Consequently, we have that

$$\vec{R}(t) = (b, v_{\rm P}t, 0)$$

Let $\vec{r}$ be the position vector of q wrt S and adopt the **distant encounter approximation**, which means that $b \gg \max[R_{\rm S}, R_{\rm P}]$, where $R_{\rm S}$ and $R_{\rm P}$ are the sizes of S and P, respectively. This means that we may treat P as a point mass $M_{\rm P}$, so that

$$\Phi_{\rm P}(\vec{r}) = -\frac{GM_{\rm P}}{|\vec{r} - \vec{R}|}$$

Using geometry, and defining ϕ as the angle between $\vec{r}$ and $\vec{R}$, we we have that

$$|\vec{r} - \vec{R}|^2 = (R - r\cos\phi)^2 + (r\sin\phi)^2$$

so that

$$|\vec{r} - \vec{R}| = \sqrt{R^2 - 2rR\cos\phi + r^2}$$

Next we use the series expansion

$$\frac{1}{\sqrt{1+x}} = 1 - \frac{1}{2}x + \frac{1}{2}\frac{3}{4}x^2 - \frac{1}{2}\frac{3}{4}\frac{5}{6}x^3 + ...$$

to write

$$\frac{1}{|\vec{r} - \vec{R}|} = \frac{1}{R}\left[1 - \frac{1}{2}\left(-2\frac{r}{R}\cos\phi + \frac{r^2}{R^2}\right) + \frac{3}{8}\left(-2\frac{r}{R}\cos\phi + \frac{r^2}{R^2}\right)^2 + ...\right]$$

Substitution in the expression for the potential of P yields

$$\Phi_{\rm P}(\vec{r}) = -\frac{GM_{\rm P}}{R} - \frac{GM_{\rm P}r}{R^2}\cos\phi - \frac{GM_{\rm P}r^2}{R^3}\left(\frac{3}{2}\cos^2\phi - \frac{1}{2}\right) + \mathcal{O}[(r/R)^3]$$

- The first term on rhs is a constant, not yielding any force (i.e., $\nabla_r \Phi_{\rm P} = 0$).
- The second term on the rhs describes how the center of mass of S changes its velocity due to the encounter with P.
- The third term on the rhs corresponds to the **tidal force** per unit mass and is the term of interest to us.

It is useful to work in a rotating coordinate frame (x', y', z') centered on S and with the x'-axis pointing towards the instantaneous location of P, i.e., x' points along $\vec{R}(t)$. Hence, we have that $x' = r'\cos\phi$, where $r'^2 = x'^2 + y'^2 + z'^2$. In this new coordinate frame, we can express the third term of $\Phi_{\rm P}(\vec{r})$ as

$$\begin{aligned}\Phi_3(\vec{r}) &= -\frac{GM_{\rm P}}{R^3}\left(\frac{3}{2}r'^2\cos^2\phi - \frac{1}{2}r'^2\right)\\ &= -\frac{GM_{\rm P}}{R^3}\left(x'^2 - \frac{1}{2}y'^2 - \frac{1}{2}z'^2\right)\end{aligned}$$

Hence, the tidal force is given by

$$\vec{F}'_{\rm tid}(\vec{r}) \equiv -\nabla\Phi_3 = \frac{GM_{\rm P}}{R^3}(2x', -y', -z')$$

We can relate the components of $\vec{F}'_{\rm tid}$ to those of the corresponding tidal force, $\vec{F}_{\rm tid}$ in the (x, y, z)-coordinate system using

$$\begin{aligned} x' &= x\cos\theta - y\sin\theta & \qquad F_x &= F_{x'}\cos\theta + F_{y'}\sin\theta\\ y' &= x\sin\theta + y\cos\theta & F_y &= -F_{x'}\sin\theta + F_{y'}\cos\theta\\ z' &= z & F_z &= F_{z'}\end{aligned}$$

where θ is the angle between the x and x' axes, with $\cos\theta = b/R$ and $\sin\theta = v_P t/R$. After some algebra one finds that

$$\begin{aligned}F_x &= \frac{GM_{\rm P}}{R^3}\left[x(2-3\sin^2\theta) - 3y\sin\theta\cos\theta\right]\\ F_y &= \frac{GM_{\rm P}}{R^3}\left[y(2-3\cos^2\theta) - 3x\sin\theta\cos\theta\right]\\ F_z &= -\frac{GM_{\rm P}}{R^3}z\end{aligned}$$

Using these, we have that

$$\Delta v_x = \int \frac{{\rm d}v_x}{{\rm d}t}\,{\rm d}t = \int F_x\,{\rm d}t = \int_{-\pi/2}^{\pi/2} F_x\,\frac{{\rm d}t}{{\rm d}\theta}\,{\rm d}\theta$$

with similar expressions for Δv_y and Δv_z. Using that $\theta = \tan^{-1}(v_{\rm P}t/b)$ one has that ${\rm d}t/{\rm d}\theta = b/(v_{\rm P}\cos^2\theta)$. Substituting the above expressions for the tidal force, and using that $R = b/\cos\theta$, one finds, after some algebra, that

$$\Delta\vec{v} = (\Delta v_x, \Delta v_y, \Delta v_z) = \frac{2GM_{\rm P}}{v_{\rm P}b^2}(x, 0, -z)$$

Substitution in the expression for $\Delta E_{\rm S}$ yields

$$\Delta E_{\rm S} = \frac{1}{2}\int |\Delta\vec{v}|^2\,\rho(r)\,{\rm d}^3\vec{r} = \frac{2\,G^2\,M_{\rm P}^2}{v_{\rm P}^2\,b^4}\,M_{\rm S}\,\langle x^2+z^2\rangle$$

Under the assumption that S is spherically symmetric we have that $\langle x^2+z^2\rangle = \frac{2}{3}\langle x^2+y^2+z^2\rangle = \frac{2}{3}\langle r^2\rangle$ and we obtain the final expression for the energy increase of S as a consequence of the impulsive encounter with P:

$$\boxed{\Delta E_{\rm S} = \frac{4}{3}G^2 M_{\rm S}\left(\frac{M_{\rm P}}{v_{\rm P}}\right)^2\frac{\langle r^2\rangle}{b^4}}$$

This derivation, which is originally due to Spitzer (1958), is surprisingly accurate for encounters with $b > 5{\rm max}[R_{\rm P}, R_{\rm S}]$, even for relatively slow encounters with $v_\infty \sim \sigma_S$. For smaller impact parameters one has to make a correction (see `Galaxy Formation and Evolution` by Mo, van den Bosch & White 2010 for details).

The **impulse approximation** shows that high-speed encounters can pump energy into the systems involved. This energy is tapped from the orbital energy of the two systems wrt each other. Note that $\Delta E_{\rm S} \propto b^{-4}$, so that close encounters are far more important than distant encounters.

Let $E_{\rm b} \propto GM_{\rm S}/R_{\rm S}$ be the **binding energy** of S. Then, it is tempting to postulate that if $\Delta E_{\rm S} > E_{\rm b}$ the impulsive encounter will lead to the **tidal disruption** of S. However, this is not at all guaranteed. What is important for disruption is how that energy $\Delta E_{\rm S}$ is distributed over the consistituent particles of S. Since $\Delta E \propto r^2$, particles in the outskirts of S typically gain much more energy than those in the center. However, particles in the outskirts are least bound, and thus require the least amount of energy to become unbound. Particles that are strongly bound (with small r) typically gain very little energy. As a consequence, a significant fraction of the mass can remain bound even if $\Delta E_{\rm S} \gg E_{\rm b}$ (see van den Bosch et al., 2019, MNRAS, **474**, 3043 for details).

After the encounter, S has gained **kinetic energy** (in the amount of $\Delta E_{\rm S}$), but its **potential energy** has remained unchanged (recall, this is the assumption that underlies the impulse approximation). As a consequence, after the encounter S will no longer be in **virial equilibrium**; S will have to readjust itself to re-establish virial equilibrium.

Let K_0 and E_0 be the initial (pre-encounter) kinetic and total energy of S. The **virial theorem** ensures that $E_0 = -K_0$. The encounter causes an increase of (kinetic) energy, so that $K_0 \rightarrow K_0 + \Delta E_{\rm S}$ and $E_0 \rightarrow E_0 + \Delta E_{\rm S}$. After S has re-established virial equilibrium, we have that $K_1 = -E_1 = -(E_0 + \Delta E_{\rm S}) = K_0 - \Delta E_{\rm S}$. Thus, we see that virialization after the encounter changes the kinetic energy of S from $K_0 + \Delta E_{\rm S}$ to $K_0 - \Delta E_{\rm S}$! The gravitational energy after the encounter is $W_1 = 2E_1 = 2E_0 + 2\Delta E_{\rm S} = W_0 + 2\Delta E_{\rm S}$, which is less negative than before the encounter. Using the definition of the **gravitational radius** (see Chapter 12), $r_{\rm g} = GM_{\rm S}^2/|W|$, from which it is clear that the (gravitational) radius of S increases due to the impulsive encounter. Note that here we have ignored the complication coming from the fact that the injection of energy $\Delta E_{\rm S}$ may result in unbinding some of the mass of S.

Dynamical Friction: Consider the motion of a subject mass $M_{\rm S}$ through a medium of individual particles of mass $m \ll M_{\rm S}$. The subject mass $M_{\rm S}$ experiences a "drag force", called dynamical friction, which transfers orbital energy and angular momentum from $M_{\rm S}$ to the sea of particles of mass m.

There are three different "views" of dynamical friction:

1. Dynamical friction arises from **two-body encounters** between the subject mass and the particles of mass m, which drives the system twowards **equipartition**. i.e., towards $\frac{1}{2}M_{\rm S}v_{\rm S}^2 = \frac{1}{2}m\langle v_m^2\rangle$. Since $M_{\rm S} \gg m$, the system thus evolves towards $v_{\rm S} \ll v_m$ (i.e., $M_{\rm S}$ slows down).

2. Due to **gravitational focussing** the subject mass $M_{\rm S}$ creates an overdensity of particles behind its path (the "wake"). The gravitational back-reaction of this wake on $M_{\rm S}$ is what gives rise to dynamical friction and causes the subject mass to slow down.

3. The subject mass $M_{\rm S}$ causes a perturbation $\delta\Phi$ in the potential of the collection of particles of mass m. The gravitational interaction between the **response density** (the density distribution that corresponds to $\delta\Phi$ according to the Poisson equation) and the subject mass is what gives rise to dynamical friction (see Fig. 28).

Although these views are similar, there are some subtle differences. For example, according to the first two descriptions dynamical friction is a **local** effect. The

third description, on the other hand, treats dynamical friction more as a **global** effect. As we will see, there are circumstances under which these views make different predictions, and if that is the case, the third and latter view presents itself as the better one.

Chandrasekhar derived an expression for the dynamical friction force which, although it is based on a number of questionable assumptions, yields results in reasonable agreement with simulations. This so-called **Chandrasekhar dynamical friction** force is given by

$$\boxed{\vec{F}_{\rm df} = M_{\rm S}\,\frac{{\rm d}\vec{v}_S}{{\rm d}t} = -\frac{4\pi\,G^2\,M_{\rm S}^2}{v_{\rm S}^2}\,\ln\Lambda\,\rho(<v_{\rm S})\,\frac{\vec{v}_{\rm S}}{v_{\rm S}}}$$

Here $\rho(<v_{\rm S})$ is the density of particles of mass m that have a speed $v_m < v_{\rm S}$, and $\ln\Lambda$ is called the **Coulomb logarithm**. It's value is uncertain (typically $3 \lesssim \ln\Lambda \lesssim 30$). One often approximates it as $\ln\Lambda \sim \ln(M_{\rm h}/M_{\rm S})$, where $M_{\rm h}$ is the total mass of the system of particles of mass m, but this should only be considered a very rough estimate at best. The uncertainties for the Coulomb logarithm derive from the oversimplified assumptions made by Chandrasekhar, which include that the medium through which the subject mass is moving is infinite, uniform and with an isotropic velocity distribution $f(v_m)$ for the sea of particles.

Similar to **frictional drag** in fluid mechanics, $\vec{F}_{\rm df}$ is always pointing in the direction **opposite** of $v_{\rm S}$.

Contrary to **frictional drag** in fluid mechanics, which always increases in strength when $v_{\rm S}$ increases, dynamical friction has a more complicated behavior: In the low-$v_{\rm S}$ limit, $F_{\rm df} \propto v_{\rm S}$ (similar to hydrodynamical drag). However, in the high-$v_{\rm S}$ limit one has that $F_{\rm df} \propto v_{\rm S}^{-2}$ (which arises from the fact that the factor $\rho(<v_{\rm S})$ saturates).

Note that $\vec{F}_{\rm df}$ is **independent** of the mass m of the constituent particles, and proportional to $M_{\rm S}^2$. The latter arises, within the second or third view depicted above, from the fact that the wake or response density has a mass that is proportional to $M_{\rm S}$, and the gravitational force between the subject mass and the wake/response density therefore scales as $M_{\rm S}^2$.

One shortcoming of Chandrasekhar's dynamical friction description is that it treats

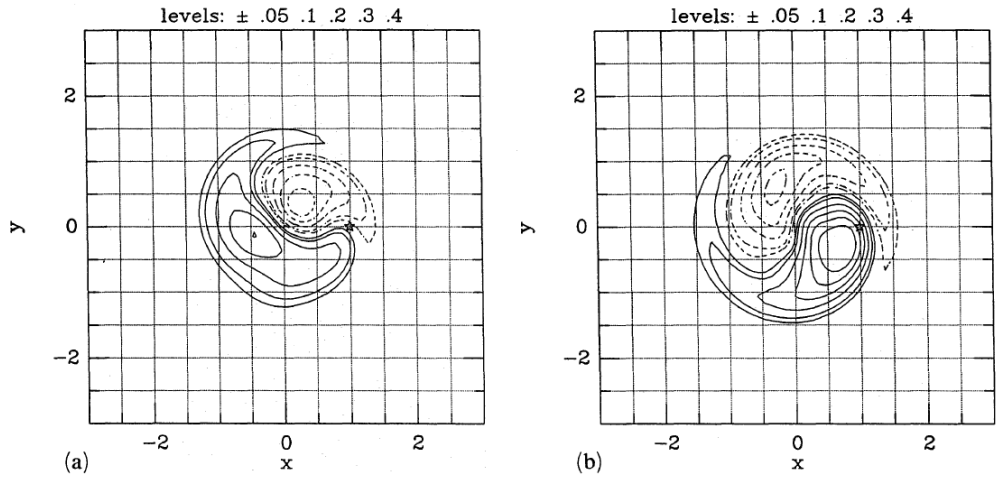


Figure 28: *Examples of the response density in a host system due to a perturber orbiting inside it. The back-reaction of this response density on the perturber causes the latter to experience dynamical friction. The position of the perturber is indicated by an asterisk. [*`Source:` *Weinberg, 1989, MNRAS, 239, 549]*

dynamical friction as a purely **local phenomenon**; it is treated as the cumulative effect of many **uncorrelated** two-body encounters between the subject masss and the individual field particles. That this local treatment is incomplete is evident from the fact that an object A orbiting **outside** of an N-body system B still experiences dynamical friction. This can be understood with the picture sketched under view 3 above, but not in a view that treats dynamical friction as a local phenomenon.

Orbital decay: As a consequence of dynamical friction, a subject mass $M_{\rm S}$ orbiting inside (or just outside) of a larger N-body system of mass $M_{\rm h} > M_{\rm S}$, will transfer its orbital energy and angular momentum to the constituent particles of the 'host' mass. As a consequence it experiences orbital decay.

Let us assume that the host mass is a **singular isothermal sphere** with density and potential given by

$$\rho(r) = \frac{V_{\rm c}^2}{4\pi G r^2} \qquad\qquad \Phi(r) = V_{\rm c}^2 \ln r$$

where $V_{\rm c}^2 = GM_{\rm h}/r_{\rm h}$ with $r_{\rm h}$ the radius of the host mass. If we further assume that this host mass has, at each point, an **isotropic** and **Maxwellian** velocity distrubution, then

$$f(v_m) = \frac{\rho(r)}{(2\pi\sigma^2)^{3/2}} \exp\left[-\frac{v_m^2}{2\sigma^2}\right]$$

with $\sigma = V_c/\sqrt{2}$.

NOTE: the assumption of a singular isothermal sphere with an isotropic, Maxwellian velocity distribution is unrealistic, but it serves the purpose of the order-of-magnitude estimate for the orbital decay rate presented below.

Now consider a subject of mass $M_{\rm S}$ moving on a **circular orbit** ($v_{\rm S} = V_{\rm c}$) through this host system of mass $M_{\rm h}$. The **Chandrasekhar dynamical friction** that this subject mass experiences is

$$F_{\rm df} = -\frac{4\pi\,\ln\Lambda\,G^2\,M_{\rm S}^2\,\rho(r)}{V_{\rm c}^2}\left[{\rm erf}(1) - \frac{2}{\sqrt{\pi}}{\rm e}^{-1}\right] \simeq -0.428\,\ln\Lambda\,\frac{GM_{\rm S}^2}{r^2}$$

The subject mass has **specific angular momentum** $L = rv_{\rm S}$, which it loses due to dynamical friction at a rate

$$\frac{{\rm d}L}{{\rm d}t} = r\frac{{\rm d}v_{\rm S}}{{\rm d}t} = r\frac{F_{\rm df}}{M_{\rm S}} \simeq -0.428\,\ln\Lambda\,\frac{GM_{\rm S}}{r}$$

Due to this angular momentum loss, the subject mass moves to a smaller radius, while it continues to move on a circular orbit with $v_{\rm S} = V_{\rm c}$. Hence, the rate at which the orbital radius changes obeys

$$V_{\rm c}\frac{{\rm d}r}{{\rm d}t} = \frac{{\rm d}L}{{\rm d}t} = -0.428\ln\Lambda\,\frac{GM_{\rm S}}{r}$$

Solving this differential equation subject to the initial condition that $r(0) = r_i$, one finds that the subject mass $M_{\rm S}$ reaches the center of the host after a time

$$\boxed{t_{\rm df} = \frac{1.17}{\ln\Lambda}\,\frac{r_i^2\,V_{\rm c}}{G\,M_{\rm S}} = \frac{1.17}{\ln\Lambda}\left(\frac{r_i}{r_{\rm h}}\right)^2\,\frac{M_{\rm h}}{M_{\rm S}}\,\frac{r_{\rm h}}{V_{\rm c}}}$$

In the case where the host system is a **virialized dark matter halo** we have that

$$\frac{r_{\rm h}}{V_{\rm c}} \simeq \frac{1}{10H(z)} = 0.1t_{\rm H}$$

where $t_{\rm H}$ is called the **Hubble time**, and is approximately equal to the age of the Universe corresponding to redshift z (the above relation derives from the fact

that virialized dark matter halos all have the same average density). Using that $\ln\Lambda \sim \ln(M_{\rm h}/M_{\rm S})$ and assuming that the subject mass starts out from an initial radius $r_i = r_{\rm h}$, we obtain a dynamical friction time

$$t_{\rm df} = 0.12\,\frac{M_{\rm h}/M_{\rm S}}{\ln(M_{\rm h}/M_{\rm S})}\,t_{\rm H}$$

Hence, the time $t_{\rm df}$ on which dynamical friction brings an object of mass $M_{\rm S}$ moving in a host of mass $M_{\rm h}$ from an initial radius of $r_i = r_{\rm h}$ to $r = 0$ is shorter than the Hubble time as long as $M_{\rm S} \gtrsim M_{\rm h}/30$. Hence, dynamical friction is only effective for fairly massive objects, relative to the mass of the host. In fact, if you take into account that the subject mass experiences mass stripping as well (due to the tidal interactions with the host), the dynamical friction time increases by a factor 2 to 3, and $t_{\rm df} < t_{\rm H}$ actually requires that $M_{\rm S} \gtrsim M_{\rm h}/10$.

For a more detailed treatment of collisions and encounters of collisionless systems, see Chapter 12 of `Galaxy Formation and Evolution` by Mo, van den Bosch & White.

CHAPTER 24

Integrability & Relaxation

NOTE: This chapter often refers to 'galaxy' as shorthand for 'a gravitational, collisionless N-body system'. It is to be understood that what holds for a galaxy will also hold for any other gravitational, collisionless N-body system.

Our treatment of the dynamics of collisionless systems has thus far focussed on the (Jeans) modeling of equilibrium systems that satisfy the Virial theorem, and on collisions and encounters of such systems. This leaves a few open questions:

- Our discussion of Hamiltonians, action-angle variables and the Jeans theorem all relied heavily on the notion of **integrable Hamiltonians**. However, most realistic galactic potentials are not fully integrable. What are the implications of this non-integrability? How is this accounted for in the modeling? This is (partially) addressed by **KAM theory**.
- How do galaxies virialize or relax, i.e., how do they reach a steady state? In a collisional gas this is achieved through the collisions, which establish equipartition of energy and momentum; the system settles in hydrostatic equilibrium and given enough time will establish thermal equilibrium through conduction. In a collisionless system, though, two-body relaxation is extremely inefficient, yet galaxies are (or at least appear) to be fairly 'relaxed'? How is this achieved? This is the topic of **collisionless relaxation**.

Integrability and KAM theory:

If the Hamiltonian of a galaxy is integrable, all its orbits are **regular**, which means that they admit three isolating integrals of motion (three actions). As a consequence the orbit is confined to a 6-3=3 dimensional manifold in phase-space, the 3-torus. In other words, a canonical transformation $(\vec{x},\vec{v}) \rightarrow (\vec{I},\vec{\theta})$ can be found such that

$$\dot{I}_i = 0\,, \qquad \dot{\theta}_i = \omega_i$$

with ω_i the natural frequencies of the orbit (see Chapter 5). The three actions define the shape of the torus, while the corresponding angles define the location on the torus.

Associated with each action, I_i, is a **fundamental frequency** $\omega_i = \partial\mathcal{H}/\partial I_i$, which is the frequency with which the corresponding angle, θ, increases from 0 to 2π. In general, the three fundamental frequencies of a regular orbit are **incommensurable**, which means that their ratios cannot be expressed as ratios of integers. Such orbits, over time, cover every point along their 3-torus (we say, they are **ergodic** on their torus). For certain orbits, though, a resonance between the fundamental frequencies occurs: i.e., $n\omega_1 + l\omega_2 + m\omega_3 = 0$, with n, l, and m integers. These orbits do not densely fill the torus, but rather they densely fill a manifold *on* the torus that has a dimensionality one less than that of the full surface of the 3 torus. If the orbit obeys two independent commensurability conditions, the orbit is a closed loops on the 3-torus. Tori for which the fundamental frequencies are resonant with each other are called **resonant tori**: they comprise a set of measure zero, although they are dense in phase-space (in the same way that rational numbers are rare compared to irrational numbers). In integrable Hamiltonians, all orbits lie on tori, and the Hamiltonian can be written as $\mathcal{H} = \mathcal{H}(\vec{I})$; and according to the Jeans theorem the same holds for the distribution function, i.e., $f = f(\vec{I})$.

However, (fully) integrable Hamiltonians are rare. First of all, after it was discovered in the mid 70's that most ellipticals rotate more slowly than needed in order to explain their flattening, it became clear that their most likely shape is **triaxial** (the flattening must be due to anisotropic 'pressure', i.e., velocity dispersion, and there is no reason to assume that it is identical along 2 directions, but different along the third). Triaxial galaxies have many box-orbits, which are orbits without a definite sense of rotation, which over time pass arbitrarily close to the center. In the late 90s, with the advent of the Hubble Space Telescope, it became clear that most ellipticals have steep, central **cusps** (an exception are the most massive ellipticals, which seem to have a central core), and it is believed that they all contain a **supermassive black hole** (SMBH) in their center. It was shown, though, that in such systems many of the box orbits are **chaotic** (aka irregular or stochastic). Such orbits have fewer isolating integrals of motion than there are degrees of freedom, and they are not confined to a 3-torus. Hence, the Hamiltonian of these systems cannot be fully integrable! As a general rule of thumb, triaxial systems are far more likely to be non-integrable than axisymmetric systems, which in turn are far more likely to be non-integrable than spherical systems.

Let's assume that we can write the Hamilonian of a non-integrable system as

$$\mathcal{H} = \mathcal{H}_0(\vec{I}) + \varepsilon\,\mathcal{H}_1(\vec{I}, \vec{\theta})$$

where $\mathcal{H}_0$ is integrable and ε is a (hopefully small) perturbation parameter. According to the **Kolmogorov-Arnold-Moser (KAM) theorem**, the tori of regular

orbits with fundamental frequencies that are sufficiently incommensurable are stable to such small perturbations, which means that they remain confined to a (slightly modified) 3-torus. Resonant tori, however, can be strongly deformed even by a very small perturbation, and often will 'break apart': the resonant orbit when perturbed becomes a chaotic orbit. Hence, according to the KAM theorem, a Hamiltonian that is **near-integrable** (i.e., can be written as above with ε small) has a phase-space that is interleaved with regions occupied by nested tori and chaotic regions. In the 'regular' regions, where the tori exist, there exists a canonical transformation to some action-angle variables such that the Hamiltonian is independent of the angles. However, this canonical transformation is only local: it does not apply to the entire system.

When the perturbation parameter ε is small, the chaotic regions are narrow and the stochastic motion of the corresponding orbits is 'bounded' by nearby regular orbits (i.e., two phase-space trajectories cannot intersect) to the extent that the stochastic orbit can mimic regular motion for a long time (i.e., it is restricted in 'executing' its stochasticity). As ε increases, the chaotic regions increase in volume and at some point chaotic zones associated with different resonant tori start to overlap. This produces large regions of interconnected phase-space in which the motion is fully stochastic.

Since stochastic orbits typically only admit one isolating integral of motion (energy), they will, over time, densely fill their energy surface (i.e., these orbits are restricted only to move on a 6-1=5 dimensional manifold in 6D phase-space). And since potential (=energy) surfaces are always rounder than density surfaces, it becomes difficult, if not impossible, to construct a **self-consistent** triaxial system that is globally stochastic'. In other words, triaxial galaxies can't be globally stochastic. It is interesting, then, to ask under what conditions a triaxial system becomes globally stochastic. Valluri & Merritt (1998) investigated this is some detail, and found that triaxial systems with a central SMBH become globally stochastic when $M_{\rm BH}/M_{\rm gal} \gtrsim 0.02$, close to the maximum observed value in real ellipticals. In models without a SMBH, but with a central cusp, the system was found to become stochastic when the central cusp slope $\mathrm{d}\ln\rho/\mathrm{d}\ln r < -2$. Once again, this is close to the steepest cusp slope seen in real ellipticals. This stongly suggests that the structure of galaxies, under certain circumstances, is regulated by the onset of stochasticity.

Collisionless relaxation:

A steady-state gravitational system obeys the Virial Theorem, $E = -K = W/2$. If it is significantly perturbed, then it will be out of virial equilibrium. For exam-

ple, immediately following an impulsive encounter, the kinetic energy of the system has increased, but its potential energy has remained fixed. The system will start to oscillate, converting kinetic energy into potential energy, and vice versa. Relaxation is the process by which such oscillations die out, and by which the system re-establishes virial equilibrium. In addition to collisional (two-body) relaxation, which is typically insignificant, there are four **collisionless relaxation** mechanisms that are responsible for this re-equilibration:

- **phase mixing:** the loss of coherence in the response due to the fact that stars have different orbital frequencies
- **chaotic mixing:** the loss of coherence in the response due to the fact that stars on stochastic orbits diverge exponentially over time
- **violent relaxation:** the loss of coherence due to scrambling of orbital energies in a time-varying potential
- **Landau damping:** the loss of coherence due to non-dissipative damping of waves due to wave-particle interactions

Note that all these mechanisms have one thing in common: 'loss of coherence'. Oscillations in a collisionless system are manifestations of coherence in the response to the perturbation that triggered the oscillations. For example, after an impulsive encounter all stars at a given position receive the same velocity impuls, independent of their actions (i.e., their orbital energy or angular momentum). This means that the response starts out extremely coherent (all stars change their velocity vector by the same amount). *Relaxation is the loss of this coherence!* Hence, any mechanism that acts to reduce the coherence in the response is a relaxation mechanism.

There is one other relaxation mechanism that is worth mentioning, which is **resonant relaxation**. It is relaxation that arises from the efficient exchange of angular momentum or orbital energy between particles (or particles and a perturber) that are in resonance with each other. Resonant relaxation is a specific example of **collisional two-body relaxation** in which the impact of two-body interactions is 'magnified' due to a specific form of coherence between the particles involved (i.e., commensurability of frequencies). An example is **dynamical friction**, which can be thought of as a resonant-driven effect; only those stars that are in resonance with the perturber (the massive object orbiting the stellar body) can efficiently exchange angular momentum with the perturber. The other particles are 'out-of-phase' with the perturber and phase-mixing assures that their *net* effect on the perturber vanishes. Hence, dynamical friction only arises from orbits that are in (near)-resonance with

the perturber (see Tremaine & Weinberg 1991 and Banik & van den Bosch 2021). Because of its different nature (and a fairly involved mathematical treatment), we will not discuss resonant relaxation in what follows.

Phase mixing:
Phase mixing is the simplest mechanism that causes **collisionless relaxation** in gravitational N-body systems. The mechanism of phase-mixing is best understood by considering a large collection of pendulums, each with a slightly different length, and hence a slightly different frequency. Imagine one initially displaces all these pendulums by the same amount (the same phase), and then let them go. Initially, they all start to swing towards their equilibrium state together; i.e., the initial response to the displacement is highly coherent. However, after a few swings back and forth, this coherence is gone due to the fact that the differences in natural frequencies imply that the pendulums cannot remain in phase with one another. After a sufficiently long time the system is 'relaxed' to the point that at each instant in time there are (roughly) equal numbers of pendulums in each equal-size phase-interval.

In galaxies, phase mixing arises from the fact that the frequencies of regular motion on adjacent tori are generally similar but different. Hence, two points on adjacent tori that are initially close together in phase-space (i.e., that initially have similar phases), will seperate linearly with time. However, two points on the *same* torus do not phase-mix; their separation remains invariant.

Note that phase-mixing *decreases* the **coarse-grained DF** around a point, by mixing in 'vacuum' (i.e., unpopulated regions of phase-space). Nevertheless, as assured by the **CBE**, the flow in phase-space of a collisionless system is perfectly incompressible: unlike the coarse-grained DF, the **fine-grained DF** is not influenced by phase-mixing and is perfectly conserved.

Although phase-mixing is a relaxation mechanism, in that it drives the system towards a state in which the phase-space density is more and more uniform, it does not cause any loss of information: the system preserves all knowledge of the initial conditions. In other words, in an integrable Hamiltonian system, phase mixing is a **time-reversible** relaxation mechanism!

Chaotic mixing:
In the parts of phase-space that are not filled with **regular**, but with **stochastic** orbits, mixing occurs naturally due to the **chaotic** behavior of the orbits. Chaos implies extreme sensitivity to initial conditions: two stars intially close together separate exponentially with time.

After a sufficiently long time, a group of stars on stochastic orbits that were initially close together will have spread over their entire accessible phase-space (sometimes referred to as the **Arnold web**). As for phase-mixing, chaotic mixing thus smooths out (i.e., relaxes) the **coarse-grained DF** (it destroyes coherence in a response), but leaves the **fine-grained DF** invariant.

Unlike phase-mixing, though, chaotic mixing is **irreversible**, in the sense that an infinitely precise fine-tuning of the phase-space coordinates is required to undo its effects.

An important quantity to characterize chaotic behavior is the **Lyapunov timescale**, which is a measure that determines, for a given point in phase space, how quickly trajectories that begin near this point diverge over time. Actually, for each point in phase-space there are 6 Lyaponov exponents, λ_i, one for each direction in phase-space. It is common, though to just refer to the largest one. Consider a small 6-dimensional sphere with radius r centered on a phase-space point $(\vec{x}, \vec{v})$. Different points on the sphere's surface evolve differently with time, causing the sphere to deform into a 6-dimensional ellipsoid with principal axes $L_i(t)$.
The **Lyapunov exponents** at $(\vec{x}, \vec{v})$ are defined as

$$\lambda_i = \lim_{t\to\infty} \frac{1}{t} \ln \left| \frac{L_i(t)}{L_i(0)} \right|$$

In a collisionless system

$$\sum_{i=1}^{2N} \lambda_i = 0$$

which expresses the **incompressibility** of the flow (conservation of volume), as dictated by the CBE. If the phase-space trajectory $\Gamma(t)$ through $(\vec{x}, \vec{v})$ corresponds to a regular orbit, then $\lambda_i = 0$ for $i = 1, ..., 2N$. On the other hand, if $\Gamma(t)$ corresponds to a stochastic orbit then $\lambda \equiv \max \lambda_i > 0$. Such a positive Lyapunov exponent implies that the neighboring trajectories diverge exponentially:

$$\delta\Gamma(t) \propto \mathrm{e}^{\lambda t} .$$

The inverse of the largest Lyapunov exponent is called the **Lyapunov time**, τ_{L}, and defines the characteristic e-folding time on which neighboring trajectories diverge. Note that $\tau_{\mathrm{L}} = \infty$ for a regular orbit.

Violent relaxation:
Since $E = \frac{1}{2}v^2 + \Phi$ and $\Phi = \Phi(\vec{x}, t)$ we have that

$$\begin{aligned}
\frac{\mathrm{d}E}{\mathrm{d}t} &= \frac{\partial E}{\partial \vec{v}} \cdot \frac{\mathrm{d}\vec{v}}{\mathrm{d}t} + \frac{\partial E}{\partial \Phi}\frac{\mathrm{d}\Phi}{\mathrm{d}t} \\
&= -\vec{v} \cdot \nabla\Phi + \frac{\mathrm{d}\Phi}{\mathrm{d}t} \\
&= -\vec{v} \cdot \nabla\Phi + \frac{\partial \Phi}{\partial t} + \frac{\partial \Phi}{\partial \vec{x}} \cdot \frac{\mathrm{d}\vec{x}}{\mathrm{d}t} \\
&= -\vec{v} \cdot \nabla\Phi + \frac{\partial \Phi}{\partial t} + \vec{v} \cdot \nabla\Phi \\
&= \frac{\partial \Phi}{\partial t}
\end{aligned}$$

Thus we see that the *only* way in which a particle's energy can change in a collisionless system, is by having a **time-dependent potential**.

An interesting case to consider is the collapse of a dark matter halo, or that of a galaxy. In this case the potential will vary as function of time, and the particles thus change their energy. Exactly how a particle's energy changes depends in a complex way on the particle's initial position and energy: particles can both gain or lose energy (see Fig. 29). Overall, however, the effect is to widen the range of energies.

The **time-scale** for violent relaxation is

$$\tau_{\rm vr} = \left\langle \frac{(\mathrm{d}E/\mathrm{d}t)^2}{E^2} \right\rangle^{-1/2} = \left\langle \frac{(\partial\Phi/\partial t)^2}{E^2} \right\rangle^{-1/2} = \frac{3}{4}\langle \dot{\Phi}^2/\Phi^2 \rangle^{-1/2}$$

where the last step follows from the time-dependent virial theorem (see Lynden-Bell 1967). Hence, $t_{\rm vr}$ is of the order of the time-scale on which the potential changes by its own amount, which is basically the collapse time. Thus, violent relaxation is very fast, which explains its name.

Note that during the collapse of a collisionless system the **CBE** is still valid, i.e., the flow in phase-space remains incompressible and $\mathrm{d}f/\mathrm{d}t = 0$. However, unlike for a 'steady-state' system, $\partial f/\partial t \neq 0$. Violent relaxation leads to efficient **fine-grained mixing** of the DF, and erases the system's memory of its initial conditions. For comparison: phase-mixing and chaotic mixing only lead to a relaxation of the coarse-grained DF and phase-mixing is even reversible. Note, though, that this erasure of the initial conditions will not be complete (i.e., some correlation between the initial and final energies of the particles will remain). The reason is that violent relaxation

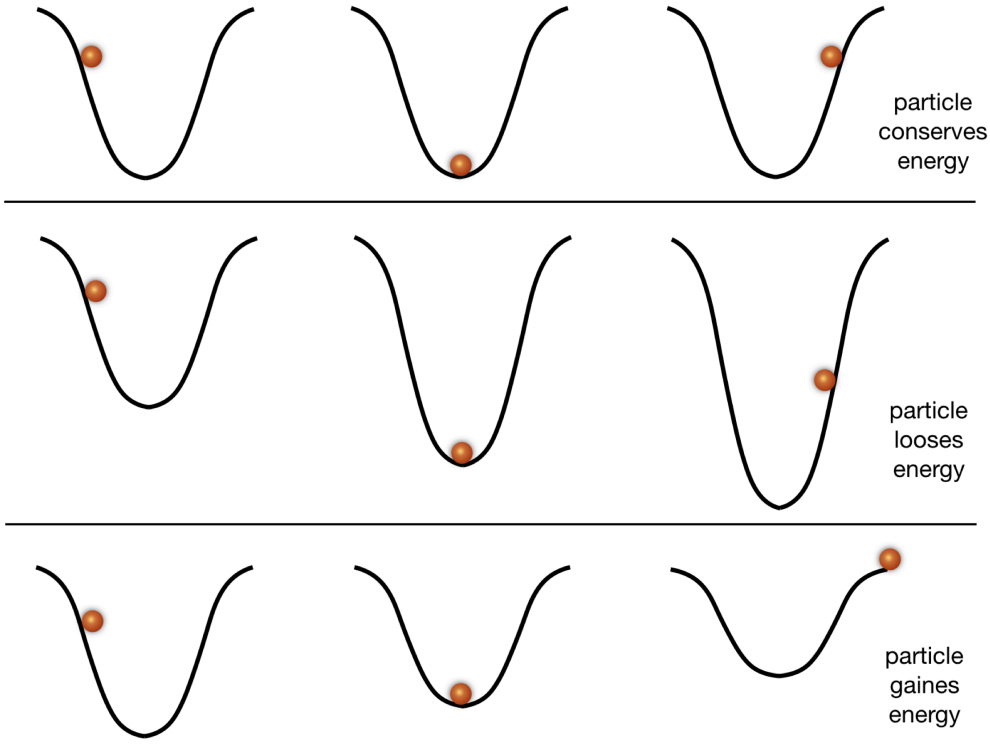


Figure 29: *Illustration of violent relaxation. Depending on the orbital phase compared to the phase of potential fluctuation, a particle can either gain or loose energy.*

is self-limiting. After all, it destroys the coherence in the oscillations of the potential that are responsible for the scrambling of the particle energies. Once the coherence disappears, the oscillations damp out and violent relaxation no longer operates.

Finally, an important aspect of violent relaxation is the fact that it changes a particle's energy in a way that is independent of the particle's mass. Thus violent relaxation drives the system to a relaxed state that is very different from the one promoted by collisional relaxation, which drives the system towards equipartition of energy and momentum (which results, among others, in mass segregation).

Landau damping:
In 1946 Landau showed that waves in a **collsionless plasma** can be damped, despite the fact that there is no dissipation. In 1962 Lynden-Bell showed that this damping mechanism, called **Landau damping**, also operates in gravitational, collisionless systems. This collisionless damping arises from the way the wave interacts with the particles that make up the medium through which the wave travels (i.e., particle-wave

interactions).

To gain insight, it is useful to start by considering a fluid. Perturbation analysis of the fluid shows that if the perturbation has a wavelength $\lambda < \lambda_{\rm J}$, with $\lambda_{\rm J}$ the **Jeans length**, then the perturbation is stable (see Chapter 19). The wave propagates with a **phase velocity** $v_{\rm p} = \omega/k$ (which can be different from the **group velocity** $v_{\rm g} = \partial\omega/\partial k$). Using that the dispersion relation is given by

$$\omega^2 = c_{\rm s}^2(k^2 - k_{\rm J}^2)$$

(see Chapter 19), we obtain that

$$v_{\rm p} = c_{\rm s}\sqrt{1 - \lambda^2/\lambda_{\rm J}^2}$$

with $c_{\rm s}$ the **sound speed**. Note that larger waves move slower, which owes to the fact that self-gravity becomes more and more important. When $\lambda = \lambda_{\rm J}$ the wave no longer propagates. Rather, the perturbation is unstable: self-gravity overpowers the pressure, causing the perturbation to grow.

One can apply a similar perturbation analysis to collisionless, gravitational systems. This yields a similar **Jeans criterion**, but with the velocity dispersion of the stars, σ, playing the role of the sound speed. Once again, perturbations with $\lambda > \lambda_{\rm J}$ are unstable and cause the perturbation to grow. For $\lambda < \lambda_{\rm J}$, however, the situation is somewhat different. While the fluid supports (gravity-modified) sound waves that are stable, the equivalent waves in gravitational systems experience Landau Damping.

Consider a density wave with $\lambda < \lambda_{\rm J}$. While for a fluid the **phase velocity** $v_{\rm p} < c_{\rm s}$, in a gravitational system we have that $v_{\rm p} < \sigma$. As explained below, stars that move *faster* than the wave (i.e., with $v > v_{\rm p}$) lose energy to the wave, while stars with $v < v_{\rm p}$ *gain* energy. If, for simplicity, we assume a Gaussian distribution of velocities, centered on $v = 0$ and with a velocity dispersion σ, we see immediately that there will be more stars with $v < v_{\rm p}$ than with $v > v_{\rm p}$. Consequently, the **net** effect is that the wave will lose energy to the background particles, and thus damp.

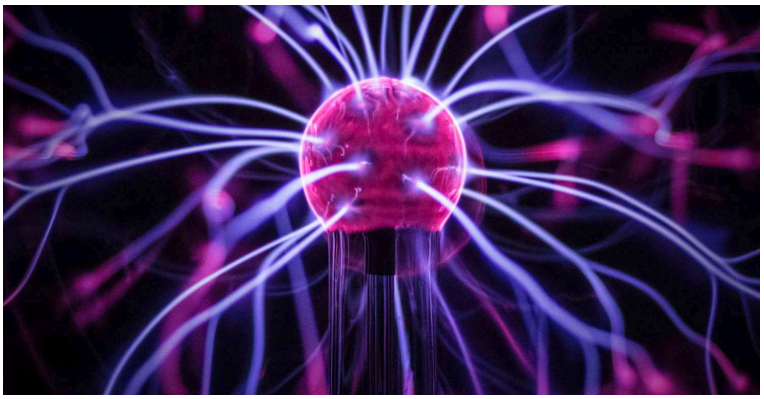
Understanding why particles with $v < v_{\rm p}$ ($v > v_{\rm p}$) gain (lose) energy to the wave, consider a surfer on the ocean. A (good) surfer sets his board such that the downhill direction of the wave that he is riding is the direction of motion of the wave. Under these conditions the wave is doing work on the surfer, propelling him forward (the force is the horizontal component of the buoyancy force that the water exerts on the surfboard).The surfer then adjusts his speed parallel to the wave crest such that this work is balanced by the friction that his board experiences moving through the

water. If the wave crest would overtake the surfer, he now finds himself pushing (doing work) on the wave, and he will slow down.

Note that Landau damping is dominated by the stars that are in near-resonance with the wave (i.e., the stars for which $|v - v_{\rm p}|$ is small). Stars that are far from resonance will pass the wave, or be overtaken by the wave, quickly, and they hence go from gaining to losing energy to the wave (or vice versa) in quick succession, the net effect of which is zero. In the other hand, stars that are only slightly slower than the wave will spend a long time 'riding the wave', draining energy from it, while the (slightly fewer) stars that are somewhat faster than the wave spend a long time pushing the wave, transferring their kinetic energy to the wave.

We end our discussion on collisionless relaxation by contrasting Landau damping and phase mixing. Both can damp waves/oscillations in a collisionless system, but do so in a different way. Phase mixing is purely a kinematic process that occurs in all systems, even those without self-gravity. Landau damping, on the other hand, is a **collective effect** that arise because of self-gravity. In addition, in phase-mixing the amplitude of the fluctuations in the DF do not decay. Rather, the fluctuations become more and more tightly wound in phase-space, and relaxation only affects the coarse-grained DF. In contract, Landau damping actually changes the energies (actions) of the particles, and thus changes the DF at the fine-grained level; it actually washes away the fluctuations in the DF.

Part IV: Plasma Physics



The following chapters give an elementary introduction into the rich topic of plasma physics. The main goal is to highlight how plasma physics differs from that the physics of neutral fluids. After introducing some characteristic time and length scales, we discuss plasma orbit theory and plasma kinetic theory before considering the dynamics of collisionless plasmas, described by the Vlasov equations, and that of collisional plasma, described (under certain circumstances) by the equations of magnetohydrodynamics.

Plasma is a rich topic, and one could easily devote an entire course to it. The following chapters therefore only scratch the surface of this rich topic. Readers who want to get more indepth information are referred to the following excellent textbooks
- *Introduction to Plasma Theory* by D.R. Nicholson
- *The Physics of Plasma* by T.J.M. Boyd & J.J. Sandweson
- *Plasma Physics for Astrophysics* by R.M. Kulsrud
- *The Physics of Fluids and Plasmas* by A.R. Choudhuri
- *Introduction to Modern Magnetohydrodynamics* by S. Galtier

CHAPTER 25

Plasma Characteristics

A **plasma** is a quasi-neutral gas of charged and neutral particles *which exhibits collective behavior*. The clause that a plasma needs to exhibit collective behavior implies that the number of particles inside a Debye volume is large.

In a plasma the motion of the charged particles is governed by the **Lorentz force** which depends on the electric and magnetic fields, which in turn are sourced, via the **Maxwell equations**, by the density distribution of the charges and their corresponding currents. This 'feedback loop' (the dynamics of the particles depends on the EM fields, which in turn are generated by the motion of the particles) is what makes plasma physics such a difficult topic.

Note though that it is not that different from a gravitational N-body systems, in which the motion of the particles is governed by the **gravitational force** which depends on the gravitational potential, which in turn is sourced, via the **Poisson equation**, by the density distribution of the particles.

NOTE ABOUT UNITS: in this and the following chapters on plasma physics we adopt the Gaussian system of units (see Chapter 0). This implies that the Coulomb force between two charges q_1 and q_2 is given by

$$F = \frac{q_1 q_2}{r^2}$$

By contrast, the same Coulomb law in the alternative SI unit system is given by

$$F = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}$$

with ε_0 the vacuum permittivity. Using Gaussian units also implies that the electric and magnetic fields have the same dimensions, and that the **Lorentz force** on a particles of charge q is given by

$$\vec{F} = q\left[\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right]$$

See any textbook on electromagnetism for details.

If the plasma is **weakly ionized**, then a charged particle is more likely to have a collision with a neutral particle. Such collisions take place when the particles are very close to each other and usually produce large deflections, similar to collisions between two neutral particles. Hence, a weakly ionized plasma can be described using the **Boltzmann equation**.

If the plasma is **highly ionized**, Coulomb interactions among the charged particles dominate. These are long-range interactions, and typically result in small deflections (see below). In addition, a particle typically has interactions with many other particles simultaneously. Hence, the collisions are not instantaneous, well-separated, and localized (i.e., short-range). Consequently, the Boltzmann equation does not apply, and we need to derive an alternative dynamical model. Unfortunately, this is a formidable problem that is not completely solved for an arbitrary, inhomogeneous magnetized plasma. Consequently, what we have are several different plasma models, each with their own ranch of validity and applicability (see Fig. 30 for a cursory overview). These include, among others, **Particle-in-Cell (PIC) simulations** in which the 'orbits' of individually charged particles, are integrated forward in time subjected to the Lorentz force resulting from the electromagnetic field due to all other particles, the **two-fluid model** that is (mainly) used to treat phenomena on small length and/or time scales for which the plasma can be treated as collisionless, and the **one-fluid model** known as **MagnetoHydroDynamics (MHD)**.

In our discussion of **neutral fluids** we have seen that a system of particles can be treated like a **continuum fluid** iff frequent collisions keep the distribution function in local regions close to a **Maxwellian**. Although not easy to proof, there is ample experimental evidence that shows that the collisions in a **plasmas** also relax to a **Maxwellian**. We therefore will seek to develop some continuum fluid model to describe our plasma. However, we need to keep in mind that such a model can only be used to describe plasma dynamics on large scales and long time scales (see below).

Characteristic Length and Time Scales:
We now describe a number of important length scales and time scales relevant for plasmas. Let $n_{\rm e}$ and $n_{\rm i}$ refer to the number densities of electrons and ions, respectively. Unless specified otherwise, we shall assume that ions are singly ionized. Since most of the matter in the Universe is hydrogen, this is a reasonable assumption to make in astrophysics (i.e., $n_{\rm i}$ is then basically the number density of free protons). In

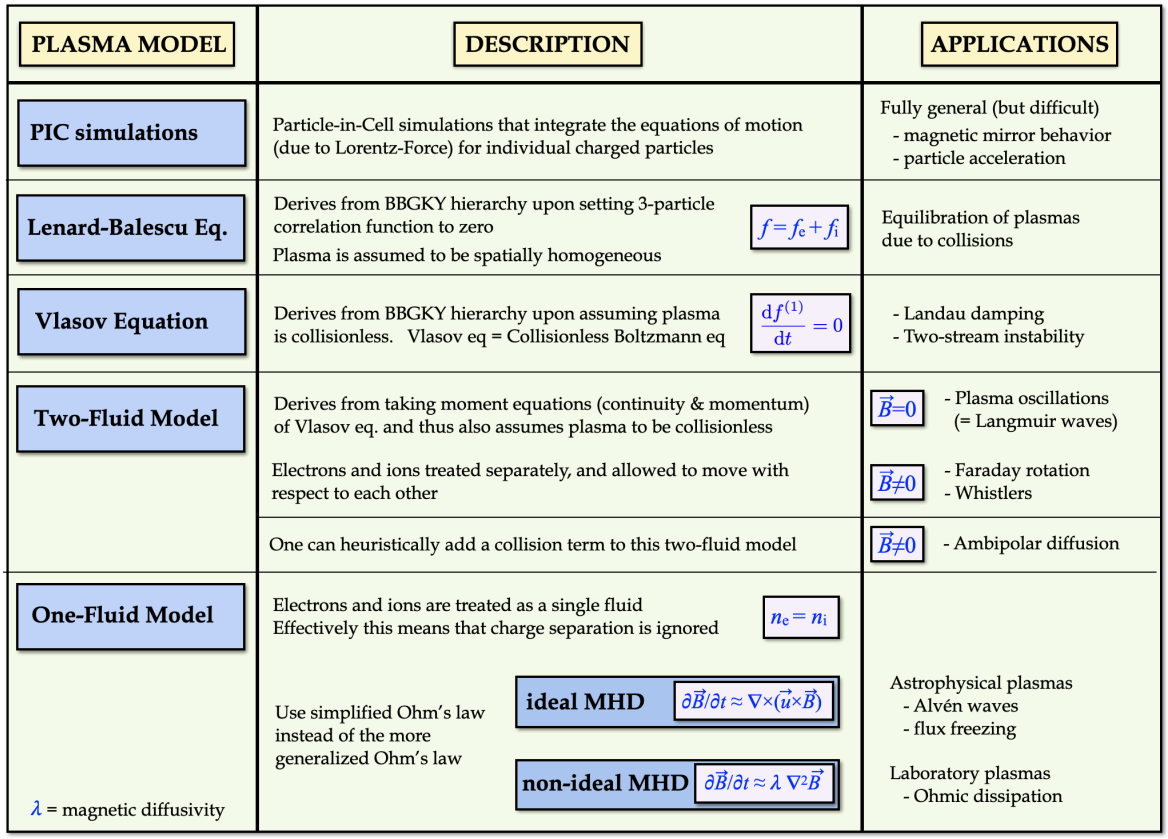


Figure 30: Overview of the various models that are used to describe plasmas.

astrophysics these number densities can span many orders of magnitudes. For example, the ISM has $n_e \sim 1\,\mathrm{cm}^{-3}$, while stellar interiors have densities $n_e \sim 10^{25}\,\mathrm{cm}^{-3}$.

The **charge density** of a plasma is defined as

$$\rho = e(n_p - n_e)$$

with e the charge of an electron. In a plasma, the strong electrostatic interactions among positively and negatively charged particles ensure that volumes with statistically large numbers of particles are **nearly neutral** (i.e., $\rho \simeq 0$). Hence, plasmas have some characteristics in common with neutral fluids. However, on small enough scales, particles feel the strong, electromagnetic forces from individual particles. Although a charged particle in a plasma in principle produces a long-range EM field, its effect is usually screened off by particles of the opposite charge within a distance

called the **Debye length**. The Debye lengths for electrons and ions are given by

$$\boxed{\lambda_{\rm e,i} = \left(\frac{k_{\rm B}\, T_{\rm e,i}}{4\pi n e^2}\right)^{1/2} \sim 740\ {\rm cm} \left(\frac{T\,[\,{\rm eV}]}{n\,[\,{\rm cm}^{-3}]}\right)^{1/2}}$$

(for a derivation, see any good textbook on Plasma Physics), while the **total Debye length**, $\lambda_{\rm D}$, is defined by

$$\lambda_{\rm D}^{-2} = \lambda_{\rm e}^{-2} + \lambda_{\rm i}^{-2}$$

Because of Debye shielding, the net electrical potential around a charge q is given by

$$\phi = \frac{q}{r}\exp(-r/\lambda_{\rm D})$$

Debye shielding is a prime example of collective behavior in a plasma; it indicates that each charged particle in a plasma, as it moves, basically carries, or better, tries to carry, a cloud of shielding electrons and ions with it.

• On scales $\lambda \gg \lambda_{\rm D}$, a plasma can be considered charge-neutral: any charge imbalance produces strong electrostatic forces which quickly restore charge-neutrality.

• On scales $\lambda \ll \lambda_{\rm D}$ particles experience strong Coulomb interactions.

The average number of particles on which a charged particle excerts an influence is roughly $n\,\lambda_{\rm D}^3$, with n the average number density of particles. Associated with this is the **Plasma Parameter**

$$\boxed{\Lambda \equiv 4\pi\, n\, \lambda_{\rm D}^3}$$

BEWARE: here is inconsistency in the literature regarding this definition; some texts define the plasma parameter as $n\lambda_{\rm D}^3$ (i.e., without the factor 4π, while, more annoyingly, other texts sometimes define the plasma parameter as $(n\lambda_{\rm D}^3)^{-1}$. And to complete the confusion, different texts also often use different symbols to denote the plasma parameter.

• When $\Lambda \ll 1$, the number of particles on which a charged particle exerts an influence is less than unity, and collisions are therefore typically well-separated in space and time (there are no collective effects). Hence, the system behaves pretty much like a neutral fluid. Such a fluid is generally not considered a plasma!

• When $\Lambda \gg 1$, many particles undergo simultaneous interactions, and as a consequence, the fluid manifests **collective behavior** (see Chapter 1 for a definition). This is known as the **plasma limit**.

NOTE: The plasma parameter $\Lambda \propto n^{-1/2}$. Hence, low density plasma's are more 'plasma-like' (display more collective phenomenology). Even though the number of particles per volume is smaller, the total number of particles within a Debye volume, $\lambda_{\rm D}^3$, is larger.

The average distance between particles is of the order $n^{-1/3}$. Hence, the average potential energy of electrostatic interactions is of the order $e^2\, n^{1/3}$. We thus see that

$$\frac{\langle \text{K.E.} \rangle}{\langle \text{P.E.} \rangle} \propto \frac{k_{\rm B}\, T}{e^2\, n^{1/3}} \propto \Lambda^{2/3}$$

In other words, *the plasma parameter is a measure for the ratio between the average kinetic energy of the particles and the average potential energy associated with their mutual interaction.*

• When $\Lambda \gg 1$, interactions among the particles are weak, but a large number of particles interact simultaneously, giving rise to plasma behavior.

• When $\Lambda \ll 1$, interactions among the particles are strong, but few particles interact collectively, and the fluid behaves like a neutral fluid. In fact, if $\Lambda < 1$ then the typical kinetic energy of the electrons is smaller then the potential energy due its nearest neighbor, and there would be a strong tendency for the electrons and ions to recombine, thus destroying the plasma. The need to keep the fluid ionized means that most plasmas have temperatures in excess of $\sim$ 1 eV ($=$ 11606K $\simeq 10^4$K). Alternatively, one can create plasmas at lower temperature by exposing them to strong ionizing radiation. An example of the latter is the Earth's ionosphere, which has temperatures in the range from 200 K to 500 K.

The plasma parameter is a measure of the dominance of collective interactions over individual collisions. The most fundamental of these collective interactions are the **plasma oscillations** that are set up in response to a charge imbalance. Consider a uniform, unmagnetized plasma in which the number density of electrons is balanced

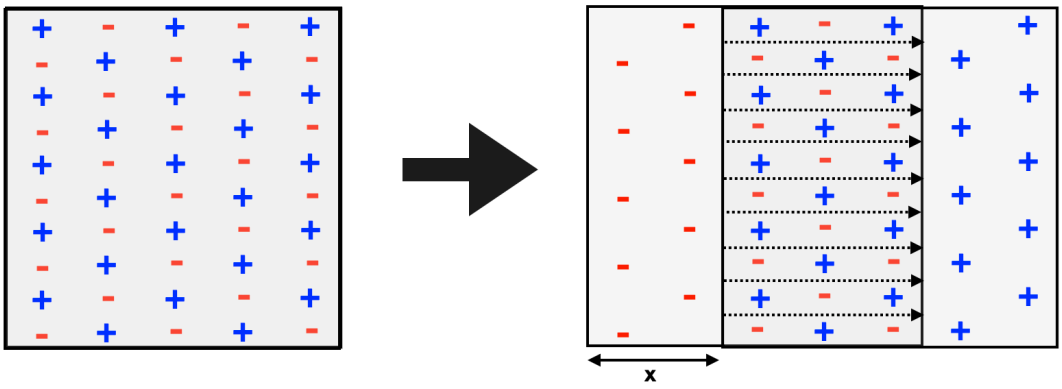


Figure 31: *Displacing electrons and ions in a plasma creates an electric field (dotted arrows in right-hand panel) that aims to restore charge neutrality, thereby causing plasma oscillations at the plasma frequency.*

by that of protons, as depicted in the left-hand panel of Fig. 31. Now let us displace the electrons by x along the x-direction, as depicted in the right hand panel. This displacement creates an electric field that is similar to that between two uniformly charged plates, which, as you hopefully recall from you EM class, is uniform and equal to the $E = 4\pi\sigma$ (in our Gaussian units), with σ the charge per unit area on the plates. In our case $\sigma = en_{\rm e}x$, and we thus have established an electric field $E = 4\pi en_{\rm e}x$ (depicted by the dashed arrows in the right-hand panel of Fig. 31). This electric field exerts a force that pulls the electrons back towards their original position. Due to their inertia, the electrons will overshoot and start to oscillate around their equilibrium position. Here we ignore the motion of the protons which is justified by the fact that their acceleration will be a factor $m_{\rm p}/m_{\rm e} = 1836$ smaller. The equation of motion for a unit volume of electron gas is given by

$$n_{\rm e}m_{\rm e}\frac{{\rm d}^2x}{{\rm d}t^2} = -n_{\rm e}eE = -4\pi n_{\rm e}^2e^2x$$

which can be written as the equation for a **harmonic oscillator**

$$\frac{{\rm d}^2x}{{\rm d}t^2} + \omega_{\rm p}^2\,x = 0$$

with $\omega_{\rm p}$ the **plasma frequency** given by

$$\boxed{\omega_{\rm p} = \left(\frac{4\pi n_{\rm e}e^2}{m_{\rm e}}\right)^{1/2} \simeq 56.5\,{\rm kHz}\,(n_{\rm e}\,[\,{\rm cm}^{-3}])^{1/2}}$$

Note that, if we define the thermal speed of the electrons to be $v_e = (k_B T/m_e)^{1/2}$, then $\omega_p = v_e/\lambda_D$ (see also Chapter 2). Hence, a thermal electron travels roughly a Debye length in a plasma oscillation period.

Because of the plasma frequency, sending EM waves (i.e., light) through a plasma is similar to how a harmonic oscillator responds to periodic forcing (see Appendix M for a refresher); if you try to force an harmonic oscillator with a frequency that is higher than its eigenfrequency, it won't respond. Similarly, EM waves with a frequency higher than the plasma frequency can traverse the plasma virtually uninhibited. However, the plasma is opaque to EM radiation of frequency $\omega < \omega_p$, as the plasma is able to respond fast enough to short the EM oscillations. This is the reason why long wavelength cyclotron radiation (the non-relativistic version of synchrotron radiation) is unable to travel through a plasma with conditions similar to the ISM. In other words, the ISM is opaque to cyclotron radiation.

Indicidently, a conducting metal is somewhat similar to a plasma, in that the electrons are moving freely relative to ions that are locked in place. However, because the electron density of such a metal is many order of magnitude higher than that of the ISM, its plasma frequency is also much higher. Hence, conducting metals are opaque to optical light; however, in many case they are transparent to UV light. This also explains why metal conductors, such as aluminium foil, do not block X-rays.

Collisions We now turn our attention to the collisions in a plasma. Our goal is twofold: to derive under what conditions collisions are important, and (ii) to demonstrate that in a plasma weak collisions (causing small deflections) are more important than strong collisions (causing large angle deflections).

As we have seen above, a particle in a plasma is feeling the Coulomb force from all Λ particles inside its Debye volume. Hence, unlike in a neutral gas, where particles have individual short-range interactions, moving freely in between, in a plasma the particle have many simultaneous, long-range (i.e., of order the Debye length) interactions.

From our definition of a plasma (i.e., $\Lambda \gg 1$) we know that the potential energy of the 'average' interaction of particles with their nearest neighbor is small. This means that the strongest of all its Λ simultaneous interactions (i.e., that with its nearest neighbor) is, on average, weak. Thus, it is safe to conclude that a charged particle in a plasma simultaneously undergoes of order Λ weak interactions (aka 'collisions'). In fact, as we shall see shortly, even the combined effect of all these Λ simultanous

collisions is still relatively weak.

The importance of collisions is most meaningfully expressed by the **collision frequency** (which is the inverse of the **two-body relaxation time**), which we now proceed to calculate.

Consider a charged particle of charge q and mass m having an encounter with impact parameter b with another particle with charge q' and mass m'. Let v_0 be the speed of the encounter when the charges are still widely separated. In what follows we assume that $m' = \infty$, and we treat the encounter from the rest-frame of m', which coincides with the center-of-mass. This is a reasonable approximation for an encounter between an electron and a much heavier ion. It makes the calculation somewhat easier, and is anyways accurate to a factor of two or so. Let $x = v_0 t$ describe the trajectory of m in the case where it would not be deflected by m'. If the scattering angle is small, then the final speed in the x-direction (i.e., the original direction of m) will be close to v_0 again. However, the particle will have gained a perpendicular momentum

$$m v_\perp = \int_{-\infty}^{\infty} F_\perp(t) \mathrm{d}t$$

where $F_\perp(t)$ is the perpendicular force experienced by the particle along its trajectory. As long as the deflection angle is small, we may approximate that trajectory as the unperturbed one (i.e., $x = v_0 t$). Note that this is exactly the same approximation as we made in our treatment of the impulse approximation in Chapter 23.

Next we use that

$$F_\perp = \frac{q\, q'}{r^2} \sin\theta$$

where $\sin\theta = b/r$. Using this to substitute for r in the above expression, we obtain that

$$v_\perp = \frac{q\, q'}{m\, b^2} \int_{-\infty}^{\infty} \sin\theta^3(t)\, \mathrm{d}t$$

Using that

$$x = -r\cos\theta = \frac{-b\cos\theta}{\sin\theta} = v_0 t$$

we see that

$$\mathrm{d}t = \frac{b}{v_0} \frac{\mathrm{d}\theta}{\sin^2\theta}$$

Substituting this in the integral expression we obtain

$$v_\perp = \frac{q\,q'}{m\,v_0\,b}\int_0^\pi \sin\theta\,\mathrm{d}\theta = \frac{2q\,q'}{m\,v_0\,b}$$

Our approximation that the collision must be weak (small deflection angle) breaks down when $v_\perp \simeq v_0$. In that case all the forward momentum is transformed into perpendicular momentum, and the deflection angle is 90°. This happens for an impact parameter

$$b_{90} = \frac{2q\,q'}{m\,v_0^2}$$

In some textbooks on plasma physics, this length scale is called the **Landau length**. Using this expression we have that

$$\frac{v_\perp}{v_0} = \frac{b_{90}}{b}$$

NOTE: although the above derivation is strictly only valid when $b \gg b_{90}$ (i.e., $v_\perp \ll v_0$), we shall consider b_{90} the border between weak and strong collisions, and compute the combined impact of all 'weak' collisions (i.e., those with $b > b_{90}$).

But first, let us compute the collision frequency for strong collisions. Let n be the number density of targets with which our particle of interest can have a collision. The cross section for each target for a strong interaction is πb_{90}^2. In a time t the particle of interest traverses a distance $v_0 t$, which implies that the expectation value for the number of strong interactions during that time is given by

$$\langle N_\mathrm{L}\rangle = n\,\pi b_{90}^2\,v_0\,t$$

where the subscript 'L' refers to Large (deflection angle). Using that the corresponding collision frequency is the inverse of the time it takes for $\langle N_\mathrm{L}\rangle = 1$, we obtain that

$$\boxed{\nu_\mathrm{L} = n\,\pi b_{90}^2\,v_0 = \frac{4\pi n q^2 q'^2}{m^2 v_0^3} = \frac{4\pi n e^4}{m^2 v_0^3}}$$

where the last step only holds if, as assumed here, all ions have $Z = 1$.

Recall that a typical charged particle is simultaneously undergoing $\Lambda \gg 1$ collisions. Only very few of these will be large angle deflections (strong collisions). To see this, we can use that the potential energy between the two colliding particles at

a separation b_{90} is e^2/b_{90}. Substituting the expression for b_{90} we see that this is equal to $\frac{1}{2}mv_0^2$, which is the kinetic energy. Thus, when $qq' < 0$ and $b < b_{90}$ we are basically in the regime of **recombination**. Furthermore, as we have seen above $\langle \text{K.E.} \rangle / \langle \text{P.E.} \rangle \propto \Lambda^{2/3}$, which for a plasma is much smaller than unity. Hence, a particle will undergo many more small angle collisions than large angle collisions.

We now proceed to compute the combined effect of all these many small angle collisions. Since the perpendicular direction in which the particle is deflected by each individual collision is random, we have that $\langle v_\perp \rangle = 0$. However, the second moment, $\langle v_\perp^2 \rangle$ will not be zero. As before, in a time t the number of collisions that our subject particle will experience with impact parameters in the range $[b, b + \mathrm{d}b]$ is given by

$$\langle N_{\rm coll} \rangle = n \, 2\pi b \, \mathrm{d}b \, v_0 \, t$$

Hence, using that each collision causes a $v_\perp = v_0(b_{90}/b)$, we can calculate the total change in $v_\perp^2$ by integrating over all impact parameters

$$\langle v_\perp^2 \rangle = \int_{b_{\rm min}}^{b_{\rm max}} \mathrm{d}b \, n \, 2\pi \, b \, v_0 \, t \, \frac{v_0^2 b_{90}^2}{b^2} = 2\pi \, n \, v_0^3 \, t \, b_{90}^2 \, \ln\left(\frac{b_{\rm max}}{b_{\rm min}}\right)$$

Substituting the expression for b_{90} and using that $b_{\rm max} \simeq \lambda_{\rm D}$ (i.e., a charged particle only experiences collisions with particles inside the Debye length) and $b_{\rm min} = b_{90}$ (i.e., collisions with $b < b_{90}$ are *strong* collisions), we find that

$$\langle v_\perp^2 \rangle = \frac{8\pi \, n \, e^4 \, t}{m^2 v_0} \, \ln\left(\frac{\lambda_{\rm D}}{b_{90}}\right)$$

Next we use that the typical velocity of charges is the thermal speed, so that $v_0^2 \simeq k_{\rm B}T/m$, to write that

$$\frac{\lambda_{\rm D}}{b_{90}} = \frac{\lambda_{\rm D} m v_0^2}{2e^2} = 2\pi \, n \, \lambda_{\rm D}^3 = \frac{\Lambda}{2}$$

where in the third step we have used the definition of the Debye length. Since for a plasma $\Lambda \gg 1$ we have that $\ln(\lambda_{\rm D}/b_{90}) \simeq \ln \Lambda$. The quantity $\ln \Lambda$ is known as the **Coulomb logarithm**.

If we now define the **collision frequency** $\nu_{\rm c}$ due to small-angle collisions as the inverse of the time it takes for $\langle v_\perp^2 \rangle$ to become of the order of v_0^2, then we obtain

$$\boxed{\nu_{\rm c} = \frac{8\pi \, n \, e^4}{m^2 v_0^3} \, \ln \Lambda}$$

Upon comparing this with the collision frequency of large-angle collisions, we see that

$$\frac{\nu_{\rm c}}{\nu_{\rm L}} = 2\ln\Lambda$$

This is a substantial factor, given that Λ is typically very large: i.e., for $\Lambda = 10^{10}$ we have the $\nu_{\rm c} \sim 46\nu_{\rm L}$ indicating that small-angle collisions are indeed much more important than large-angle collisions.

Let us now compare this collision frequency to the plasma frequency. By once again using that v_0 is of order the thermal velocity, and ignoring the small difference between $\lambda_{\rm D}$ and $\lambda_{\rm e}$, we find that

$$\frac{\omega_{\rm c}}{\omega_{\rm p}} = \frac{2\pi\,\nu_{\rm c}}{\omega_{\rm p}} = \frac{\ln\Lambda}{n\,\lambda_{\rm D}^3} = 4\pi\frac{\ln\Lambda}{\Lambda} \ll 1$$

Hence, we see that the collision frequency is much, much smaller than the plasma frequency, which basically indicates that, in a plasma, particle collisions are far less important than collective effects: a plasma wave with frequency near $\omega_{\rm p}$ will oscillate many times before experiencing significant damping due to collisions. Put differently, **collisional relaxation** mechanisms in a plasma are far less important than **collective relaxation** mechanisms, such as, for example, **Landau damping** (to be discussed in a later chapter).

Finally, we point out that, since both the Coulomb force and the gravitational force scale as r^{-2}, the above derivation also applies to gravitational systems. All that is required is to replace $q\,q' = e^2 \rightarrow Gm^2$, and the derivation of the collision frequencies now apply to gravitational N-body systems. This is the calculation that is famously used to derive the **relaxation time** of a gravitational system. The only non-trivial part in that case is what to use for $b_{\rm max}$; since there is no equivalent to Debye shielding for gravity, there is no Debye length. It is common practice (though contentious) to therefore adopt $b_{\rm max} \simeq R$ with R a characteristic length or maximum extent of the gravitational system under consideration. In a gravitational system, we also find that the two-body, collisional relaxation time is very long, which is why we approximate such systems as 'collisionless'. Similar to a plasma, in a gravitational system relaxation is not due to two-particle collisions, but due to collective effects (i.e., violent relaxation) and to more subtle relaxation mechanisms such as phase mixing.

Let's take a closer look at this comparison. If we define the **two-body relaxation time** as the inverse of the collision frequency, we see that for a plasma

$$\boxed{\tau_{\rm relax}^{\rm plasma} = \frac{1}{4\pi}\frac{\Lambda}{\ln\Lambda}\,\tau_{\rm p} \simeq \frac{\Lambda}{\ln\Lambda}\omega_{\rm p}^{-1} \simeq \frac{\Lambda}{\ln\Lambda}10^{-4}{\rm s}\,\left(\frac{n_{\rm e}}{\rm cm^{-3}}\right)^{-1/2}}$$

(see also Chapter 1). Here we have used the ratio between the collision frequency and plasma frequency derived above, and we have used the expression for $\omega_{\rm p} \equiv 2\pi/\tau_{\rm p}$ in terms of the electron density. We thus see that the two-body relaxation time for a plasma is very short. Even for a plasma with $\Lambda = 4\pi n\lambda_{\rm D}^3 = 10^{10}$, the relaxation time for a plasma at ISM densities ($\sim 1\,{\rm cm}^{-3}$) is only about 12 hours, which is much shorter than any hydrodynamical time scale in the plasma (but much longer than the characteristic time scale on which the plasma responds to a charge-imbalance, which is $\tau_{\rm p} = 2\pi/\omega_{\rm p} \simeq 0.1$ms). Hence, although a plasma can often be considered **collisionless** (in which case its dynamics are described by the **Vlasov equation**, see Chapter 28), on astrophysical time scales, plasmas are collisionally relaxed, and thus well described by a **Maxwell-Boltzmann distribution**.

In the case of a gravitational N-body system, the two-body relaxation time is given by

$$\tau_{\rm relax}^{\rm Nbody} \simeq \frac{\Lambda}{\ln\Lambda}\,\frac{\tau_{\rm c}}{10}$$

with the **crossing time** $\tau_{\rm c} \simeq R/V \simeq (2/\pi)\tau_{\rm dyn}$ (for a detailed derivation, see Binney & Tremaine 2008). If, as discussed above, we set $\Lambda = b_{\rm max}/b_{90}$ with $b_{\rm max} \simeq R$ the size of the gravitational system and $b_{90} = 2Gm/\sigma^2$, with σ the characteristic velocity dispersion, then it is easy to see that $\Lambda = [NR\,\sigma^2]/[2G(Nm)] \sim N$, where we have used the **virial relation** $\sigma^2 = GM/R$ with $M = Nm$ (see Chapter 22). Hence, we obtain the well-known expression for the two-body relaxation time of a gravitational N-body system

$$\boxed{\tau_{\rm relax}^{\rm Nbody} \simeq \frac{N}{10\ln N}\,\tau_{\rm c}}$$

And since $\tau_{\rm c}$ can easily be of order a Gyr or larger in astrophysical systems like dark matter halos or galaxies, while $N \gg 1$, we see that $\tau_{\rm relax}^{\rm Nbody}$ is typically much larger than the Hubble time. Hence, gravitational N-body systems are much better approximations of truly collisionless systems than plasmas, and their velocity distributions can thus *not* be assumed to be Maxwellian.

As a final discussion of our comparison of plasmas and gravitational systems, let's consider the force fields. In the latter case, the forces are gravitational forces which are gradients of the gravitational potential: $\vec{F} \propto \nabla\Phi$. Note that Φ is the sum of N point-mass potentials. When N is sufficiently large, this total Φ is sufficiently smooth that a test-particle only feels the combined (collective) force of all N particles; it only notices the coarsiness of the potential when it comes sufficiently close to an individual particle (i.e., with b_{90}) to experience a large angle deflection, which is very rare. In fact, it is more likely to feel the coarsiness in terms of the cumulative effect of many small-angle deflections. When N becomes smaller the potential becomes coarser (larger small-scale fluctuations) and the system becomes more collisional in nature.

In a plasma the forces are Lorentz forces $\vec{F} = q\left[\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right]$. The E and B fields are due to the positions and motions of the individual charges. On large scales ($\lambda > \lambda_{\rm D}$), the electromagnetic (EM) fields, and hence the resulting Lorentz forces, are always smooth (no manifestation of coarsiness). However, on sufficiently small scales ($\lambda < \lambda_{\rm D}$), the level of coarsiness of the EM fields depends on the number of particles inside the Debye volume, which is basically equal to the plasma parameter. If Λ is sufficiently small, individual charged particles feel the microscopic E and B fields due to individual charges within the Debye volume, giving rise to coarsiness and thus collisionality. Only if Λ is sufficiently large, will the impact of coarsiness be negligible (that is, will the corresponding two-body relaxation time be sufficiently long).

CHAPTER 26

Plasma Orbit Theory

In describing a fluid, and plasmas are no exception, we are typically not interested in the trajectories of individual particles, but rather in the behaviour of the statistical ensemble. Nevertheless, one can obtain some valuable insight as to the behavior of a plasma if one has some understanding of the typical orbits that charged particles take. In this chapter we therefore focus on **plasma orbit theory**, which is the study of the motion of individual particles in a plasma.

A particle of mass m and charge q moving in an EM field is subject to the **Lorentz force** and therefore satisfies the following equation of motion:

$$m\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = q\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right)$$

Each particle is subjected to the EM field produced by the other particles. In addition, there may be an external *magnetic* field imposed on the plasma. The interior of a plasma is usually shielded from external *electric* fields.

A charged particle moving in a **uniform, magnetic field**, $\vec{B}$, has a motion that can be decomposed into

1. a circular motion with **gyrofrequency** $\omega_{\mathrm{c}} = |q|B/mc$ and **gyroradius** $r_{\mathrm{L}} = mv_{\perp}c/|q|B$ around a central point called the **guiding center**. Here $v_{\perp}$ is the component of velocity perpendicular to the magnetic field lines.
2. a translatory motion of the guiding center.

This combination gives rise to a **helical motion** along the magnetic field lines.

NOTE: The gyroradius is also known as the **Larmor radius** or the **cyclotron radius**.

What about the motion in a **non-uniform, magnetic field**, $\vec{B}(\vec{x})$? As long as the non-uniformities in $\vec{B}(\vec{x})$ are small over the scale of the gyroradius, i.e.,

$|\vec{B}/(\mathrm{d}\vec{B}/\mathrm{d}r)| < r_\mathrm{L}$, one can still meaningfully decompose the motion into a circular motion around the guiding center and the motion of the guiding center itself. The latter can be quite complicated, though. The aim of plasma orbit theory is to find equations describing the motion of the guiding center. Unfortnately, there is no general equation of motion for the guiding center in an arbitrary EM field. Rather, plasma orbit theory provides a 'bag of tricks' to roughly describe what happens under certain circumstances. In what follows we discuss five examples: four circumstances under which the guiding center experiences a **drift**, and one in which the guiding center is **reflected**.

In what follows we shall discuss four cases in which the guiding center experiences a 'drift':

(A) drift due to the effect of a perpendicular force
(B) drift due to a gradient in the magnetic field
(C) drift due to curvature in the magnetic field
(D) drift due to temporal oscillations in the electric field

We now discuss these in turn.

(A) The effect of a perpendicular force:
Consider the case of an **external force**, $\vec{F}$ (e.g., gravity), acting on a charged particle in a direction **perpendicular** to a **uniform magnetic field**. The equation of motion then is

$$m\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = \vec{F} + \frac{q}{c}\,\vec{v}\times\vec{B}$$

In the limit where $\vec{B}$ vanishes, the particle simply moves in the direction of $\vec{F}$. In the limit where $\vec{F}$ vanishes, the particle makes a circular motion around the magnetic field line. When *both* $\vec{F}$ and $\vec{B}$ are present, the guiding center of the circular motion will drift in a direction perpendicular to both $\vec{F}$ and $\vec{B}$. To understand where this comes from, consider the ion's trajectory depicted in Fig. 32. When the ion is moving in the direction of $\vec{F}$, it is accelerated by the external force, which causes an increase of $v_\perp$ (afterall $\vec{F} \perp \vec{B}$). As a consequence, the Larmor radius, r_L will also increase causing a reduced curvature in the circular motion. Conversely, when the particle is moving opposite to the direction of $\vec{F}$, its $v_\perp$ decrease, r_L decreases, and the curvature of gyration becomes larger (more strongly bent). The net outcome of this is a drift

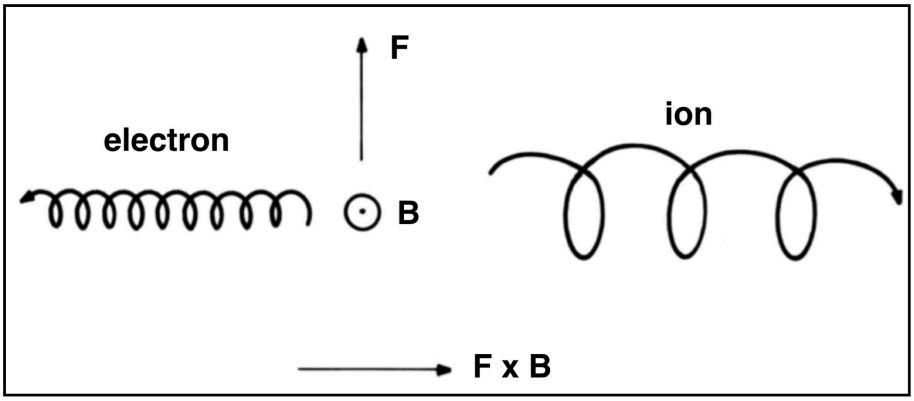


Figure 32: *Drift of a gyrating particle in crossed gravitational and magnetic fields. The magnetic field is pointing out of the page, while the gravitational force is pointing upward. Note that the positively and negatively charged particles drift in opposite directions, giving rise to a non-zero current in the plasma.*

of the guiding center (as depicted in Fig. 32) with a velocity

$$\vec{v}_{\rm GC} = \frac{c}{q}\,\frac{\vec{F}\times\vec{B}}{B^2}$$

Note that positively and negatively charged particles will drift in opposite directions, thereby giving rise to a non-zero **current** in the plasma. In the case where the external force is gravity, such that $\vec{F} = m\vec{g}$, this drift current is give by

$$\vec{j} = n\,q\,(v_{\rm i} - v_{\rm e}) = n\,(m_{\rm i} + m_{\rm e})\,c\,\frac{\vec{g}\times\vec{B}}{B^2}$$

where we have assumed that all ions are protons.

Note that *no current* will arise if the external force is the electrical force, i.e., if $\vec{F} = q\vec{E}$. In that case

$$\vec{v}_{\rm GC} = c\,\frac{\vec{E}\times\vec{B}}{B^2}$$

which does not depend on the charge; hence, all particles drift in the same direction and no current arises.

(B) Gradient Drift:

If the magnetic field has a gradient ∇B in the direction perpendicular to $\vec{B}$, there will also be a drift of the guiding center. The origin is very similar to that of the drift

in the presence of an external force discussed above. Assume the same geometry as in Fig. 32, but now imagine there being a gradient ∇B in the same direction as $\vec{F}$, but with $F = 0$. If an electron moves in the direction of increasing (decreasing) $\vec{B}$, its Larmor radius will decrease (increase), and the curvature of its gyration increases (decreases). Hence, the curvature is different at the top and bottom of the gyration, resulting in a net drift.

It can be shown that the resulting drift of the guiding center is given by:

$$\vec{v}_{\rm GC} = \pm\frac{1}{2} v_\perp r_{\rm L} \frac{\vec{B} \times \nabla B}{B^2}$$

where, as throughout this chapter, $v_\perp$ is the component of velocity perpendicular to the magnetic field lines, $r_{\rm L}$ is the Larmor radius, and the + and − signs correspond to positive and negative charges. Hence, particles of opposite charge drift in opposite directions, once again giving rise to a non-zero current in the plasma.

(C) Curvature Drift:

If the magnetic field is curved, with a radius of curvature $R_{\rm c}$, once again the guiding center experiences a drift. Physically this arises because as a charged particle gyrates around a curved field, a centrifugal force arises given by

$$\vec{F}_{\rm c} = -m v_\parallel^2 \frac{\vec{R}_{\rm c}}{R_{\rm c}^2}$$

where $v_\parallel$ is the velocity component parallel to $\vec{B}$, and $\vec{R}_{\rm c}$ is the curvature *vector* directed towards the curvature center. Substituting this as the external force in the expression for the drift under (A) yields a curvature drift velocity

$$\vec{v}_{\rm GC} = -\frac{c\, m\, v_\parallel^2}{q R_{\rm c}^2} \frac{\vec{R}_{\rm c} \times \vec{B}}{B^2}$$

Like the gradient drift, the curvature drift is also in opposite directions for positively and negatively charged particles, and can thus give rise to a non-zero current. As an interesting aside, because of unavoidable curvature drift, bending a magnetic field into a torus for the purpose of confining a (thermonuclear) plasma, always results in losses (leaky confinement). This is the reason why **tokamaks** make use of twisted, helical magnetic field lines that wrap around the torus. Understanding why this prevents losses is a bit too involved to address here.

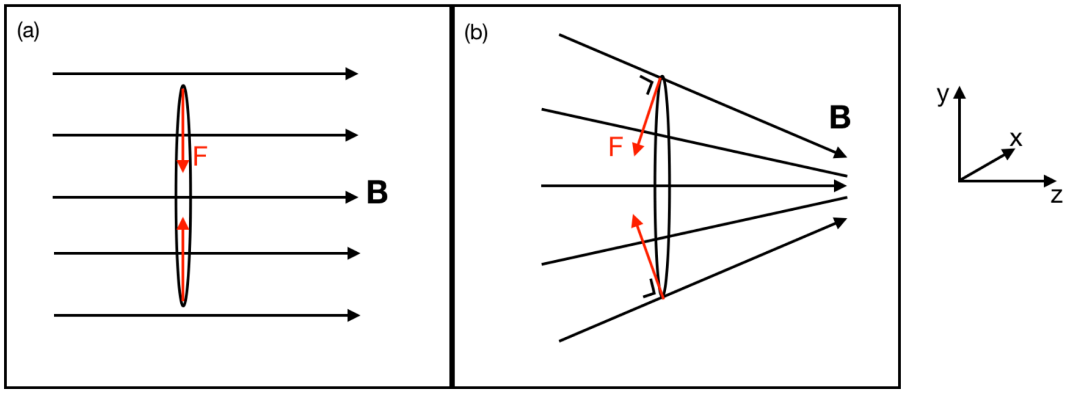


Figure 33: *Illustration of Lorentz force (red arrows) acting on a gyrating particle (indicated by ellipse) in two magnetic field configurations. In panel (a) the field lines are parallel, and the Lorentz force has no component along the z-direction. In the configuration shown in panel (b), though, the Lorentz force, which is always perpendicular to the local magnetic field, now has a component in the z-direction, pointing away from where the magnetic field is stronger.*

(D) Polarization Drift:

The final drift that we discuss here arises due to a time-varying electric field. Assume that both $\vec{E}$ and $\vec{B}$ are uniform in space, but let the electric field vary periodically with time according to $\vec{E} = E_0 \exp[i\omega t]\vec{e}_x$. We shall assume that the oscillation is slow compared to the cyclotron frequency (the frequency of gyration, see Chapter 2), i.e., $\omega \ll \omega_c$. As before, we assume that the magnetic field points in the z-direction. It can be shown (see most books on plasma physics) that this results in a drift of the guiding center that consists of two components. Along the y-direction, perpendicular to both the electric and magnetic fields, the particles experience the usual $\vec{E}\times\vec{B}$ drift discussed under (A). Note, though, that here the drift velocity $v_{GC} = c(\vec{E}\times\vec{B})/B^2$ will oscillate slowly at frequency ω. Along the x-direction a new drift now appears, which is called the **polarization drift** given by

$$v_p = \frac{mc}{qB^2}\frac{d\vec{E}}{dt}$$

Hence, we see that the polarization drift is in opposite directions for the electrons and ions, thus giving rise to a **polarization current**. If the ions are protons, this current is

$$\vec{j}_p = n\,e\,(\vec{v}_i - \vec{v}_e) = \frac{ne}{eB^2}(m_i + m_e)\frac{d\vec{E}}{dt} = \frac{\rho}{B^2}\frac{d\vec{E}}{dt}$$

with ρ the mass density. Note that $v_p \propto m$, indicating that the polarization drift of the electrons is negligible compared to that for the much heavier protons (which is why polarization drift is sometimes called **inertia drift**).

The name polarization drift comes from the fact that the electric field inside most plasmas derives not from an externally applied source (such an electric field is easily shorted) but from the polarization of the plasma due to charge separation, which in this case is driven by the time-variable electric field. To see this, consider an ion at rest in a magnetic field pointing in the z direction. Since it is a rest, and there is no electric field, there is no force acting on the ion. Now turn on an electric field $\vec{E}$ in the x-direction (increase the field strength from 0 to some constant value E). Initially, the ion now reacts by accelerating in the x-direction due to the electric force $\vec{F} = q\vec{E}$. Due to the fact that it now develops a non-zero velocity, the force it experiences shifts from being purely electric to being electromagnetic; i.e., $\vec{F} = q\vec{E} \rightarrow q(\vec{E} + \vec{v} \times \vec{B})$. Once E becomes constant the Lorentz force is fixed and the particle now experiences the regular $\vec{E} \times \vec{B}$ drift discussed under (A) which is in the y-direction; the polarization drift has disappeared. But if E oscillates back and forth, the ion experiences opposing accelerations in the $+x$ and $-x$ directions, causing an oscillatory polarization drift. Due to the large inertia of the ions, they overshoot, and one typically finds that the oscillations in the electric field (if sufficiently slow) result in a polarization drift oscillating 90 degrees out of phase.

This is an important result. It shows that a plasma behaves somewhat like a solid **dielectric** (aka an insulator), which also responds to an induced electric field, $\vec{E}$, by setting up a polarization field, $\vec{P} = 4\pi\chi\vec{E}$, with χ the **electric susceptibility**. Note, though, that if the electric field is constant in time, a plasma acts like a conductor. Only when the applied electric field is time-variable does the plasma reveal dielectric behavior, which can be quantified by the **dielectric function** (see Appendix N).

Magnetic Mirrors:

In all four examples above the drift arises from a 'force' perpendicular to the magnetic field. There are also forces that are parallel to the magnetic field, and these can give rise to the concept of magnetic mirrors.

Let us first introduce the concept of **magnetic moment**. As you may recall from a course on electromagnetism, the current loop with area A and current I (i.e., the current flowing along a closed loop that encloses an area A) has an associated

magnetic moment given by

$$\mu = \frac{I\,A}{c}$$

A charged particle moving along its Larmor radius is such a loop with $A = \pi r_{\rm L}^2$ and $I = q(\Omega/2\pi)$. Using that $\Omega = v_\perp/r_{\rm L}$ and substituting the definition of the Larmor radius, we find that the magnetic moment of a charge gyrating in a magnetic field B is given by

$$\mu = \frac{\pi r_{\rm L}^2 q v_\perp}{2\pi c r_{\rm L}} = \frac{\frac{1}{2}m v_\perp^2}{B}$$

The magnetic moment is an **adiabatic invariant**, which means that it is conserved under slow changes in an external variable. In other words, if B only changes slowly with position and time, then the magnetic momentum of a charged particle gyrating is conserved!

Now consider the magnetic field topologies shown in Fig. 33. In panel (a), on the left, the field is uniform and all field lines run parallel. The ellipse represents a gyration of a particle whose guiding center moves along the central field line. The Lorentz force is perpendicular to both $\vec{B}$ and $v_\perp$ and pointing towards the guiding center. Now consider the topology in panel (b). The field lines converge towards the right. At the top of its gyro-radius, the magnetic field now makes an angle wrt the magnetic field line corresponding to the guiding center, and as a result the Lorentz force (indicated in red), now has a non-zero component in the z-direction. Hence, the particle will be accelerated away from the direction in which the field strength increases!

To make this quantitative, let us compute the Lorentz force in the z-direction:

$$F_z = \frac{q}{c}\left(\vec{v}\times\vec{B}\right)_z = \frac{q}{c}v_\perp B_R$$

where B_R is the magnetic field component in the cylindrical R-direction, and the z-direction is as indicated in Fig. 33. Using the Maxwell equation $\nabla\cdot\vec{B} = 0$, we have that

$$\frac{1}{R}\frac{\partial}{\partial R}\left(RB_R\right) + \frac{\partial B_z}{\partial z} = 0$$

which implies

$$R\,B_R = -\int_0^R R\,\frac{\partial B_z}{\partial z}(R)\mathrm{d}R$$

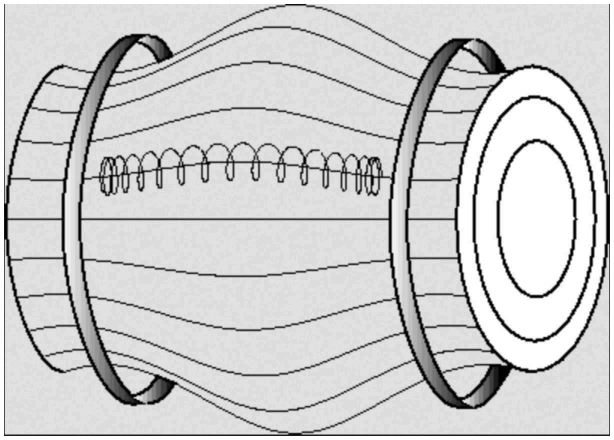


Figure 34: *Illustration of a magnetic bottle. The black horizontal curved lines depict magnetic field lines, which are 'squeezed' together at the ends by two electrical coils. As a result the magnetic field strength is larger at the ends than in the middle, creating magnetic mirrors in between charged particles can be trapped. An example of a particle trajectory is shown.*

If we take into account that $\frac{\partial B_z}{\partial z}$ does not vary significantly over one Larmor radius, we thus find that

$$B_R = -\frac{1}{2} R \frac{\partial B_z}{\partial z}$$

Substituting this in our expression for the Lorentz force in the z-direction, with R equal to the Larmor radius, we find that

$$F_z = -\frac{1}{2} m v_\perp^2 \frac{1}{B} \frac{\partial B_z}{\partial z} = -\mu \frac{\partial B_z}{\partial z}$$

This makes it clear that the Lorentz force has a non-zero component, proportional to the magnetic moment of the charged particle, in the direction opposite to that in which the magnetic field strength increases.

Now we are ready to address the concept of magnetic **mirror confinement**. Consider a magnetic field as depicted in Fig. 34. Close to the coils, the magnetic field is stronger than in between. Now consider a particle gyrating along one of these field lines, as shown. Suppose the particle starts out with kinetic energy $K_0 = \frac{1}{2} m (v_\perp^2 + v_\parallel^2)$ and magnetic moment μ. Both of these quantities will be conserved as the charged particle moves. As the particle moves in the direction along which the strength of

$\vec{B}$ increases (i.e., towards one of the coils), $v_\perp$ must increase in order to guarantee conservation of the magnetic moment. However, the transverse kinetic energy can never exceed the total kinetic energy. Therefore, when the particle reaches a region of sufficiently strong $\vec{B}$, where the transverse kinetic energy equals the total kinetic energy, it is not possible for the particle to penetrate further into regions of even stronger magnetic field: the particle will be reflected back, and the region of increasing magnetic field thus acts as a reflector, known as a **magnetic mirror**.

The contraption shown in Fig. 34 is known as a **magnetic bottle** as it can be used to 'store' charged particles. Note, though, that a magnetic bottle is inherently 'leaky'. To see this, let B_0 denote the magnetic field strength in the middle of the bottle, and $B_{\rm max}$ the maximum magnetic field strength, which arises at the positions of the two coils. Let $\vec{v}_0$ be the velocity of the particle at the middle of the bottle, and let $v_{\perp,0}$ be its corresponding transverse speed: $v_{\perp,0} = v_0 \sin\theta$. Since the transverse velocity at the time of reflection has to equal v_0, we see that only those particles will be reflected for which $\sin^2\theta > B_0/B_{\rm max}$. Particles for which θ is smaller make up a **loss cone**, as these particles will leak out of the magnetic bottle.

Magnetic bottles are not only found in laboratories; the Earth's magnetic field creates its own magnetic bottles due to its toroidal topology. The charged particles that are trapped give rise to what are called the Van Allen belts (electrons and protons have their own belts, as depicted in Fig. 35). As the trapped particles move back and forth between the North and South poles of the Earth's magnetic field, they experience **curvature drift** (in opposite directions for the electrons and protons). The resulting currents are called the **ring currents**. Note that collisions among the charged particles causes charges to be kicked into the loss cone (i.e., the loss cone is constantly refilled due to two-body interactions), which causes them to leak out of the Earth's magnetosphere. Over time a quasi-equilibrium is established in which the loss rate equals the rate of replenishment due to the Solar wind. Note that this is only a quasi-equilibrium due to the time-variability of the Solar wind.

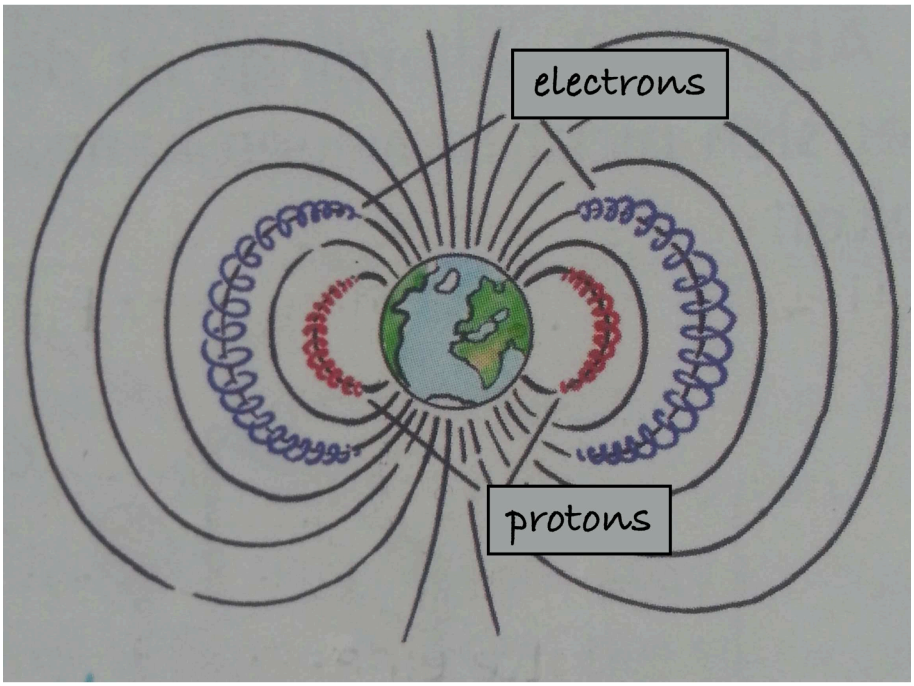


Figure 35: *Illustration of the Van Allen belts of trapped, charged particles in the toroidal magnetic field of the Earth.*

CHAPTER 27

Plasma Kinetic Theory

In Chapter 6 we discussed the kinetic theory of fluids. Starting from the Liouville theorem we derived the BBGKY hierarchy of equations, which we repeat here for convenience:

$$\begin{aligned}
\frac{\partial f^{(1)}}{\partial t} &= \{\mathcal{H}^{(1)}, f^{(1)}\} + \int \mathrm{d}^3\vec{q}_2\,\mathrm{d}^3\vec{p}_2\,\frac{\partial U(|\vec{q}_1-\vec{q}_2|)}{\partial \vec{q}_1}\cdot\frac{\partial f^{(2)}}{\partial \vec{p}_1} \\
&\cdot \\
&\cdot \\
&\cdot \\
\frac{\partial f^{(k)}}{\partial t} &= \{\mathcal{H}^{(k)}, f^{(k)}\} + \sum_{i=1}^{k}\int \mathrm{d}^3\vec{q}_{k+1}\,\mathrm{d}^3\vec{p}_{k+1}\,\frac{\partial U(|\vec{q}_i-\vec{q}_{k+1}|)}{\partial \vec{q}_i}\cdot\frac{\partial f^{(k+1)}}{\partial \vec{p}_i}
\end{aligned}$$

Here $k = 1, 2, ..., N$, $f^{(k)}$ is the k-particle DF, which relates to the N-particle DF ($N > k$) according to

$$f^{(k)}(\vec{w}_1, \vec{w}_2, ..., \vec{w}_k, t) \equiv \frac{N!}{(N-k)!}\int \prod_{i=k+1}^{N} \mathrm{d}^6\vec{w}_i\, f^{(N)}(\vec{w}_1, \vec{w}_2, ..., \vec{w}_N, t)\,,$$

and $\mathcal{H}^{(k)}$ is the k-particle Hamiltonian given by

$$\mathcal{H}^{(k)} = \sum_{i=1}^{k}\frac{\vec{p}_i^{\,2}}{2m} + \sum_{i=1}^{k} V(\vec{q}_i) + \frac{1}{2}\sum_{i=1}^{k}\sum_{\substack{j=1\\ j\neq i}}^{k} U(|\vec{q}_i - \vec{q}_j|)$$

with $V(\vec{q})$ the potential associated with an external force, and $U(r)$ the two-body interaction potential between two (assumed equal) particles separated by a distance $r = |\vec{q}_i - \vec{q}_j|$.

All of this is completely general: it holds for any Hamiltonian system consisting of N particles, and therefore also applies to plasmas. However, we also saw that in order

to make progress, one needs to make certain assumptions that allow one to truncate the BBGKY hierarchy at some point (in order to achieve closure).

If one can ignore two-body collisions, then the phase-space coordinates of the particles will be uncorrelated, such that

$$f^{(2)}(\vec{q}_1, \vec{q}_2, \vec{p}_1, \vec{p}_2) = f^{(1)}(\vec{q}_1, \vec{p}_1)\, f^{(1)}(\vec{q}_2, \vec{p}_2)$$

which is equivalent to setting the **correlation function** $g(1,2) = 0$ (see Chapter 6 for details). The first equation in the BBGKY hierarchy is now closed, and yields the **Collisionless Boltzmann Equation** (CBE), which can be written as

$$\boxed{\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \dot{\vec{x}} \cdot \frac{\partial f}{\partial \vec{x}} + \dot{\vec{v}} \cdot \frac{\partial f}{\partial \vec{v}} = 0}$$

and is the fundamental evolution equation for collisionless systems. As we have discussed in Chapter 25, collective effects are typically more important for plasmas than collisional effects. Hence, as long as one considers plasma effects for which collisions are NOT important (i.e., high frequency plasma waves), then one is justified in using the CBE. It is common, though, to refer to this as the **Vlasov equation**, when applied to a plasma, and we will follow that nomenclature.

In a gravitational N-body system the acceleration in the third term of the CBE $\dot{\vec{v}} = -\nabla\Phi$, where Φ follows from the **Poisson equation**

$$\nabla^2\Phi = 4\pi G\rho$$

with

$$\rho(\vec{x}, t) = m \int f(\vec{x}, \vec{v}, t)\mathrm{d}^3\vec{v}$$

In the case of a plasma, the acceleration is given by

$$\dot{\vec{v}} = \frac{q}{m}\left[\vec{E}(\vec{x}, t) + \frac{\vec{v}}{c} \times \vec{B}(\vec{x}, t)\right]$$

And since the effects of collisions are ignored here, the fields $\vec{E}$ and $\vec{B}$ are the *smooth*,

ensemble-averaged fields that satisfy the **Maxwell equations**

$$\begin{aligned}
\nabla \cdot \vec{E} &= 4\pi\rho \\
\nabla \cdot \vec{B} &= 0 \\
\nabla \times \vec{E} &= -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \\
\nabla \times \vec{B} &= \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}
\end{aligned}$$

Here $\rho = \rho_{\rm i} + \rho_{\rm e}$ is the total charge density and $\vec{J} = \vec{J}_{\rm e} + \vec{J}_{\rm i}$ the total current density, which are related to the distribution function according to

$$\begin{aligned}
\rho_{\rm s}(\vec{x},t) &= q_{\rm s} \int {\rm d}^3\vec{v}\, f_{\rm s}(\vec{x},\vec{v},t) \\
\vec{J}_{\rm s}(\vec{x},t) &= q_{\rm s} \int {\rm d}^3\vec{v}\, \vec{v}\, f_{\rm s}(\vec{x},\vec{v},t)
\end{aligned}$$

for species 's'. Thus we see that the Maxwell equations are for a plasma, what the Poisson equation is for a gravitational system[6]. Note that the distribution function in the Vlasov equation is the sum of $f_{\rm i}(\vec{x},\vec{v},t)$ plus $f_{\rm e}(\vec{x},\vec{v},t)$.

As we discussed at great length in Chapter 6, if one wants to describe a dilute, neutral fluid in which the particles only have short-range interactions (such that $U(r) \simeq 0$ outside of some small distance $r_{\rm coll}$), then we can make the assumption of **molecular chaos**, which states that

$$f^{(2)}(\vec{q},\vec{q},\vec{p}_1,\vec{p}_2) = f^{(1)}(\vec{q},\vec{p}_1)\, f^{(1)}(\vec{q},\vec{p}_2)$$

(note that the collisions are assumed to be perfectly localized, such that we only need to know the 2-particle DF for $\vec{q}_1 = \vec{q}_2 = \vec{q}$). This assumption allows us to close the BBGKY hierarchy, yielding the **Boltzmann Equation**:

$$\boxed{\frac{{\rm d}f}{{\rm d}t} = \frac{\partial f}{\partial t} + \dot{\vec{x}} \cdot \frac{\partial f}{\partial \vec{x}} + \dot{\vec{v}} \cdot \frac{\partial f}{\partial \vec{v}} = I[f]}$$

Here $I[f]$ is the **collision integral**, which describes how the phase-space density around a particle (or fluid element) changes with time due to short-range collisions.

[6]Actually, the first of the Maxwell equations can be written in the form of a Poisson equation when using that the electric field can be written as the gradient of the scalar, electric potential $\phi(r)$

Upon taking the moment equations of this Boltzmann equation we obtain a hierarchy of 'fluid equations', which we can close upon supplementing them with **constitutive equations** for various **transport coefficients** (i.e., viscosity and conductivity) that can be computed using the **Chapman-Enskog expansion** (something we did not cover in these lecture notes).

In the case of a plasma, though, we cannot use the assumption of molecular chaos, as individual particles have many (of order Λ) simultaneous long-range Coulomb interactions. This situation is very different from that of a neutral gas, and the Boltzmann equation can therefore NOT be used to describe a plasma.

So what assumption can we make for a plasma that allows us to truncate the BBGKY hierarchy? The standard approach is to assume that $h(1,2,3) = 0$ (i.e., assume that the three-body correlation function is zero). This is a very reasonable assumption to make, as it basically asserts that two-body interactions are more important than three-body interactions. However, even with $h(1,2,3) = 0$ the BBGKY hierarchy yields a set of two equations (for $\partial f^{(1)}/\partial t$ and $\partial f^{(2)}/\partial t$) that is still extremely difficult to solve. Hence, additional assumptions are necessary. The two assumptions that are typically made to arrive at a manageable equation are

1. the plasma is spatially homogeneous.
2. the two-point correlation function $g(1,2)$ relaxes much faster than the one-point distribution function $f(1)$.

The latter of these is known as **Bogoliubov's hypothesis**, and is a reasonable assumption under certain conditions. Consider for example injecting an electron into a plasma. The other electrons will adjust to the presence of this new electron in roughly the time it takes for them to have a collision with the new electron. Using that the typical speed of the electrons is $v_{\rm e} \propto k_{\rm B}T$ and using the Debye length as the typical length scale, the time scale for the injected electron to relax is $\lambda_{\rm e}/v_{\rm e} \sim \omega_{\rm p}^{-1}$. In contrast, the time for $f(1)$ to relax to the newly injected electron is $\sim \Lambda\omega_{\rm p}^{-1}$, as all the Λ particles within the Debye volume need to undergo mutual collisions.

Using the BBGKY hierarchy with $h(1,2,3) = 0$, assuming the plasma to be spatially homogeneous, and adopting Bogoliubov's hypothesis yields, after some tedious algebra the **Lenard-Balescu** equation. Although the student is not required to know

or comprehend this equation, it is given here for the sake of completeness:

$$\frac{\partial f(\vec{v},t)}{\partial t} = -\frac{8\pi^4 n_{\rm e}}{m_{\rm e}^2}\frac{\partial}{\partial \vec{v}}\int \mathrm{d}\vec{k}\,\mathrm{d}\vec{v}'\,\vec{k}\vec{k}\cdot\frac{\phi^2(k)}{|\varepsilon(\vec{k},\vec{k}\cdot\vec{v})|^2}\,\delta[\vec{k}\cdot(\vec{v}-\vec{v}')]\,\left[f(\vec{v})\,\frac{\partial f}{\partial \vec{v}'} - f(\vec{v}')\frac{\partial f}{\partial \vec{v}}\right]$$

Here

$$\phi(k) = \frac{e^2}{2\pi^2 k^2}$$

is the Fourier transform of the Coulomb potential $\phi(x) = e^2/|x|$, and

$$\varepsilon(\vec{k},\omega) = 1 + \frac{\omega_{\rm p,e}^2}{k^2}\int \mathrm{d}\vec{v}\,\frac{\vec{k}\cdot(\partial f/\partial \vec{v})}{\omega - \vec{k}\cdot\vec{v}}$$

is called the **dielectric function**, which basically represents the plasma shielding of a test particle. Note that $\vec{x}$ does not appear as an argument of the distribution function, which reflects the assumption of a homogeneous plasma. And the term in square brackets has no explicit time-dependence, which reflects Bogoliubov's hypothesis. We emphasize that because of the assumptions that underly the Lenard-Balescu equation, it is NOT applicable to all plasma processes. Although it can be used to describe, say, the collisional relaxation of an electron beam in a plasma, it cannot be used to describe for example the collisional damping of *spatially inhomogeneous* wave motion.

The rhs of the Lenard-Balescu equation represents the physics of two-particle collisions. This is evident from the fact that the term $\phi(k)/\varepsilon(\vec{k},\vec{k}\cdot\vec{v})$ appears squared. This term represents the Coulomb potential of a charged particle (the $\phi(k)$-part) together with its shielding cloud (represented by the dielectric function). Hence, the fact that this term appears squared represents the collision of two shielded particles. It may be clear that this is not an easy equation to deal with. However, one can obtain a simplified but fairly accurate form of the Lenard-Balescu equation that can be recast in the form of a **Fokker-Planck equation**

$$\boxed{\frac{\partial f(\vec{v},t)}{\partial t} = -\frac{\partial}{\partial v_i}\left[A_i\, f(\vec{v})\right] + \frac{1}{2}\frac{\partial^2}{\partial v_i \partial v_j}\left[B_{ij}\, f(\vec{v})\right]}$$

Here

$$A_i = \frac{8\pi n_{\rm e} e^4 \ln\Lambda}{m_{\rm e}^2}\,\frac{\partial}{\partial v_i}\int \mathrm{d}\vec{v}'\,\frac{f(\vec{v}',t)}{|\vec{v}-\vec{v}'|}$$

is called the **coefficient of dynamical friction**, which represents the slowing down of a typical particle because of many small angle collisions, and

$$B_{ij} = \frac{4\pi n_{\rm e} e^4 \ln\Lambda}{m_{\rm e}^2} \frac{\partial^2}{\partial v_i \partial v_j} \int {\rm d}\vec{v}' \, |\vec{v} - \vec{v}'| \, f(\vec{v}', t)$$

is the **diffusion coefficient**, which represents the increase of a typical particle's velocity (in the direction perpendicular to its instantaneous velocity) due to the many small angle collisions.

If the two terms on the rhs of the Fokker-Planck equation balance each other, such that $\partial f(\vec{v}, t)/\partial t = 0$, then the plasma has reached an equilibrium. It can be shown that this is only the case if $f(\vec{v})$ follows a **Maxwell-Boltzmann distribution**. This is another way of stating that two-body collisions drive the systems towards a Maxwellian.

As discussed in Chapter 9, the **Fokker-Planck equation** is a very general equation in physics; it describes the evolution of a distribution function due to any phenomenon that in some approximate sense can be considered **Markovian**. A well known example is **Brownian motion**. The Fokker-Planck equation is also used for describing the collisional evolution of gravitational N-body systems (i.e., globular clusters), while the first-order diffusion coefficient A_i is used to describe the orbital decay of a massive body due to dynamical friction (cf. Chapter 16). Hence, once more we see the strong similarity between gravitational N-body systems and plasmas.

CHAPTER 28

Vlasov Equation & Two-Fluid Model

In Chapter 25 we have seen that the two-body collision frequency of a plasma is much smaller than the plasma frequency (by roughly a factor Λ). Hence, there are plasma phenomena that have a characteristic time scale that is much shorter than the two-body relaxation time. For such phenomena, collisions can be ignored, and we may consider the plasma as being **collisionless**.

And as we have seen in the previous chapter, the equation that governs the dynamics of a collisionless plasma is the **Vlasov equation**.

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \frac{\partial f}{\partial \vec{x}} + \frac{\vec{F}}{m} \cdot \frac{\partial f}{\partial \vec{v}} = 0$$

with

$$\vec{F} = n \int \mathrm{d}\vec{x}_2 \mathrm{d}\vec{v}_2 \vec{F}_{12} f^{(1)}(\vec{x}_2, \vec{v}_2, t)$$

the smooth force acting on particle 1 due to the long-range interactions of all other particles (effectively only those within the Debye length). This equation derives from the BBGKY hierarchy upon neglecting the two-particle correlation function, $g(1,2)$, which, as we have seen in Chapter 6, is equivalent to assuming that the system is collisionless.

An important application of the Vlasov equation is the so-called **two-fluid model of plasma physics**, in which the plasma is regarded as an inter-penetrating mixture of a negatively charged fluid of electrons, and a positively charged fluid of ions. In that case,

$$f(\vec{x}, \vec{v}, t) = f_{\mathrm{e}}(\vec{x}, \vec{v}, t) + f_{\mathrm{i}}(\vec{x}, \vec{v}, t)$$

where the subscripts 'e' and 'i' refer to electrons and ions, respectively. Since the **Vlasov equation** is linear, both f_{e} and f_{i} obey the Vlasov equation. If the force $\vec{F}$ is purely electromagnetic (i.e., we can ignore the gravitational force), then we have that

$$\boxed{\frac{\partial f_{\mathrm{a}}}{\partial t} + \vec{v} \cdot \frac{\partial f_{\mathrm{a}}}{\partial \vec{x}} + \frac{q_{\mathrm{a}}}{m} \left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B} \right) \cdot \frac{\partial f_{\mathrm{a}}}{\partial \vec{v}} = 0}$$

where 'a' is either 'e' or 'i'.

Rather than solving the Vlasov equation, we follow the same approach as with our neutral fluids, and our collisionless fluids, and solve instead the moment equations, by multiplying the Vlasov equation with $\chi(\vec{v})$ and integrating over all of velocity (momentum) space (cf. Chapter 7).

For $\chi = 1$ this yields the **continuity equation**

$$\frac{\partial n_{\rm a}}{\partial t} + \nabla \cdot (n_{\rm a}\, \vec{u}_{\rm a}) = 0$$

while $\chi = m_{\rm a}\vec{v}_{\rm a}$ yields the **momentum equations**

$$m_{\rm a}\, n_{\rm a} \left[\frac{\partial \vec{u}_{\rm a}}{{\rm d}t} + (\vec{u}_{\rm a} \cdot \nabla)\, \vec{u}_{\rm a} \right] = -\nabla P_{\rm a} + q_{\rm a} n_{\rm a} \left(\vec{E} + \frac{\vec{u}_{\rm a}}{c} \times \vec{B} \right)$$

Note that the continuity equation is exactly the same as for a neutral fluid or a collisionless fluid, while the momentum equations are the same as the **Euler equations** for a **neutral, inviscid fluid** or the **Jeans equations** for a **collisionless fluid**, except that the gravitational force is now replaced by the electromagnetic **Lorentz force**.

We emphasize that we have ignored **viscosity** here, something we will continue to do throughout our discussion of plasma physics. More accurately, the above momentum equations should be the **Navier-Stokes equations**, i.e., there should be an additional term $\mu[\nabla^2 \vec{u}_{\rm a} + \frac{1}{3}\nabla(\nabla \cdot \vec{u}_{\rm a})]$ on the rhs. As we are mainly concerned with astrophysical flows, for which the Reynolds number is large, ignoring **viscosity** when discussing astrophysical plasmas is a reasonable thing to do.

As for neutral fluids, we need to complement these moment equations with an equation of state (EoS) in order to close the equations. Without going into detail, in most cases the EoS of a plasma can be taken to have one of the following three forms:

$$\begin{array}{ll} P_{\rm a} = 0 & (\text{"cold plasma"}) \\ P_{\rm a} = n_{\rm a}\, k_{\rm B}\, T_{\rm a} & (\text{"ideal plasma"}) \\ P_{\rm a} = C n_{\rm a}^{\gamma} & (\text{"adiabatic processes"}) \end{array}$$

A 'cold plasma' is a plasma in which the random motions of the particles are not important.

NOTE: in the presence of strong magnetic fields, the thermodynamic properties of the plasma can be very different in directions parallel and perpendicular to $\vec{B}$; in those cases the pressure cannot be fully described by a scalar, but requires a stress-tensor-equivalent instead. We will not consider such situations here, but it will be discussed in the next Chapter on MHD.

Since the momentum equations for our plasma contain the electric and magnetic fields, we need to complement the moment equations and EoS with the **Maxwell equations**

$$\begin{aligned} \nabla \cdot \vec{E} &= 4\pi(n_\mathrm{i} - n_\mathrm{e})\, e \\ \nabla \cdot \vec{B} &= 0 \\ \nabla \times \vec{E} &= -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \\ \nabla \times \vec{B} &= \frac{4\pi}{c}(n_\mathrm{i}\vec{u}_\mathrm{i} - n_\mathrm{e}\vec{u}_\mathrm{e})\, e + \frac{1}{c}\frac{\partial \vec{E}}{\partial t} \end{aligned}$$

Upon inspection, this complete set of 18 equations (8 Maxwell eqs, 2×3 momentum equations, 2 continuity equations, and 2 equations of state) describes the evolution of a total of 16 scalar quantitie: $\vec{E}$ (3), $\vec{B}$ (3), $\vec{u}_\mathrm{i}$ (3), $\vec{u}_\mathrm{e}$ (3), n_i, n_e, P_i, and P_e. This set of equations constitutes the **two-fluid model** of plasma physics. Note that this model derives from the **Vlasov equation**, and can therefore only be used to describe plasma phenomena in which collisions can be neglected.

As an example of an application of the two-fluid model, consider **electro-magnetic oscillations** in a plasma.

Let's assume the plasma to be 'cold' (i.e., $P_\mathrm{e} = P_\mathrm{i} = 0$), and consider perturbations in a uniform, homogeneous plasma. The perturbation analysis treatment is exactly analogous to that of accoustic waves in Chapter 17: First, apply small perturbations to the dynamical quantities (i.e., $n_0 \rightarrow n_0 + n_1$, $\vec{E}_0 \rightarrow \vec{E}_0 + \vec{E}_1$, etc, where subscripts '0' refer to the unperturbed equilibrium solution. In what follows, we assume the plasma to be unmagnetized, i.e., $\vec{B}_0 = 0$. Next, linearize the equations, which implies that we ignore all higher-order terms. For the momentum equations this yields

$$m_\mathrm{e}\, n_0\, \frac{\partial \vec{v}_1}{\partial t} = -e\, n_0\, \vec{E}_1$$

Note that the magnetic force $\vec{v}_1 \times \vec{B}_1$ is second-order in the perturbed quantities and therefore neglected. For the Maxwell equations we obtain that

$$\nabla \times \vec{B}_1 = -\frac{4\pi}{c}\, n_0\, e\, \vec{v}_1 + \frac{1}{c}\frac{\partial \vec{E}_1}{\partial t}$$

and

$$\nabla \times \vec{E}_1 = -\frac{1}{c}\frac{\partial \vec{B}_1}{\partial t}.$$

These three equations suffice to solve for the evolution of $\vec{v}_1$, $\vec{E}_1$ and $\vec{B}_1$. In particular, upon combining these equations, and assuming all perturbations to be of the form EXP$[-i(\vec{k}\cdot\vec{x}-\omega t)]$, which implies that $\partial/\partial t \rightarrow -i\omega$ and $\nabla \rightarrow -i\vec{k}$, one obtains, after some algebra, the **dispersion relation** $\omega(\vec{k})$. In the case of our two-fluid model, this dispersion relation has the form

$$\vec{k} \times (\vec{k} \times \vec{E}_1) = -\frac{\omega^2}{c^2}\left(1 - \frac{\omega_{\rm p}^2}{\omega^2}\right)\vec{E}_1$$

Here

$$\omega_{\rm p} = \left(\frac{4\pi\, n_0\, e^2}{m_{\rm e}}\right)^{1/2}$$

is the **plasma frequency** in the undisturbed plasma. As a rule of thumb, it is useful to remember that

$$f_{\rm p} = \frac{\omega_{\rm p}}{2\pi} \simeq 10^4 {\rm Hz}\, \left(\frac{n_{\rm e}}{{\rm cm}^{-3}}\right)^{1/2}$$

Hence, in the ISM, which has densities of the order of $n_{\rm e} \sim 0.01 - 1.0\,{\rm cm}^{-3}$, the plasma frequency is roughly 10^3 to 10^4Hz.

If, without loss of generality, we choose our z-axis to be in the direction of the wave-vector, such that $\vec{k} = k\vec{e}_z$, then the dispersion relation takes the following matrix form

$$\begin{pmatrix} \omega^2 - \omega_{\rm p}^2 - k^2c^2 & 0 & 0 \\ 0 & \omega^2 - \omega_{\rm p}^2 - k^2c^2 & 0 \\ 0 & 0 & \omega^2 - \omega_{\rm p}^2 \end{pmatrix} \begin{pmatrix} E_{1,x} \\ E_{1,y} \\ E_{1,z} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

This dispersion relation corresponds to two physically distinct types of wave modes:

Plasma Oscillations:

These are oscillation modes for which

$$E_{1x} = E_{1y} = 0\,, \qquad \omega^2 = \omega_{\rm p}^2$$

where the z-direction is taken to be along $\vec{k}$. Hence, since the group velocity $v_{\rm g} = \partial\omega/\partial k = 0$ we see that these correspond to **non-propagating**, **longitudinal** oscillations with a frequency equal to the plasma frequency. These are called **plasma waves**, or **Langmuir waves**. Physically, they are waves in which perturbations in $\vec{E}$, cause a separation between electrons and ions, which results in an electrostatic restoring force.

Electromagnetic waves:
These are oscillation modes for which

$$E_{1z} = 0\,, \qquad \omega^2 = \omega_{\rm p}^2 + k^2c^2$$

where as before the z-direction is taken to be along $\vec{k}$. Hence, these are **transverse** waves. In fact, these are simply electromagnetic waves, but modified by the plasma. Note that if the plasma density approaches zero, $\omega_{\rm p} \propto n_{\rm e}^{1/2} \to 0$, and the dispersion relation becomes that of light in vacuum (as it should be). The group velocity (i.e., the velocity with which information propagates) is given by

$$v_{\rm g} \equiv \frac{\partial\omega}{\partial k} = c\sqrt{1 - \frac{\omega_{\rm p}^2}{\omega^2}}$$

which is less than the speed of light, c. For comparison, the **phase velocity** is given by

$$v_{\rm ph} \equiv \frac{\omega}{k} = \frac{c}{\sqrt{1 - \frac{\omega_{\rm p}^2}{\omega^2}}}$$

which is larger than c. Note, though, that this does not violate special relativity, as no physical signal is travelling at this speed (i.e., it does not carry any information).

• If $\omega \gg \omega_{\rm p}$, we have that $\omega^2 = k^2c^2$ which is the usual dispersion relation for EM waves in a vacuum, and $v_{\rm g} = c$. The frequency of these EM waves is too high for the plasma to respond, and plasma effects are negligible. When $\omega \downarrow \omega_{\rm p}$ then the EM waves slow down, and the phase velocity increases. The **refractive index** of a medium is defined as $n \equiv c/v_{\rm ph} = ck/\omega$, which for a plasma is given by $n^2 = 1 - (\omega_{\rm p}/\omega)^2$. Hence, when $\omega \downarrow \omega_{\rm p}$ we have that $n \downarrow 0$.

• If $\omega < \omega_{\rm p}$, then k and $v_{\rm g}$ become imaginary. This indicates that the EM waves simply cannot penetrate the plasma; they are reflected back. The reason is that the plasma can counteract the oscillations in the EM field at a rate that is faster, thereby

shorting the fluctuations, and thus the EM wave. This explains why low-frequency radio signals can be reflected from the ionispheric plasma, and why cyclotron radiation cannot travel through the plasma that permeates the Universe (unless it derives from very strong magnetic fields, in which case the frequency can be larger than the plasma frequency).

In the analysis above we have assumed that the plasma is 'cold', which implies that we could ignore its thermal pressure (i.e., $P_{\rm a} \neq 0$). We now revisit this analysis, this time assuming that the pressure is non-negligble (i.e., a 'warm' plasma) and obeys the adiabatic relation. Linearizing the Euler equation for the electron fluid yields

$$m_{\rm e}\, n_0\, \frac{\partial \vec{v}_1}{\partial t} = -e\, n_0\, \vec{E}_1 - \frac{\gamma P_0}{n_0} \nabla n_1$$

which differs from the one for the cold plasma in that it contains an additional pressure perturbation term. Since our Euler equation now depends on n_1, we need to complement it with the perturbed continuity equation, which reads

$$\frac{\partial n_1}{\partial t} + n_1 \nabla \cdot \vec{v}_1 = 0$$

Using the first of the Maxwell equations, we have that

$$\nabla \cdot \vec{E}_1 = -4\pi e n_1$$

Thus, we have three equations for the three perturbed quantities, n_1, $\vec{v}_1$, and $\vec{E}_1$. As before, assuming all perturbations to be of the form EXP$[-i(\vec{k}\cdot\vec{x} - \omega t)]$, the combination of these three equations, applied to longitudinal waves, yields the famous **dispersion relation for Langmuir waves**:

$$\omega^2 = \omega_{\rm p}^2 + k^2 \gamma \frac{P_0}{m_{\rm e} n_0} = \omega_{\rm p}^2 + \gamma k^2 \frac{k_{\rm B} T_{\rm e}}{m_{\rm e}} = \omega_{\rm p}^2 + 3k^2 \langle v_{\rm e}^2 \rangle$$

Here, in the second step, we have assumed that $P_0 = n_0 k_{\rm B} T_{\rm e}$, while in the third step we have used the fact that the electrons only move in one direction (that of $\vec{k}$) to set $\gamma = 3$. Note that this is only valid for adiabatic compression, which will be a good approximation as long as the typical electron only travels a small fraction of wavelength in one wave period; Hence, the above dispersion relation is only valid as long as $\langle v_{\rm e} \rangle\, \omega^{-1} \ll \lambda$, i.e., $\omega/k \gg \langle v_{\rm e} \rangle$.

Using the dispersion relation for Langmuir waves, we see that the group velocity is

$$v_{\rm g} \equiv \frac{\partial \omega}{\partial k} = \sqrt{3} \langle v_{\rm e} \rangle \sqrt{1 - \frac{\omega_{\rm p}^2}{\omega^2}}$$

which is equivalent to that for the EM waves in a cold plasma, but with c replaced by $\sqrt{3}\langle v_{\rm e}\rangle$. Contrary to the dispersion relation for Langmuir waves in a cold plasma, we now have a non-zero group velocity (i.e., the Langmuir waves in a warm plasma are travelling) with a dependence on k (i.e., a warm plasma is a dispersive medium to Langmuir waves).

The above analysis is based on a perturbation analysis of the two-fluid model, which is based on moment equations of the Vlasov equation. Landau performed a more thorough analysis, by actually perturbing the **Vlasov equation** itself. He found that the Langmuir waves will damp, a process known as **Landau damping**.

This damping may come as a surprise (as it did to Landau, when he first derived this result). After all, damping is typically associated with dissipation, and hence requires either radiation, or collisions that convert wave energy into random, thermal energy. But the Vlasov equation includes neither radiation nor collisions. So where does this damping come from? Without going through a rigorous derivation, which is somewhat tedious, involving nasty complex contour integrals, we merely sketch how Landau damping arises from the energy exchange between a Langmuir wave with phase velocity $v_{\rm ph} \equiv \omega/k$ and particles in the plasma with velocities approximately equal to $v_{\rm ph}$; these particles can interact strongly with the wave (similar to how particles that are in near-resonance with a perturber can exchange energy with it). Particles that have a velocity $v \lesssim v_{\rm ph}$ will be *accelerated* (i.e., gaining energy) by the electric field of the Langmuir wave to move with the phase velocity of the wave. Particles with $v \gtrsim v_{\rm ph}$, on the other hand, will be *decelerated* (losing energy). All in all, the particles have a tendency to synchronize with the wave. An imbalance between energy gainers and energy losers arises from the fact that the velocity distribution of a plasma is typically a Maxwell-Boltzmann distribution; hence, there will be slightly more particles with $v < v_{\rm ph}$ (energy gainers) than particles with $v > v_{\rm ph}$ (energy losers). Hence, there is a net tranfer of energy from the wave to the particles, causing the former to damp.

A famous metaphor for Landau damping involves surfing. One can view Langmuir waves as waves in the ocean, and the particles as surfers trying to catch the wave, all moving in the same direction. If the surfer is moving on the water surface at a velocity slightly less than the waves he will eventually be caught and pushed along by the wave (gaining energy). On the other hand, a surfer moving slightly faster than a wave will be pushing on the wave as he moves uphill (losing energy to the

wave). Within this metaphor, it is also clear that if the surfer is not moving at all, no exchange of energy happens as the wave simply moves the surfer up and down as it goes by. Also a wind-surfer, who is moving much faster than the wave won't interact much with the wave either.

Hence, **Landau damping** arises from gradients in the distribution function at the phase velocity of the wave, which can cause a transfer of energy from the wave to the particles; Landau damping is a prime example of a **wave-particle interaction**. As first pointed out by Lynden-Bell, it is similar to **violent relaxation** for a purely collisionless, gravitational system, in which the energy in potential fluctuations (i.e., oscillations in the gravitational system, for example due to gravitational collapse) are transferred into random motions, ultimately leading to virialization (relaxation).

CHAPTER 29

Magnetohydrodynamics

As discussed in the previous chapter, when discussing phenomena in which electrons and ions respond differently (such as the response of a plasma to electromagnetic oscillations), the two-fluid model (or Vlasov equation) has to be applied.

When we consider phenomena with length scales much larger than the **Debye length**, and time scales much longer than the inverse of the **plasma frequency**, charge separation is small, and can typically be neglected. In that case we don't need to treat electrons and ions separately. Rather, we treat the plasma as a single fluid. Note, though, that as we are considering phenomena with longer time scales, our **one-fluid model** of plasma physics will have to account for collisions (i.e., we won't be able to use the Vlasov equation as our starting point). As we will see, the main effect of these collisions is to transfer momentum between electrons and ions, which in turn manifests as an electrical current.

A formal derivation of the MHD equation is a formidable task. We instead follow a more heuristic approach in what follows. In the previous chapter we derived a two-fluid model by taking moment equations of the Vlasov equations for the electron and ion species. We follow the same approach here. However, since in MHD we cannot ignore collisions, we have to supplement the Vlasov equation with a collision term:

$$\boxed{\frac{\partial f_\mathrm{a}}{\partial t} + \vec{v}\cdot\frac{\partial f_\mathrm{a}}{\partial \vec{x}} + \frac{q_\mathrm{a}}{m}\left(\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right)\cdot\frac{\partial f_\mathrm{a}}{\partial \vec{v}} = \left(\frac{\partial f_\mathrm{a}}{\partial t}\right)_\mathrm{coll}}$$

where as before 'a' refers to a species, either 'e' or 'i'. As we will see, we can obtain the necessary insight to develop our one-fluid model, without regard of what this collision term looks like in detail.

By integrating the above 'Boltzmann-like' equation over velocity space, we obtain the **continuity equation**

$$\frac{\partial n_{\rm a}}{\partial t} + \nabla \cdot (n_{\rm a}\, \vec{u}_{\rm a}) = 0$$

where we have used that

$$\int {\rm d}\vec{v} \left(\frac{\partial f_{\rm a}}{\partial t}\right)_{\rm coll} = 0$$

This term represents the change in the number of particles of species 'a' in a small volume of configuration space due to collisions. To good approximation this is zero, which follows from the fact that while Coulomb interactions can cause large changes in momentum (when $b < b_{90}$), they do not cause much change in the positions of the particles. Hence, the collision term leaves the continuity equation unaltered.

For the momentum equation, we multiply the above 'Boltzmann-like' equation with velocity and again integrate over velocity space. If we once again ignore viscosity, this yields exactly the same equation as for the two-fluid model discussed in the previous chapter, but with one additional, *collisional term*:

$$m_{\rm a}\, n_{\rm a} \left[\frac{\partial \vec{u}_{\rm a}}{{\rm d}t} + (\vec{u}_{\rm a} \cdot \nabla)\, \vec{u}_{\rm a}\right] = -\nabla P_{\rm a} + q_{\rm a} n_{\rm a} \left(\vec{E} + \frac{\vec{u}_{\rm a}}{c} \times \vec{B}\right) + \vec{C}_{\rm a}$$

where

$$\boxed{\vec{C}_{\rm a} = m_{\rm a} \int {\rm d}\vec{v}\, \vec{v} \left(\frac{\partial f_{\rm a}}{\partial t}\right)_{\rm coll}}$$

This term represents the change in the momentum of species 'a' at position $\vec{x}$ due to Coulomb interactions. Note that a given species cannot change its momentum through collisions with members of its own species (i.e., the center of mass of two electrons is not changed after they have collided with each other). Hence, $\vec{C}_{\rm e}$ represents the change in the momentum of the electrons due to collisions with the ions, and $\vec{C}_{\rm i}$ represents the change in the momentum of the ions due to collisions with the electrons. And, since the total momentum is a conserved quantity, we have that $\vec{C}_{\rm e} = -\vec{C}_{\rm i}$.

Since in MHD we treat the plasma as a single fluid, we now define the relevant quanties:

total mass density	$\rho \equiv m_{\rm e}\, n_{\rm e} + m_{\rm i}\, n_{\rm i}$
total charge density	$\rho_{\rm c} \equiv q_{\rm e}\, n_{\rm e} + q_{\rm i}\, n_{\rm i} = e(n_{\rm i} - n_{\rm e})$
com fluid velocity	$\vec{u} \equiv \dfrac{1}{\rho}\,(m_{\rm i}\, n_{\rm i}\, \vec{u}_{\rm i} + m_{\rm e}\, n_{\rm e}\, \vec{u}_{\rm e})$
current density	$\vec{J} = q_{\rm e}\, n_{\rm e}\, \vec{u}_{\rm e} + q_{\rm i}\, n_{\rm i}\, \vec{u}_{\rm i}$
total pressure	$P = P_{\rm e} + P_{\rm i}$

By multiplying the continuity equations for the electrons with $m_{\rm e}$, and adding it to the continuity equation for the ions multiplied by $m_{\rm i}$, one obtains the **MHD continuity equation**,

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0}$$

This equation, which expresses **mass conservation**, is identical to the continuity equation for a neutral fluid.

In addition, we also have **charge conservation** which is given by

$$\boxed{\frac{\partial \rho_{\rm c}}{\partial t} + \nabla \cdot \vec{J} = 0}$$

For the **momentum equation**, it is common to assume that $\partial n_{\rm a}/\partial t$ and $u_{\rm a}$ are small compared to other terms. This allows one to neglect terms that contain products of these small quantities, which in turn allows one to add the momentum equations for electrons and ions, yielding:

$$\boxed{\rho \frac{\partial \vec{u}}{\partial t} = -\nabla P + \rho_{\rm c} \vec{E} + \frac{1}{c} \vec{J} \times \vec{B}}$$

In general, in MHD one assumes that $n_{\rm e} \simeq n_{\rm i}$, which implies that the charge density, $\rho_{\rm c}$, is typically (very) small. We adopt that assumption here as well, which implies that the $\rho_{\rm c}\vec{E}$ term in the momentum equations vanishes and that we no longer need to consider the charge conservation equation.

In MHD, the **energy equation** is the same as for a neutral fluid, except that there is an additional term to describe **Ohmic dissipation** (aka Ohmic loss). In the absence of radiation, viscosity and conduction, we therefore have

$$\boxed{\rho \frac{{\rm d}\varepsilon}{{\rm d}t} = -P \nabla \cdot \vec{u} - \frac{J^2}{\sigma}}$$

Here $J^2 = \vec{J} \cdot \vec{J}$ and σ is the **electric conductivity**. The Ohmic dissipation term describes how collisions convert magnetic energy into thermal energy. Hence, it is similar in nature to the **viscous dissipation rate**, $\mathcal{V}$, which describes how collisions manifest as viscosity in converting bulk motion (shear) in thermal energy.

Since both the momentum and energy equations contain the current density, we need to complement them with an equation for the time-evolution of $\vec{J}$. This relation, called the **generalized Ohm's law**, derives from multiplying the momentum equations for the individual species by $q_{\rm a}/m_{\rm a}$, adding the versions for the electrons and ions, while once again ignoring terms that contain products of small quanties (i.e., $\partial n_{\rm a}/\partial t$ and $u_{\rm a}$). Using that $\vec{C}_{\rm e} = -\vec{C}_{\rm i}$, that $n_{\rm i} \approx n_{\rm e}$, that $P_{\rm e} \approx P_{\rm i} \approx P/2$, and that $m_{\rm i}^{-1} \ll m_{\rm e}^{-1}$, one can show that

$$\boxed{\frac{m_{\rm e}\, m_{\rm i}}{\rho e^2} \frac{\partial \vec{J}}{\partial t} = \frac{m_{\rm i}}{2\rho e} \nabla P + \vec{E} + \frac{1}{c}\vec{u} \times \vec{B} - \frac{m_{\rm i}}{\rho e c} \vec{J} \times \vec{B} + \frac{m_{\rm i}}{\rho e} \vec{C}_{\rm i}}$$

(for a derivation see the excellent textbook *Introduction to Plasma Theory* by D.R. Nicholson).

The above generalized Ohm's law is rather complicated. But fortunately, in most circumstances certain terms are significantly smaller than others and can thus be ignored. Before discussing which terms can be discarded, though, we first give a heuristic derivation of the **collision term** $\vec{C}_{\rm i}$.

As already mentioned above, $\vec{C}_{\rm e} = -\vec{C}_{\rm i}$ describes the transfer of momentum from the electrons to the ions (and vice-versa). Let's consider the **strong** interactions, i.e., those with an impact parameter $b \simeq b_{90}$. Since the electron basically loses all its forward momentum in such a collision, we have that the electron fluid loses an average momentum $m_{\rm e}(\vec{u}_{\rm e} - \vec{u}_{\rm i})$ to the ion fluid per strong electron-ion encounter. Hence, the *rate* of momentum transfer is approximately given by

$$\vec{C}_{\rm e} = -m_{\rm e}\, n_{\rm e}\, \nu_{\rm L}\, (\vec{u}_{\rm e} - \vec{u}_{\rm i})$$

where $\nu_{\rm L}$ is the collision frequency for strong collisions. Since the current density is $\vec{J} = q_{\rm e}\, n_{\rm e}\, \vec{u}_{\rm e} + q_{\rm i}\, n_{\rm i}\, \vec{u}_{\rm i} \simeq n_{\rm e}\, e\, (\vec{u}_{\rm i} - \vec{u}_{\rm e})$, where we have used that $n_{\rm i} \simeq n_{\rm e}$, we can write this as

$$\vec{C}_{\rm e} = +n_{\rm e}\, e\, \eta\, \vec{J}$$

where

$$\eta = \frac{m_{\rm e}\, \nu_{\rm L}}{n_{\rm e}\, e^2}$$

This parameter is called the **electric resistivity**, and is the inverse of the **electric conductivity**, σ. Substituting the expression for $\nu_{\rm L}$ derived in Chapter 27 (and using $v_0 \sim v_{\rm e} \sim (3k_{\rm B}T/m_{\rm e})^{1/2}$), we find that

$$\eta = \frac{4\pi}{3\sqrt{3}}\, \frac{m_{\rm e}^{1/2}\, e^2}{(k_{\rm B}\, T)^{3/2}} \approx 2.4\, \frac{m_{\rm e}^{1/2}\, e^2}{(k_{\rm B}\, T)^{3/2}}$$

Using a much more rigorous derivation of the electrical resistivity, accounting for the whole range of impact parameters, Spitzer & Härm (1953) obtain (assuming that all ions have $Z = 1$)

$$\eta = 1.69 \ln \Lambda\, \frac{m_{\rm e}^{1/2}\, e^2}{(k_{\rm B}\, T)^{3/2}}$$

in reasonable agreement with our crude estimate.

Using the above expression for $\vec{C}_{\rm e}$, the collision term in the generalized Ohm's law reduces to $\vec{J}/\sigma$. Typically, especially when considering low frequency phenomena, which we tend to do with MHD, the $\partial\vec{J}/\partial t$ term is small compared to this collision term and can thus be ignored. In addition, for a sufficiently 'cold' plasma the ∇P terms can also be neglected. Finally, since currents are typically small the $\vec{J} \times \vec{B}$ term, which describes the **Hall effect**, is typically small compared to $\vec{u}/c \times \vec{B}$. Hence,

we can simplify the generalized Ohm's law to read

$$\boxed{\vec{J} = \sigma \left(\vec{E} + \frac{\vec{u}}{c} \times \vec{B} \right)}$$

This equation is usually referred to as **Ohm's law**.

The MHD equations derived thus far (mass continuity, charge continuity, momentum conservation and Ohm's law) need to be complemented with the **Maxwell equations**. Fortunately, these can also be simplified. Let's start with Ampère's circuital law

$$\nabla \times \vec{B} = \frac{4\pi}{c} \vec{J} + \frac{1}{c} \frac{\partial \vec{E}}{\partial t}$$

It can be shown (see §3.6 in *The Physics of Fluids and Plasma* by A.R. Choudhuri) that

$$\left[\frac{1}{c} \frac{\partial \vec{E}}{\partial t} \right] / \left[\nabla \times \vec{B} \right] \sim \frac{v^2}{c^2}$$

Hence, in the *non-relativistic* regime considered here, the **displacement current** is negligble, which implies that $\vec{J} = \frac{c}{4\pi} \nabla \times \vec{B}$. Combined with Ohm's law, we therefore have that

$$\vec{E} = \frac{c}{4\pi\sigma} (\nabla \times \vec{B}) - \frac{\vec{u}}{c} \times \vec{B}$$

Hence, we see that, in MHD, the electric field does not have to be considered an independent variable; instead, it can be obtained from $\vec{u}$ and $\vec{B}$.

Plugging the above expression for $\vec{E}$ in Faraday's law of induction

$$\nabla \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}$$

which is another Maxwell equation, yields that

$$\frac{\partial \vec{B}}{\partial t} = -\frac{c^2}{4\pi\sigma} \nabla \times (\nabla \times \vec{B}) + \nabla \times (\vec{u} \times \vec{B})$$

Using the vector identity $\nabla \times (\nabla \times \vec{B}) = \nabla(\nabla \cdot \vec{B}) - \nabla^2 \vec{B}$ (see Appendix A), and the fact that $\nabla \cdot \vec{B} = 0$ (yet another Maxwell equation), we finally obtain the **induction equation**

$$\boxed{\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{u} \times \vec{B}) + \lambda \nabla^2 \vec{B}}$$

where

$$\lambda \equiv \frac{c^2}{4\pi\sigma}$$

is called the **magnetic diffusivity**. As is evident from the induction equation, it describes the rate at which the magnetic field diffuses due to collisions in the plasma.

Before we finally summarize our set of MHD equations, we apply one final modification by expanding the Lorentz force term, $\vec{J} \times \vec{B}$, in the moment equations. Using Ampère's circuital law without the displacement current, we have that

$$\frac{1}{c}(\vec{J} \times \vec{B}) = \frac{1}{4\pi}(\nabla \times \vec{B}) \times \vec{B} = \frac{1}{4\pi}\left[(\vec{B} \cdot \nabla)\vec{B} - \nabla\left(\frac{B^2}{2}\right)\right]$$

where the last step is based on a standard vector identity (see Appendix A).
Next, using that

$$\left(\vec{B} \cdot \nabla\right)\vec{B} = B_j \frac{\partial B_i}{\partial x_j} = \frac{\partial B_i B_j}{\partial x_j} - B_i \frac{\partial B_j}{\partial x_j} = \frac{\partial B_i B_j}{\partial x_j}$$

where the last step follows from the fact that $\nabla \cdot \vec{B} = 0$, we can now write the momentum equations in *index form* as

$$\rho \frac{\partial u_i}{\partial t} = -\frac{\partial P}{\partial x_i} - \frac{\partial}{\partial x_i}\left(\frac{B^2}{8\pi}\right) + \frac{\partial}{\partial x_j}\left(\frac{B_i B_j}{4\pi}\right) = +\frac{\partial}{\partial x_j}\left[\sigma_{ij} - M_{ij}\right]$$

Here $\sigma_{ij} = -P\delta_{ij}$ is the **stress tensor** (in the absence of viscosity) and

$$M_{ij} \equiv \frac{B^2}{8\pi}\delta_{ij} - \frac{B_i B_j}{4\pi}$$

is the **magnetic stress tensor**. Its' diagonal elements represent the **magnetic pressure**, while its off-diagonal terms arise from **magnetic tension**.

The following table summarizes the full set of **resistive MHD** equations. These are valid to describe low-frequency plasma phenomena for a relatively cold plasma in which $n_\mathrm{e} \simeq n_\mathrm{i}$, such that the charge density can be neglected. Note also that conduction, viscosity, radiation and gravity are all neglected (the corresponding terms are trivially added). A fluid that obeys these MHD equations is called a **magnetofluid**.

Continuity Eq.	$\dfrac{\mathrm{d}\rho}{\mathrm{d}t} = -\rho\,\nabla\cdot\vec{u}$
Momentum Eqs.	$\rho\,\dfrac{\mathrm{d}\vec{u}}{\mathrm{d}t} = -\nabla P + \dfrac{1}{c}\vec{J}\times\vec{B}$
Energy Eq.	$\rho\,\dfrac{\mathrm{d}\varepsilon}{\mathrm{d}t} = -P\,\nabla\cdot\vec{u} - \dfrac{J^2}{\sigma}$
Ohm's Law	$\vec{J} = \sigma\left(\vec{E} + \dfrac{\vec{u}}{c}\times\vec{B}\right)$
Induction Eq.	$\dfrac{\partial\vec{B}}{\partial t} = \nabla\times(\vec{u}\times\vec{B}) + \lambda\,\nabla^2\vec{B}$
Constitutive Eqs.	$\lambda = \dfrac{c^2}{4\pi\sigma}\,, \qquad \sigma^{-1} = \eta \propto \dfrac{m_\mathrm{e}^{1/2}e^2}{(k_\mathrm{B}T)^{3/2}}$

The equations of resistive MHD

Note that in the momentum equations we have written the Lagrangian derivative $\mathrm{d}\vec{u}/\mathrm{d}t$, rather than the Eulerian $\partial\vec{u}/\partial t$ that we obtained earlier in our derivation. This is allowed, since we had assumed that both $(\vec{u}_\mathrm{e}\cdot\nabla)\vec{u}_\mathrm{e}$ and $(\vec{u}_\mathrm{i}\cdot\nabla)\vec{u}_\mathrm{i}$ are small compared to other terms, which therefore also applies to $(\vec{u}\cdot\nabla)\vec{u}$.

Note also that although we have written the above set of MHD equations including the electric field $\vec{E}$, this is not an independent dynamical quantity. After all, as

already mentioned above, it follows from $\vec{B}$ and $\vec{u}$ according to

$$\vec{E} = \frac{c}{4\pi\sigma}\nabla \times \vec{B} - \frac{\vec{u}}{c} \times \vec{B}$$

In fact, **Ohm's law** is not required to close this set of equations as the current density $\vec{J}$ can be computed directly from the magnetic field $\vec{B}$ using Ampère's circuital law without displacement current; $\nabla \times \vec{B} = (4\pi/c)\vec{J}$.

Hence, we see that in the end MHD is actually remarkably similar to the hydrodynamics of neutral fluids. The 'only' additions are the magnetic field, which adds an additional pressure and an (anisotropic) tension, and the Coulomb collisions, which cause Ohmic dissipation and a diffusion of the magnetic fields. To further strengthen the similarities with regular fluid hydrodynamics, note that the **induction equation** is very similar to the **vorticity equation**

$$\frac{\partial \vec{w}}{\partial t} = \nabla \times (\vec{u} \times \vec{\omega}) - \nabla \times \left(\frac{\nabla P}{\rho}\right) + \nu \nabla^2 \vec{\omega}$$

(see Chapter 13). Here $\vec{w}$ is the **vorticity** and ν the **kinetic viscosity**. Except for the **baroclinic** term, which is absent in the induction equation, vorticity and magnetic field (in the MHD approximation) behave very similar (cf., magnetic field lines and vortex lines).

Motivated by this similarity, we now define the **magnetic Reynolds number**

$$\mathcal{R}_{\rm m} = \frac{U\,L}{\lambda}$$

with U and L the characteristic velocity and length scales of the plasma flow. Recall (from Chapter 16), the definition of the **Reynolds number** $\mathcal{R} = U\,L/\nu$. We thus merely replaced the **kinetic viscosity** with the **magnetic diffusivity**, which is proportional to the **electric resistivitiy** (and thus inversely proportional to the **conductivity**).

• When $\mathcal{R}_{\rm m} \ll 1$, the second term in the induction equation dominates, which therefore becomes

$$\frac{\partial B}{\partial t} \simeq \lambda\,\nabla^2 \vec{B}$$

This is the situation one typically encounters in laboratory plasmas, where U and L are small. The implication is that, the magnetic field in laboratory plasmas, when

left to itself, decays away due to magnetic diffusion. This can be understood from the fact that magnetic fields are directly related to currents, which die away due to **Ohmic dissipation** unless one applies a source of voltage.

• When $\mathcal{R}_{\rm m} \gg 1$, the first term in the induction equation dominates, which therefore becomes

$$\frac{\partial B}{\partial t} \simeq \nabla \times (\vec{u} \times \vec{B})$$

This is the situation we typically encounter in astrophysics, where U and L are large. In the limit of infinite conductivity (i.e., zero electrical resistivity, and thus zero magnetic diffusivity), the above equation is exact, and we talk of **ideal MHD**. Obviously, with infinite conductivity there is also no Ohmic dissipation, and the energy equation in ideal MHD is therefore identical to that for a neutral fluid. Hence, for ideal MHD the full, closed set of equations reduces to

Continuity Eq.	$\dfrac{\mathrm{d}\rho}{\mathrm{d}t} = -\rho\,\nabla\cdot\vec{u}$
Momentum Eqs.	$\rho\,\dfrac{\mathrm{d}\vec{u}}{\mathrm{d}t} = -\nabla P + \dfrac{1}{c}\vec{J}\times\vec{B}$
Energy Eq.	$\rho\,\dfrac{\mathrm{d}\varepsilon}{\mathrm{d}t} = -P\,\nabla\cdot\vec{u}$
Induction Eq.	$\dfrac{\partial\vec{B}}{\partial t} = \nabla\times(\vec{u}\times\vec{B})$
Ampère's law	$\nabla\times\vec{B} = \dfrac{4\pi}{c}\,\vec{J}$

The equations of ideal MHD

The reader may wonder what happens to Ohm's law in the limit where $\sigma \to \infty$; In order to assure only finite currents, we need to have that $E + \vec{u}/c \times \vec{B} = 0$, and thus $\vec{E} = \frac{1}{c}\vec{u} \times \vec{B}$. However, since $\vec{E}$ is not required (is not an independent dynamical quantity), this is of little relevance.

An important implication of ideal MHD is that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_S \vec{B}\cdot\mathrm{d}^2 s = 0$$

This expresses that the magnetic flux is conserved as it moves with the fluid. This is known as **Alfvén's theorem of flux freezing**. It is the equivalent of **Helmholtz' theorem** that $\mathrm{d}\Gamma/\mathrm{d}t = 0$ for an inviscid fluid (with Γ the circulation). An implication is that, in the case of ideal MHD, two fluid elements that are connected by a magnetic flux line, will remain connected by that same magnetic flux line.

(Ideal) MHD is used to describe many astrophysical processes, from the magnetic field topology of the Sun, to angular momentum transfer in accretion disks, and from the formation of jets in accretion disks, to the magnetic breaking during star formation. One can also apply linear perturbation theory to the ideal MHD equations, to examine what happens to a magnetofluid if it is perturbed. If one ignores viscosity, heat conduction, and electric resistivity (i.e., we are in the ideal MHD regime), then the resulting **dispersion relation** is given by

$$\omega^2\vec{u}_1 = (c_\mathrm{s}^2 + u_\mathrm{A}^2)(\vec{k}\cdot\vec{u}_1)\vec{k} + \vec{u}_\mathrm{A}\cdot\vec{k}\left[(\vec{u}_\mathrm{A}\cdot\vec{k})\vec{u}_1 - (\vec{u}_\mathrm{A}\cdot\vec{u}_1)\vec{k} - (\vec{k}\cdot\vec{u}_1)\vec{u}_\mathrm{A}\right]$$

Here $\vec{u}_\mathrm{A}$ is a characteristic velocity, called the **Alfvén velocity**, given by

$$\boxed{\vec{u}_\mathrm{A} = \frac{\vec{B}_0}{\sqrt{4\pi\rho_0}}}$$

The above dispersion relation, $\omega(\vec{k})$, for given sound speed, c_s, and Alvén velocity, $\vec{u}_\mathrm{A}$, of the magnetofluid, is the basic dispersion relation for **hydromagnetic waves**. Although it has a complicated looking form, there is one solution that is rather simple. It corresponds to a purely tranverse wave in which the displacement, and therefore the velocity perturbation $\vec{u}_1(\vec{x},t)$, is perpendicular to both the wave vector $\vec{k}$ and the magnetic field (which is in the direction of $\vec{u}_\mathrm{A}$). Under those conditions the dispersion relation reduces to

$$\omega^2 = (\vec{u}_\mathrm{A}\cdot\vec{k})^2$$

These waves, called **Alfvén waves**, have a group velocity $v_\mathrm{g} = \partial\omega/\partial\vec{k} = \vec{u}_\mathrm{A}$ and are moving, with that velocity, along the magnetic field lines.

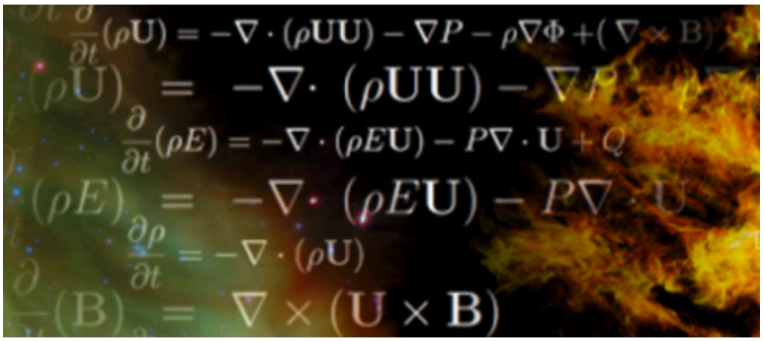
Any wave is driven by some restoring force. In the case of accoustic waves these are pressure gradients, while the restoring force in the case of the plasma oscillations (**Langmuir waves**) discussed in the previous chapter arise from the electrical field that results from a separation of electrons and ions. In the case of perturbations to a magnetofluid, there are two restoring forces that play a role; pressure gradients and magnetic tension. In the case of Alfvèn waves the restoring force is purely the tension in the magnetic field lines (pressure plays no role). Hence, Alfvén waves are similar to the waves in a rope or string, which are also transverse waves. The group velocity of these waves is proportional to $\sqrt{\text{tension/density}}$. Since the magnetic tension is given by $B^2/4\pi$, we see that the Alfvén velocity has exactly the same form.

Note that in the case of ideal MHD the resistivity is zero, and there is thus no diffusion or dissipation of the magnetic fields, which instead are 'frozen' into the fluid. In the case of resistive MHD (i.e., if the magnetic resistivity is non-zero) the Alfvén waves will experience damping, thereby transferring the energy stored in the magnetic wave to random, thermal energy.

Alfvèn waves, though, are not the only solution to the dispersion relation given above. There are two additional solutions, corresponding to **fast mode** and **slow mode** waves. Contrary to the Alfvèn waves, the restoring force for these modes is a combination of magnetic tension *and* pressure (i.e., they are mixtures of acoustic and magnetic waves). Without going into any detail, we only mention in closing that any disturbance of a magnetofluid can be represented as a superposition of the Alfvèn, fast and slow modes.

Supplemental Material

Appendices



The following appendices provide supplemental material, including some calculus and radiation essentials that is relevant for this course. Students are assumed to be familiar with this. An exception are Appendices H to K, which provide details that are NOT considered part of this course's curriculum. They are included to provide background information for those readers that want to know a bit more.

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Appendix A

Vector Calculus

Vector: $\vec{A} = (a_1, a_2, a_3) = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

Amplitude of vector: $|\vec{A}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

Unit vector: $|\vec{A}| = 1$

Basis: In the above example, the unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$ form a vector basis.
Any 3 vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ can form a vector basis
as long as $\det(\vec{A}, \vec{B}, \vec{C}) \neq 0$.

Determinant:
$$\det(\vec{A}, \vec{B}) = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

$$\det(\vec{A}, \vec{B}, \vec{C}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} + a_2 \begin{vmatrix} b_3 & b_1 \\ c_3 & c_1 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Geometrically:
$\det(\vec{A}, \vec{B}) = \pm$ area of parallelogram
$\det(\vec{A}, \vec{B}, \vec{C}) = \pm$ volume of parallelepiped

Multiplication by scalar:
$\alpha\,\vec{A} = (\alpha a_1, \alpha a_2, \alpha a_3)$
$|\alpha\,\vec{A}| = |\alpha|\,|\vec{A}|$

Summation of vectors: $\vec{A} + \vec{B} = \vec{B} + \vec{A} = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$

Einstein Summation Convention: $a_i\, b_i = \sum_i a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3 = \vec{a} \cdot \vec{b}$
$\partial A_i / \partial x_i = \partial A_1/\partial x_1 + \partial A_2/\partial x_2 + \partial A_3/\partial x_3 = \nabla \cdot \vec{A}$
$A_{ii} = A_{11} + A_{22} + A_{33} = \mathrm{Tr}\, \vec{A}$ (trace of $\vec{A}$)

Dot product (aka scalar product): $\vec{A} \cdot \vec{B} = a_i\, b_i = |\vec{A}|\, |\vec{B}| \cos\theta$
$\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$

Useful for:

- computing angle between two vectors: $\cos\theta = \vec{A} \cdot \vec{B} / (|\vec{A}|\, |\vec{B}|)$
- check orthogonality: two vectors are orthogonal if $\vec{A} \cdot \vec{B} = 0$
- compute projection of $\vec{B}$ in direction of $\vec{A}$, which is given by $\vec{A} \cdot \vec{B} / |\vec{A}|$

Cross Product (aka vector product): $\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \varepsilon_{ijk} a_i\, b_j\, \hat{e}_k$
$|\vec{A} \times \vec{B}| = |\vec{A}|\, |\vec{B}| \sin\theta = \det(\vec{A}, \vec{B})$

NOTE: ε_{ijk} is called the **Levi-Civita tensor**, which is described in Appendix G.

In addition to the dot product and cross product, there is a third vector product that one occasionally encounters in dynamics;

Tensor product: $\vec{A} \otimes \vec{B} = \overline{\overline{\mathbf{AB}}}$ $\qquad (\overline{\overline{\mathbf{AB}}})_{ij} = a_i b_j$
$\vec{A} \otimes \vec{B} \neq \vec{B} \otimes \vec{A}$

The tensor product $\overline{\overline{\mathbf{AB}}}$ is a tensor of rank two and is called a **dyad**. It is best to define a dyad by what it *does*: it transforms a vector $\vec{C}$ into another vector with the direction of $\vec{A}$ according to the rule

$$(\vec{A} \otimes \vec{B})\, \vec{C} = \vec{A}\, (\vec{B} \cdot \vec{C})$$

A **dyadic** is a linear combination of dyads (i.e., $4(\vec{a} \otimes \vec{b} + 2(\vec{c} \otimes \vec{d}) - 7(\vec{e} \otimes \vec{f})$). Dyadics are important because *each tensor of rank two can be written as a dyadic!*.

$\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$	$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$
$(\alpha\vec{A}) \cdot \vec{B} = \alpha(\vec{A} \cdot \vec{B}) = \vec{A} \cdot (\alpha\vec{B})$	$(\alpha\vec{A}) \times \vec{B} = \alpha(\vec{A} \times \vec{B}) = \vec{A} \times (\alpha\vec{B})$
$\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$	$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$
$\vec{A} \cdot \vec{B} = 0 \quad \rightarrow \quad \vec{A} \perp \vec{B}$	$\vec{A} \times \vec{B} = 0 \quad \rightarrow \quad \vec{A} \parallel \vec{B}$
$\vec{A} \cdot \vec{A} = \lvert\vec{A}\rvert^2$	$\vec{A} \times \vec{A} = 0$

Triple Scalar Product: $\vec{A} \cdot (\vec{B} \times \vec{C}) = \det(\vec{A}, \vec{B}, \vec{C}) = \varepsilon_{ijk} a_i \, b_j \, c_k$
$\vec{A} \cdot (\vec{B} \times \vec{C}) = 0 \quad \rightarrow \quad \vec{A}, \vec{B}, \vec{C}$ are coplanar
$\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A}) = \vec{C} \cdot (\vec{A} \times \vec{B})$

Triple Vector Product: $\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\, \vec{B} - (\vec{A} \cdot \vec{B})\, \vec{C}$
as is clear from above, $\vec{A} \times (\vec{B} \times \vec{C})$ lies in plane of $\vec{B}$ and $\vec{C}$.

Useful to remember: $(\vec{A} \times \vec{B}) \cdot (\vec{C} \times \vec{D}) = (\vec{A} \cdot \vec{C})\, (\vec{B} \cdot \vec{D}) - (\vec{A} \cdot \vec{D})\, (\vec{B} \cdot \vec{C})$
$(\vec{A} \times \vec{B}) \times (\vec{C} \times \vec{D}) = \left[\vec{A} \cdot (\vec{B} \times \vec{D})\right] \vec{C} - \left[\vec{A} \cdot (\vec{B} \times \vec{C})\right] \vec{D}$

Gradient Operator: $\nabla = \vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$
This **vector operator** is sometimes called the **nabla** or **del** operator.

Laplacian operator: $\nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$
This is a **scalar operator**.

Differential: $f = f(x, y, z) \quad \rightarrow \quad \mathrm{d}f = \frac{\partial f}{\partial x}\, \mathrm{d}x + \frac{\partial f}{\partial y}\, \mathrm{d}y + \frac{\partial f}{\partial z}\, \mathrm{d}z$

Chain Rule: If $x = x(t)$, $y = y(t)$ and $z = z(t)$ then $\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial f}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t}$
If $x = x(s, t)$, $y = y(s, t)$ and $z = z(s, t)$ then $\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$

Gradient Vector: $\nabla f = \text{grad} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$
the gradient vector at (x, y, z) is normal to the level surface through the point (x, y, z).

Directional Derivative: The derivative of $f = f(x, y, z)$ in direction of $\vec{u}$ is
$\mathrm{D}_u f = \nabla f \cdot \frac{\vec{u}}{|\vec{u}|} = |\nabla f| \cos\theta$

Vector Field: $\vec{F}(\vec{x}) = (F_x, F_y, F_z) = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$
where $F_x = F_x(x, y, z)$, $F_y = F_y(x, y, z)$, and $F_z = F_z(x, y, z)$.

Divergence of Vector Field: $\text{div}\vec{F} = \nabla \cdot \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$
A vector field for which $\nabla \cdot \vec{F} = 0$ is called **solenoidal** or **divergence-free**.

Curl of Vector Field: $\text{curl}\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_x & F_y & F_z \end{vmatrix}$
A vector field for which $\nabla \times \vec{F} = 0$ is called **irrotational** or **curl-free**.

Laplacian of Vector Field: $\nabla^2\vec{F} = (\nabla \cdot \nabla)\vec{F} = \nabla(\nabla \cdot \vec{F}) - \nabla \times (\nabla \times \vec{F})$
Note that $\nabla^2\vec{F} \neq \nabla(\nabla \cdot \vec{F})$: **do not make this mistake**.

Let $S(\vec{x})$ and $T(\vec{x})$ be scalar fields, and let $\vec{A}(\vec{x})$ and $\vec{B}(\vec{x})$ be vector fields:

$\nabla S = \text{grad} S = \text{vector}$	$\nabla^2 S = \nabla \cdot (\nabla S) = \text{scalar}$
$\nabla \cdot \vec{A} = \text{div}\vec{A} = \text{scalar}$	$\nabla^2\vec{A} = (\nabla \cdot \nabla)\,\vec{A} = \text{vector}$
$\nabla \times \vec{A} = \text{curl}\vec{A} = \text{vector}$	

$$\nabla \times (\nabla S) = 0 \qquad \operatorname{curl} \operatorname{grad} S = 0$$

$$\nabla \cdot (\nabla \times \vec{A}) = 0 \qquad \operatorname{div} \operatorname{curl} \vec{A} = 0$$

$$\nabla(ST) = S\,\nabla T + T\,\nabla S$$

$$\nabla \cdot (S\vec{A}) = S(\nabla \cdot \vec{A}) + \vec{A} \cdot \nabla S$$

$$\nabla \times (S\vec{A}) = (\nabla S) \times \vec{A} + S(\nabla \times \vec{A})$$

$$\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B})$$

$$\nabla \times (\vec{A} \times \vec{B}) = \vec{A}(\nabla \cdot \vec{B}) - \vec{B}(\nabla \cdot \vec{A}) + (\vec{B} \cdot \nabla)\vec{A} - (\vec{A} \cdot \nabla)\vec{B}$$

$$\nabla(\vec{A} \cdot \vec{B}) = (\vec{A} \cdot \nabla)\vec{B} + (\vec{B} \cdot \nabla)\vec{A} + \vec{A} \times (\nabla \times \vec{B}) + \vec{B} \times (\nabla \times \vec{A})$$

$$\vec{A} \times (\nabla \times \vec{A}) = \tfrac{1}{2}\nabla(\vec{A} \cdot \vec{A}) - (\vec{A} \cdot \nabla)\vec{A}$$

$$\nabla \times (\nabla^2 \vec{A}) = \nabla^2(\nabla \times \vec{A})$$

Appendix B

Conservative Vector Fields

Line Integral of a Conservative Vector Field: Consider a curve γ running from location $\vec{x}_0$ to $\vec{x}_1$. Let $\mathrm{d}\vec{l}$ be the directional element of length along γ (i.e., with direction equal to that of the tangent vector to γ), then, for any scalar field $\Phi(\vec{x})$,

$$\int_{\vec{x}_0}^{\vec{x}_1} \nabla\Phi \cdot \mathrm{d}\vec{l} = \int_{\vec{x}_0}^{\vec{x}_1} \mathrm{d}\Phi = \Phi(\vec{x}_1) - \Phi(\vec{x}_0)$$

This implies that the line integral is independent of γ, and hence

$$\oint_c \nabla\Phi \cdot \mathrm{d}\vec{l} = 0$$

where c is a closed curve, and the integral is to be performed in the counter-clockwise direction.

Conservative Vector Fields:
A conservative vector field $\vec{F}$ has the following properties:

- $\vec{F}(\vec{x})$ is a gradient field, which means that there is a scalar field $\Phi(\vec{x})$ so that $\vec{F} = \nabla\Phi$
- Path independence: $\oint_c \vec{F} \cdot \mathrm{d}\vec{l} = 0$
- Irrotational = curl-free: $\nabla \times \vec{F} = 0$

Appendix C

Integral Theorems

Green's Theorem: Consider a 2D vector field $\vec{F} = F_x\hat{i} + F_y\hat{j}$

$$\oint \vec{F} \cdot \mathrm{d}\vec{l} = \int\int_A \nabla \times \vec{F} \cdot \hat{n}\, \mathrm{d}A = \int\int_A |\nabla \times \vec{F}|\, \mathrm{d}A$$

$$\oint \vec{F} \cdot \hat{n}\, \mathrm{d}l = \int\int_A \nabla \cdot \vec{F}\, \mathrm{d}A$$

NOTE: in the first equation we have used that $\nabla \times \vec{F}$ is always pointing in the direction of the normal $\hat{n}$.

Gauss' Divergence Theorem: Consider a 3D vector field $\vec{F} = (F_x, F_y, F_z)$
If S is a closed surface bounding a region D with normal pointing outwards, and $\vec{F}$ is a vector field defined and differentiable over all of D, then

$$\int\int_S \vec{F} \cdot \mathrm{d}\vec{S} = \int\int\int_D \nabla \cdot \vec{F}\, \mathrm{d}V$$

Stokes' Curl Theorem: Consider a 3D vector field $\vec{F} = (F_x, F_y, F_z)$
If C is a closed curve, and S is *any* surface bounded by C, then

$$\oint_c \vec{F} \cdot \mathrm{d}\vec{l} = \int\int_S (\nabla \times \vec{F}) \cdot \hat{n}\, \mathrm{d}S$$

NOTE: The curve of the line intergral must have positive orientation, meaning that $\mathrm{d}\vec{l}$ points counterclockwise when the normal of the surface points towards the viewer.

Appendix D

Curvi-Linear Coordinate Systems

In astrophysics, one often works in **curvi-linear**, rather than **Cartesian** coordinate systems. The two most often encountered examples are the **cylindrical** (R, ϕ, z) and **spherical** (r, θ, ϕ) coordinate systems.

In this chapter we describe how to handle **vector calculus** in non-Cartesian coordinate systems (Euclidean spaces only). After giving the 'rules' for arbitrary coordinate systems, we apply them to cylindrical and spherical coordinate systems, respectively.

Vector Calculus in an Arbitrary Coordinate System:

Consider a vector $\vec{x} = (x, y, z)$ in Cartesian coordinates. This means that we can write

$$\vec{x} = x\,\vec{e}_x + y\,\vec{e}_y + z\,\vec{e}_z$$

where $\vec{e}_x$, $\vec{e}_y$ and $\vec{e}_z$ are the unit directional vectors. Now consider the same vector $\vec{x}$, but expressed in another **general** (arbitrary) coordinate system; $\vec{x} = (q_1, q_2, q_3)$. It is tempting, **but terribly wrong**, to write that

$$\vec{x} = q_1\,\vec{e}_1 + q_2\,\vec{e}_2 + q_3\,\vec{e}_3$$

where $\vec{e}_1$, $\vec{e}_2$ and $\vec{e}_3$ are the unit directional vectors in the new (q_1, q_2, q_3)-coordinate system. In what follows we show how to properly treat such generalized coordinate systems.

In general, one expresses the distance between (q_1, q_2, q_3) and $(q_1 + \mathrm{d}q_1, q_2 + \mathrm{d}q_2, q_3 + \mathrm{d}q_3)$ in an arbitrary coordinate system as

$$\mathrm{d}s = \sqrt{h_{ij}\,\mathrm{d}q_i\,\mathrm{d}q_j}$$

Here h_{ij} is called the **metric tensor**. In what follows, we will only consider **orthogonal** coordinate systems for which $h_{ij} = 0$ if $i \neq j$, so that $\mathrm{d}s^2 = h_i^2\,\mathrm{d}q_i^2$ (Einstein summation convention) with $h_i = \sqrt{h_{ii}}$.

An example of an orthogonal coordinate system are the Cartesian coordinates, for which $h_{ij} = \delta_{ij}$. After all, the distance between two points separated by the infinitesimal displacement vector $\mathrm{d}\vec{x} = (\mathrm{d}x, \mathrm{d}y, \mathrm{d}z)$ is $\mathrm{d}s^2 = |\mathrm{d}\vec{x}|^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$.

The coordinates (x, y, z) and (q_1, q_2, q_3) are related to each other via the **transformation relations**

$$\begin{aligned} x &= x(q_1, q_2, q_3) \\ y &= y(q_1, q_2, q_3) \\ z &= z(q_1, q_2, q_3) \end{aligned}$$

and the corresponding inverse relations

$$\begin{aligned} q_1 &= q_1(x, y, z) \\ q_2 &= q_2(x, y, z) \\ q_3 &= q_3(x, y, z) \end{aligned}$$

Hence, we have that the **differential vector** is:

$$\mathrm{d}\vec{x} = \frac{\partial \vec{x}}{\partial q_1}\,\mathrm{d}q_1 + \frac{\partial \vec{x}}{\partial q_2}\,\mathrm{d}q_2 + \frac{\partial \vec{x}}{\partial q_3}\,\mathrm{d}q_3$$

where

$$\frac{\partial \vec{x}}{\partial q_i} = \frac{\partial}{\partial q_i}(x, y, z)$$

The **unit directional vectors** are:

$$\vec{e}_i = \frac{\partial \vec{x}/\partial q_i}{|\partial \vec{x}/\partial q_i|}$$

which allows us to rewrite the expression for the **differential vector** as

$$\mathrm{d}\vec{x} = \left|\frac{\partial \vec{x}}{\partial q_1}\right|\,\mathrm{d}q_1\,\vec{e}_1 + \left|\frac{\partial \vec{x}}{\partial q_2}\right|\,\mathrm{d}q_2\,\vec{e}_2 + \left|\frac{\partial \vec{x}}{\partial q_3}\right|\,\mathrm{d}q_3\,\vec{e}_3$$

and thus

$$|\mathrm{d}\vec{x}|^2 = \left|\frac{\partial \vec{x}}{\partial q_i}\right|^2\,\mathrm{d}q_i^2$$

(Einstein summation convention). Using the definition of the metric, according to which $|\mathrm{d}\vec{x}|^2 = h_i^2 \mathrm{d}q_i^2$ we thus infer that

$$\boxed{h_i = \left|\frac{\partial \vec{x}}{\partial q_i}\right|}$$

Using this expression for the metric allows us to write the **unit directional vectors** as

$$\vec{e}_i = \frac{1}{h_i}\frac{\partial \vec{x}}{\partial q_i}$$

and the **differential vector** in the compact form as

$$\mathrm{d}\vec{x} = h_i \, \mathrm{d}q_i \, \vec{e}_i$$

From the latter we also have that the **infinitesimal volume element** for a general coordinate system is given by

$$\mathrm{d}^3\vec{x} = |h_1 \, h_2 \, h_3| \, \mathrm{d}q_1 \, \mathrm{d}q_2 \, \mathrm{d}q_3$$

Note that the absolute values are needed to assure that $\mathrm{d}^3\vec{x}$ is positive.

Now consider a vector $\vec{A}$. In the Cartesian **basis** $\mathcal{C} = \{\vec{e}_x, \vec{e}_y, \vec{e}_z\}$ we have that

$$[\vec{A}]_{\mathcal{C}} = A_x\,\vec{e}_x + A_y\,\vec{e}_y + A_z\,\vec{e}_z$$

In the **basis** $\mathcal{B} = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$, corresponding to our generalized coordinate system, we instead have that

$$[\vec{A}]_{\mathcal{B}} = A_1\,\vec{e}_1 + A_2\,\vec{e}_2 + A_3\,\vec{e}_3$$

We can rewrite the above as

$$[\vec{A}]_{\mathcal{B}} = A_1 \begin{pmatrix} e_{11} \\ e_{12} \\ e_{13} \end{pmatrix} + A_2 \begin{pmatrix} e_{21} \\ e_{22} \\ e_{23} \end{pmatrix} + A_3 \begin{pmatrix} e_{31} \\ e_{32} \\ e_{33} \end{pmatrix} = \begin{pmatrix} A_1 e_{11} + A_2 e_{21} + A_3 e_{31} \\ A_2 e_{12} + A_2 e_{22} + A_3 e_{32} \\ A_3 e_{13} + A_2 e_{23} + A_3 e_{33} \end{pmatrix}$$

and thus

$$[\vec{A}]_{\mathcal{B}} = \begin{pmatrix} e_{11} & e_{21} & e_{31} \\ e_{12} & e_{22} & e_{32} \\ e_{13} & e_{23} & e_{33} \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} \equiv \mathbf{T} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}$$

Using similar logic, one can write

$$[\vec{A}]_{\mathcal{C}} = \begin{pmatrix} e_{x1} & e_{y1} & e_{z1} \\ e_{x2} & e_{y2} & e_{z2} \\ e_{x3} & e_{y3} & e_{z3} \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = \mathbf{I} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

and since $\vec{A}$ is the same object independent of its basis we have that

$$\mathbf{I} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = \mathbf{T} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}$$

and thus, we see that the relation between $[\vec{A}]_{\mathcal{B}}$ and $[\vec{A}]_{\mathcal{C}}$ is given by

$$[\vec{A}]_{\mathcal{C}} = \mathbf{T}\,[\vec{A}]_{\mathcal{B}}\,, \qquad\qquad [\vec{A}]_{\mathcal{B}} = \mathbf{T}^{-1}\,[\vec{A}]_{\mathcal{C}}$$

For this reason, $\mathbf{T}$ is called the **transformation of basis matrix**. Note that the columns of $\mathbf{T}$ are the unit-direction vectors $\vec{e}_i$, i.e., $T_{ij} = e_{ij}$. Since these are orthogonal to each other, the matric $\mathbf{T}$ is said to be **orthogonal**, which implies that $\mathbf{T}^{-1} = \mathbf{T}^{\,\mathrm{T}}$ (the inverse is equal to the transpose), and $\det(T) = \pm 1$.

Now we are finally ready to determine how to write our position vector $\vec{x}$ in the new basis $\mathcal{B}$ of our generalized coordinate system. Let's write $\vec{x} = a_i\,\vec{e}_i$, i.e.

$$[\vec{x}]_{\mathcal{B}} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

We started this chapter by pointing out that it is tempting, but wrong, to set $a_i = q_i$ (as for the Cartesian basis). To see this, recall that $|\vec{x}| = \sqrt{(a_1)^2 + (a_2)^2 + (a_3)^2}$, from which it is immediately clear that each a_i needs to have the dimension of length. Hence, when q_i is an angle, clearly $a_i \neq q_i$. To compute the actual a_i you need to use the transformation of basis matrix as follows:

$$[\vec{x}]_{\mathcal{B}} = \mathbf{T}^{-1}\,[\vec{x}]_{\mathcal{C}} = \begin{pmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} e_{11}x + e_{12}y + e_{13}z \\ e_{21}x + e_{22}y + e_{23}z \\ e_{31}x + e_{32}y + e_{33}z \end{pmatrix}$$

Hence, using our expression for the unit direction vectors, we see that

$$\boxed{\boxed{a_i = \frac{1}{h_i}\left(\frac{\partial x_j}{\partial q_i}\,x_j\right) = \frac{1}{h_i}\left(\frac{\partial \vec{x}}{\partial q_i}\cdot\vec{x}\right)}}$$

Hence, the position vector in the generalized basis $\mathcal{B}$ is given by

$$[\vec{x}]_{\mathcal{B}} = \sum_i \frac{1}{h_i}\left(\frac{\partial \vec{x}}{\partial q_i}\cdot\vec{x}\right)\vec{e}_i$$

and by operating $\mathrm{d}/\mathrm{d}t$ on $[\vec{x}]_{\mathcal{B}}$ we find that the corresponding velocity vector in the $\mathcal{B}$ basis is given by

$$[\vec{v}]_{\mathcal{B}} = \sum_i h_i\,\dot{q}_i\,\vec{e}_i$$

with $\dot{q}_i = \mathrm{d}q_i/\mathrm{d}t$. Note that the latter can also be inferred more directly by simply dividing the expression for the **differential vector** ($\mathrm{d}\vec{x} = h_i\,q_i\,\vec{e}_i$) by $\mathrm{d}t$.

Next we write out the gradient, the divergence, the curl and the Laplacian for our generalized coordinate system:

The gradient:

$$\nabla\psi = \frac{1}{h_i}\frac{\partial\psi}{\partial q_i}\vec{e}_i$$

The divergence:

$$\nabla\cdot\vec{A} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}(h_2h_3A_1) + \frac{\partial}{\partial q_2}(h_3h_1A_2) + \frac{\partial}{\partial q_3}(h_1h_2A_3)\right]$$

The curl (only one component shown)**:**

$$(\nabla\times\vec{A})_3 = \frac{1}{h_1h_2}\left[\frac{\partial}{\partial q_1}(h_2A_2) - \frac{\partial}{\partial q_2}(h_1A_1)\right]$$

The Laplacian:

$$\nabla^2\psi = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\psi}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{h_3h_1}{h_2}\frac{\partial\psi}{\partial q_2}\right) + \frac{\partial}{\partial q_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\psi}{\partial q_3}\right)\right]$$

The Convective operator:

$$(\vec{A}\cdot\nabla)\,\vec{B} = \left[\frac{A_i}{h_i}\frac{\partial B_j}{\partial q_i} + \frac{B_i}{h_i\,h_j}\left(A_j\frac{\partial h_j}{\partial q_i} - A_i\frac{\partial h_i}{\partial q_j}\right)\right]\,\vec{e}_j$$

Vector Calculus in Cylindrical Coordinates:

For cylindrical coordinates (R, ϕ, z) we have that

$$x = R\cos\phi \qquad y = R\sin\phi \qquad z = z$$

The **scale factors** of the **metric** therefore are:

$$h_R = 1 \qquad h_\phi = R \qquad h_z = 1$$

and the position vector is $\vec{x} = R\vec{e}_R + z\vec{e}_z$.

Let $\vec{A} = A_R\vec{e}_R + A_\phi\vec{e}_\phi + A_z\vec{e}_z$ an arbitrary vector, then

$$\begin{aligned} A_R &= A_x\cos\phi - A_y\sin\phi \\ A_\phi &= -A_x\sin\phi + A_y\cos\phi \\ A_z &= A_z \end{aligned}$$

In cylindrical coordinates the **velocity vector** becomes:

$$\begin{aligned} \vec{v} &= \dot{R}\,\vec{e}_R + R\,\dot{\vec{e}}_R + \dot{z}\,\vec{e}_z \\ &= \dot{R}\,\vec{e}_R + R\,\dot{\phi}\,\vec{e}_\phi + \dot{z}\,\vec{e}_z \end{aligned}$$

The Gradient:

$$\nabla\psi = \frac{\partial\psi}{\partial R}\,\vec{e}_R + \frac{1}{R}\frac{\partial\psi}{\partial\phi}\,\vec{e}_\phi + \frac{\partial\psi}{\partial z}\,\vec{e}_z$$

The Divergence:

$$\nabla\cdot\vec{A} = \frac{1}{R}\,\frac{\partial}{\partial R}(RA_R) + \frac{1}{R}\,\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z}$$

The Curl:

$$\nabla\times\vec{A} = \left(\frac{1}{R}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right)\vec{e}_R + \left(\frac{\partial A_R}{\partial z} - \frac{\partial A_z}{\partial R}\right)\vec{e}_\phi + \frac{1}{R}\left(\frac{\partial}{\partial R}(RA_\phi) - \frac{\partial A_R}{\partial\phi}\right)\vec{e}_z$$

The Convective Operator:

$$\begin{aligned}(\vec{A}\cdot\nabla)\,\vec{B} &= \left(A_R\frac{\partial B_R}{\partial R}+\frac{A_\phi}{R}\frac{\partial B_R}{\partial \phi}+A_z\frac{\partial B_R}{\partial z}-\frac{A_\phi B_\phi}{R}\right)\vec{e}_R \\ &+ \left(A_R\frac{\partial B_\phi}{\partial R}+\frac{A_\phi}{R}\frac{\partial B_\phi}{\partial \phi}+A_z\frac{\partial B_\phi}{\partial z}+\frac{A_\phi B_R}{R}\right)\vec{e}_\phi \\ &+ \left(A_R\frac{\partial B_z}{\partial R}+\frac{A_\phi}{R}\frac{\partial B_z}{\partial \phi}+A_z\frac{\partial B_z}{\partial z}\right)\vec{e}_z\end{aligned}$$

The Laplacian:

$$\text{scalar}: \quad \nabla^2\psi = \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial \psi}{\partial R}\right)+\frac{1}{R^2}\frac{\partial^2\psi}{\partial \phi^2}+\frac{\partial^2\psi}{\partial z^2}$$

$$\begin{aligned}\text{vector}: \quad \nabla^2\vec{F} &= \left(\nabla^2 F_R-\frac{F_R}{R^2}-\frac{2}{R^2}\frac{\partial F_\theta}{\partial \theta}\right)\vec{e}_R \\ &+ \left(\nabla^2 F_\theta+\frac{2}{R^2}\frac{\partial F_R}{\partial \theta}-\frac{F_\theta}{R^2}\right)\vec{e}_\theta \\ &+ \left(\nabla^2 F_z\right)\vec{e}_z\end{aligned}$$

Vector Calculus in Spherical Coordinates:

For spherical coordinates (r, θ, ϕ) we have that

$$x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi \qquad z = r\cos\theta$$

The **scale factors** of the **metric** therefore are:

$$h_r = 1 \qquad h_\theta = r \qquad h_\phi = r\sin\theta$$

and the position vector is $\vec{x} = r\vec{e}_r$.

Let $\vec{A} = A_r\vec{e}_r + A_\theta\vec{e}_\theta + A_\phi\vec{e}_\phi$ an arbitrary vector, then

$$\begin{aligned} A_r &= A_x\sin\theta\cos\phi + A_y\sin\theta\sin\phi + A_z\cos\theta \\ A_\theta &= A_x\cos\theta\cos\phi + A_y\cos\theta\sin\phi - A_z\sin\theta \\ A_\phi &= -A_x\sin\phi + A_y\cos\phi \end{aligned}$$

In spherical coordinates the **velocity vector** becomes:

$$\begin{aligned} \vec{v} &= \dot{r}\,\vec{e}_r + r\,\dot{\vec{e}}_r \\ &= \dot{r}\,\vec{e}_r + r\,\dot{\theta}\,\vec{e}_\theta + r\,\sin\theta\,\dot{\phi}\,\vec{e}_\phi \end{aligned}$$

The Gradient:

$$\nabla\psi = \frac{\partial\psi}{\partial r}\,\vec{e}_r + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\,\vec{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}\,\vec{e}_\phi$$

The Divergence:

$$\nabla\cdot\vec{A} = \frac{1}{r^2}\,\frac{\partial}{\partial r}(r^2A_r) + \frac{1}{r\sin\theta}\,\frac{\partial}{\partial\theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\,\frac{\partial A_\phi}{\partial\phi}$$

The Curl:

$$\nabla\times\vec{A} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_\phi) - \frac{\partial A_\theta}{\partial\phi}\right]\vec{e}_r + \left[\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\right]\vec{e}_\theta + \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\phi) - \frac{\partial A_r}{\partial\theta}\right]\vec{e}_\phi$$

The Convective Operator:

$$
\begin{aligned}
(\vec{A}\cdot\nabla)\,\vec{B} &= \left(A_r\frac{\partial B_r}{\partial r}+\frac{A_\theta}{r}\frac{\partial B_r}{\partial \theta}+\frac{A_\phi}{r\sin\theta}\frac{\partial B_r}{\partial \phi}-\frac{A_\theta B_\theta + A_\phi B_\phi}{r}\right)\vec{e}_r \\
&+ \left(A_r\frac{\partial B_\theta}{\partial r}+\frac{A_\theta}{r}\frac{\partial B_\theta}{\partial \theta}+\frac{A_\phi}{r\sin\theta}\frac{\partial B_\theta}{\partial \phi}+\frac{A_\theta B_r}{r}-\frac{A_\phi B_\phi \cot\theta}{r}\right)\vec{e}_\theta \\
&+ \left(A_r\frac{\partial B_\phi}{\partial r}+\frac{A_\theta}{r}\frac{\partial B_\phi}{\partial \theta}+\frac{A_\phi}{r\sin\theta}\frac{\partial B_\phi}{\partial \phi}+\frac{A_\phi B_r}{r}+\frac{A_\phi B_\theta \cot\theta}{r}\right)\vec{e}_\phi
\end{aligned}
$$

The Laplacian:

$$
\text{scalar}: \quad \nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \psi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial \phi}{\partial \theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial \psi^2}
$$

$$
\begin{aligned}
\text{vector}: \quad \nabla^2\vec{F} &= \left(\nabla^2 F_r-\frac{2F_r}{r^2}-\frac{2}{r^2\sin\theta}\frac{\partial(F_\theta\sin\theta)}{\partial\theta}-\frac{2}{r^2\sin\theta}\frac{\partial F_\phi}{\partial\phi}\right)\vec{e}_r \\
&+ \left(\nabla^2 F_\theta+\frac{2}{r^2}\frac{\partial F_r}{\partial\theta}-\frac{F_\theta}{r^2\sin\theta}-\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial F_\phi}{\partial\phi}\right)\vec{e}_\theta \\
&+ \left(\nabla^2 F_\phi+\frac{2}{r^2\sin\theta}\frac{\partial F_r}{\partial\phi}+\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial F_\theta}{\partial\phi}-\frac{F_\phi}{r^2\sin^2\theta}\right)\vec{e}_\phi
\end{aligned}
$$

Appendix E

Legendre Transforms

Consider a function of only two variables, $f(x, y)$, so that the total derivative has the form

$$\mathrm{d}f = u\,\mathrm{d}x + v\,\mathrm{d}y$$

where

$$u = \frac{\partial f}{\partial x}\,, \qquad v = \frac{\partial f}{\partial y}$$

Suppose we want to transition to a new function $g = g(u, y)$. If we define

$$g = ux - f$$

then we have that

$$\mathrm{d}g = u\,\mathrm{d}x + x\,\mathrm{d}u - \mathrm{d}f = x\,\mathrm{d}u - v\,\mathrm{d}y$$

which is exactly the form for a differential of a function $g = g(u, y)$. The quantities x and v are now functions of the variables u and y given by the relations

$$x = \frac{\partial g}{\partial u}\,, \qquad v = -\frac{\partial g}{\partial y}$$

We say that the **Legendre transformation**

$$f \to g = \frac{\partial f}{\partial x}\,x - f$$

transforms a function $f = f(x, y)$ to a new function $g = g(u, y)$ where $u = \frac{\partial f}{\partial x}$. The key here is that no information has been lost. After all, we can always use the inverse Legendre transform

$$g \to f = \frac{\partial g}{\partial u}\,u - g$$

to transform $g(u, y)$ back to the original $f(x, y)$.

In physics, Legendre transformations are used to convert functions of a particular quantity into a function of its **conjugate quantity**. The product of two quantities

that are conjugate has units of energy, or sometime power. Examples are velocity which is conjugate to momentum, pressure which is conjugate to volume, or temperature which is conjugate to entropy.

As an explicit example encountered in Chapter 3, we can convert from the Lagrangian $\mathcal{L}(q_i, \dot{q}_i, t)$ to a new function, the Hamiltonian $\mathcal{H}(q_i, p_i, t)$, where $p_i = \partial\mathcal{L}/\partial\dot{q}_i$ is the conjugate variable to $\dot{q}_i$ (i.e., the generalized momentum p is conjugate to the generalized velocity $\dot{q}$). Using the relations above, we immediately see that

$$\mathcal{H} = \dot{q}_i \, p_i - \mathcal{L}$$

which indeed is the expression that defines the Hamiltonian.

Other examples of the use of Legendre transforms in physics are in thermodynamics. In thermodynamics one encounters a variety of thermodynamic potentials, which have units of energy and are always expressed as functions of two variables. Examples are the energy $U = U(S, V)$, the enthalpy $H = H(S, P)$, and the Gibbs free energy $G = G(T, V)$, among others. These thermodynamic potentials are related to each other via Legendre transformations. For example, since V and P are conjugate to each other (their product has units of energy), $U(S, V)$ and $H(S, P)$ are related via a Legendre transformation

$$H = U + P\,V$$

To see this, use the first law of thermodynamics, according to which

$$\mathrm{d}U = T\,\mathrm{d}S - P\,\mathrm{d}V$$

which implies that

$$T = \frac{\partial U}{\partial S}\,, \qquad P = -\frac{\partial U}{\partial V}$$

Along similar lines, we have that

$$G = H - T\,S$$

For details, see any good textbook on thermodynamics.

Appendix F

Differential Equations

The equations of fluid dynamics are all differential equations. In order to provide the necessary background, this appendix gives a very brief overview of the basics.

A differential equation is an **ordinary differential equation (ODE)** if the unknown function depends on only 1 independent variable.

If the unknown function depends on two or more independent variables, then the differential equation is a **partial differential equation (PDE)**.

The **order** of a differential equation is that of the highest derivative appearing in the equation.

Consider the following examples:

$$\begin{aligned} [\mathbf{a}] \quad & \tfrac{\mathrm{d}u}{\mathrm{d}x} = 2x^2 - 4 \\ [\mathbf{b}] \quad & \mathrm{e}^u \tfrac{\mathrm{d}^2u}{\mathrm{d}x^2} + 3\left(\tfrac{\mathrm{d}u}{\mathrm{d}x}\right)^4 = 2 \\ [\mathbf{c}] \quad & \tfrac{\partial^2 u}{\partial t^2} - 4\tfrac{\partial^2 u}{\partial x^2} = 0 \end{aligned}$$

Equation [**a**] is an ODE of order 1, equation [**b**] is an ODE of order 2, and equation [**c**] is a PDE of order 2.

In what follows we shall often use the following **shorthand notation:**

$$u' \equiv \frac{\mathrm{d}u}{\mathrm{d}x}\,, \qquad u'' \equiv \frac{\mathrm{d}^2u}{\mathrm{d}x^2}\,, \qquad u^{(n)} \equiv \frac{\mathrm{d}^n u}{\mathrm{d}x^n}\,.$$

When the independent variable is time, we often use a dot rather than a hyphen, i.e., $\dot{u} = \mathrm{d}u/\mathrm{d}t$, $\ddot{u} = \mathrm{d}^2u/\mathrm{d}t^2$, etc.

When dealing with PDEs, we use the following shorthand:

$$u_{,x} \equiv \frac{\partial u}{\partial x}\,, \qquad u_{,xy} \equiv \frac{\partial^2 u}{\partial x \partial y}\,, \qquad u_{,tt} \equiv \frac{\partial^2 u}{\partial^2 t}\,,$$

etc. Consider the following examples

$$\begin{aligned} \nabla^2 u = 0 \quad &\leftrightarrow \quad u_{,xx} + u_{,yy} + u_{,zz} = 0 \\ \nabla(\nabla \cdot u) + \nabla^2 u + u = 0 \quad &\leftrightarrow \quad u_{k,ki} + u_{i,jj} + u_i = 0 \end{aligned}$$

Note that in the latter we have adopted the Einstein summation convention.

A differential equation along with subsidiary conditions on the unknown function and its derivatives, all given at the same value of the independent variable, consistitute an **initial value problem**.

If the subsidiary conditions are given at more than one value of the independent variable, the problem is a **boundary value problem** and the conditions are **boundary conditions**.

There are three broad classes of boundary conditions:

- **Dirichlet boundary conditions:** The value of the dependent variable is specified on the boundary.
- **Neumann boundary conditions:** The normal derivative of the dependent variable is specified on the boundary.
- **Cauchy boundary conditions:** Both the value and the normal derivative of the dependent variable are specified on the boundary.

Cauchy boundary conditions are analogous to the initial conditions for a second-order ODE. These are given at one end of the interval only.

Linear and non-linear PDEs: A linear PDE is one that is of first degree in all of its field variables and partial derivatives.

Consider the following examples:

$$\begin{aligned}
&[\mathbf{a}] \quad \tfrac{\partial u}{\partial x} + \tfrac{\partial u}{\partial y} = 0 \\
&[\mathbf{b}] \quad \tfrac{\partial u}{\partial x} + \left(\tfrac{\partial u}{\partial y}\right)^2 = 0 \\
&[\mathbf{c}] \quad \tfrac{\partial^2 u}{\partial x^2} + \tfrac{\partial^2 u}{\partial y^2} = x + y \\
&[\mathbf{d}] \quad \tfrac{\partial u}{\partial x} + \tfrac{\partial u}{\partial y} = u^2 \\
&[\mathbf{e}] \quad \tfrac{\partial^2 u}{\partial x^2} + u\,\tfrac{\partial^2 u}{\partial y^2} = 0
\end{aligned}$$

Equations **[a]** and **[c]** are linear, while **[b]**, **[d]** and **[e]** are all non-linear.

We can write the above equations in **operator notation** as:

$$\begin{aligned}
&[\mathbf{a}] \quad L(u) = 0 \quad \text{with} \quad L := \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \\
&[\mathbf{b}] \quad L(u) = 0 \quad \text{with} \quad L := \frac{\partial}{\partial x} + \left(\frac{\partial}{\partial y}\right)^2 \\
&[\mathbf{c}] \quad L(u) = x + y \quad \text{with} \quad L := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \\
&[\mathbf{d}] \quad L(u) = 0 \quad \text{with} \quad L := \frac{\partial}{\partial x} + \frac{\partial}{\partial y} - u^2 \\
&[\mathbf{e}] \quad L(u) = 0 \quad \text{with} \quad L := \frac{\partial^2}{\partial x^2} + u\,\frac{\partial^2}{\partial y^2} = 0
\end{aligned}$$

Homogeneous and non-homogeneous PDEs: Let L be a **linear** operator. Then, a linear PDE can be written in the form

$$L(u) = f(x_1, x_2, x_3, ..., x_n, t)$$

The PDE is said to be homogeneous iff $f(x_1, x_2, x_3, ..., x_n, t) = 0$. Thus, in the examples above, equation **[a]** is homogeneous, while **[c]** is non-homogeneous (aka inhomogeneous).

In (hydro-)dynamics, we typically encounter three types of second-order PDEs, classified as **elliptic**, **hyperbolic**, and **parabolic**. Each type has certain characteristics that help determine if a particular finite element approach is appropriate to the problem being described by the PDE. Interestingly, just knowing the type of PDE can give us insight into how smooth the solution is, how fast information propagates, and the effect of initial and boundary conditions.

Consider a second-order PDE for the unknown function $u(x, y)$ of the form

$$a\,u_{,xx} + b\,u_{,xy} + c\,u_{,yy} + d\,u_{,x} + e\,u_{,y} + f\,u + g = 0$$

where each of a, b,...,g are allowed to be functions of x and/or y.

Elliptic: The above PDE is called elliptic if $b^2 - 4ac < 0$.
An example is the 2D **Poisson equation** $u_{,xx} + u_{,yy} = f$ (which has $a = c = 1$ and $b = 0$). The solutions of elliptic PDEs are always smooth, and boundary data at any point affect the solution at all points in the domain. There is no temporal propagation, yet elliptic PDEs convey the effect of objects on each other. Newtonian mechanics is an example of this, which is why the Poisson equation is elliptic.

Parabolic: The above PDE is called parabolic if $b^2 - 4ac = 0$.
An example is the **heat equation** $u_{,t} = u_{,xx}$ (which has $a = 1$ and $b = c = 0$) which describes heat flow in a 1D system. Parabolic PDEs are usually time dependent and represent diffusion-like processes (i.e., dissipation, convection). Solutions are smooth in space but may possess singularities.

Hyperbolic: The above PDE is called hyperbolic if $b^2 - 4ac > 0$.
An example is the **wave equation** $u_{,xx} - \frac{1}{c_s^2}\,u_{,tt} = f$ (which has $b = 0$, $a = 1$ and $c = -1/c_s^2 < 0$). In a system modeled with a hyperbolic PDE, information travels at a finite speed referred to as the wavespeed (c_s in the example here). Information is not transmitted until the wave arrives. The smoothness of the solution to a hyperbolic PDE depends on the smoothness of the initial and boundary conditions. For instance, if there is a jump in the data at the start or at the boundaries, then the jump will propagate as a shock in the solution. If, in addition, the PDE is nonlinear, then shocks may develop even though the initial conditions and the boundary conditions are smooth.

Finally, since solving PDEs can often be reduced to solving (sets) of ODEs, a few words about solving the latter. Problems involving ODEs can always be reduced to a set of first-order ODEs! For example, the 2nd order ODE

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + s(x)\,\frac{\mathrm{d}u}{\mathrm{d}x} = t(x)$$

can be rewritten as two first-order ODEs

$$\frac{\mathrm{d}u}{\mathrm{d}x} = v(x)\,, \qquad \frac{\mathrm{d}v}{\mathrm{d}x} = t(x) - s(x)v(x)$$

Consider the general n^{th}-order initial value problem

$$\frac{\mathrm{d}^nu}{\mathrm{d}x^n} = a_{n-1}(x)\,\frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}} + \ldots\ldots + a_1(x)\,\frac{\mathrm{d}u}{\mathrm{d}x} + a_0(x)\,u(x) + f(x)$$

with $u(0) = c_0$, $u'(0) = c_1$, $u''(0) = c_2$, ... ,$u^{(n-1)}(0) = c_{n-1}$ as the initial values. In general, this can be written in matrix form as

$$\mathbf{u}' = \mathbf{A}(x)\,\mathbf{u}(x) + \mathbf{f}(x)$$

with the initial values given by $\mathbf{u}(0) = \mathbf{c}$. Here the elements of $\mathbf{u}$ are given by $u_1 = u(x)$, $u_2 = u'(x)$, ..., $u_n = u^{(n-1)}(x)$. Theses are interrelated with the elements of $\mathbf{u}'$ by the equations $u_1' = u_2$, $u_2' = u_3$, ... ,$u_{n-1}' = u_n$, $u_n' = u^{(n)}(x)$. The matrices $\mathbf{A}(x)$ and $\mathbf{f}(x)$ are related to $a_i(x)$ and $f(x)$ according to

$$\mathbf{A}(x) = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ a_0(x) & a_1(x) & a_2(x) & a_3(x) & \cdots & a_{n-1}(x) \end{pmatrix}$$

and

$$\mathbf{f}(x) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ f(x) \end{pmatrix}$$

Hence, solving an ODE of order N reduces to solving a set of N coupled first-order differential equations for the functions u_i $(i = 1, 2, ..., N)$ having the general form

$$\frac{\mathrm{d}u_i}{\mathrm{d}x} = f_i(x, u_1, u_2, ..., u_n)$$

where the functions f_i on the rhs are known.

Appendix G

The Levi-Civita Symbol

The **Levi-Civita symbol**, also known as the **permutation symbol** or the **anti-symmetric symbol**,is a collection of numbers, defined from the sign of a permutation of the natural numbers $1, 2, 3, ..., n$. It is often encountered in linear algebra, vector and tensor calculus, and differential geometry.

The n-dimensional Levi-Civita symbol is indicated by $\varepsilon_{i_1 i_2 ... i_n}$, where each index $i_1, i_2, ..., i_n$ takes values $1, 2, ..., n$, ans has the defining property that the symbol is total antisymmetric in all its indices: when any two indices are interchanged, the symbol is negated:

$$\varepsilon_{...i_p...i_q...} = -\varepsilon_{...i_q...i_p...}$$

If any two indices are equal, the symbol is zero, and when all indices are unequal, we have that

$$\varepsilon_{i_1 i_2 ... i_n} = (-1)^p \varepsilon_{1,2,...n}$$

where p is called the **parity of the permutation**. It is the number of pairwise interchanges necessary to unscramble $i_1, i_2, ..., i_n$ into the order $1, 2, ..., n$. A permutation is said to be even (odd) if its parity is an even (odd) number.

Example: what is the parity of $\{3, 4, 5, 2, 1\}$?

$$\{1, 2, 3, 4, 5\}$$
$$\{\mathbf{3}, 2, \mathbf{1}, 4, 5\}$$
$$\{3, \mathbf{4}, 1, \mathbf{2}, 5\}$$
$$\{3, 4, \mathbf{5}, 2, \mathbf{1}\}$$

Answer: $p = 3$, since three pairwise interchanges are required.

In three dimensions the Levi-Civita symbol is defined by

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{if } (i,j,k) \text{ is (1,2,3), (2,3,1), or (3,1,2)} \\ -1 & \text{if } (i,j,k) \text{ is (3,2,1), (1,3,2), or (2,1,3)} \\ 0 & \text{if } i = j, \text{ or } j = k, \text{ or } k = i \end{cases}$$

Appendix H

The BBGKY Hierarchy

In this Appendix we derive the BBGKY hierarchy of evolution equations for the k-particle distribution function $f^{(k)}(\vec{w}_1, \vec{w}_2, ...\vec{w}_k)$ starting from the **Liouville equation** for the N-particle distribution function $f^{(N)}(\vec{w}_1, \vec{w}_2, ...\vec{w}_N)$, where $N > k$. Here $\vec{w}_i \equiv (\vec{q}_i, \vec{p}_i)$ is the 6D phase-space vector of particle i,

As we have seen in Chapter 6, the Liouville equation, which is given by

$$\frac{\mathrm{d}f^{(N)}}{\mathrm{d}t} = \frac{\partial f^{(N)}}{\partial t} + \{f^{(N)}, \mathcal{H}^{(N)}\} = 0$$

expresses the incompressibilty of Hamiltonian flow in Γ-space. Here we have adopted the notation based on **Poisson brackets** (see Chapter 4), and we have used the index '(N)' on the Hamiltonian to emphasize that this is the N-particle Hamiltonian

$$\mathcal{H}^{(N)}(\vec{q}_i, \vec{p}_i) = \sum_{i=1}^{N} \frac{\vec{p}_i^{\,2}}{2m} + \sum_{i=1}^{N} V(\vec{q}_i) + \frac{1}{2}\sum_{i=1}^{N}\sum_{\substack{j=1 \\ j\neq i}}^{N} U_{ij}$$

Here $V(\vec{q})$ is the potential corresponding to an **external force**, and we have used U_{ij} as shorthand notation for

$$U_{ij} \equiv U(|\vec{q}_i - \vec{q}_j|)$$

the potential energy associated with the two-body interaction between particles i and j. Note that $U_{ij} = U_{ji}$. The factor $1/2$ in the above expression for the Hamiltonian is to correct for double-counting of the particle pairs.

We can relate the N-particle Hamiltonian, $\mathcal{H}^{(N)}$ to the k-particle Hamiltonian, $\mathcal{H}^{(k)}$, which is defined in the same way as $\mathcal{H}^{(N)}$ but with N replaced by $k < N$, according to

$$\mathcal{H}^{(N)} = \mathcal{H}^{(k)} + \mathcal{H}^{(k,N)} + \sum_{i=1}^{k}\sum_{j=k+1}^{N} U_{ij}$$

Here

$$\mathcal{H}^{(k)} = \sum_{i=1}^{k} \frac{\vec{p}_i^{\,2}}{2m} + \sum_{i=1}^{k} V(\vec{q}_i) + \frac{1}{2} \sum_{i=1}^{k} \sum_{\substack{j=1 \\ j \neq i}}^{k} U_{ij}$$

and

$$\mathcal{H}^{(k,N)} = \sum_{i=k+1}^{N} \frac{\vec{p}_i^{\,2}}{2m} + \sum_{i=k+1}^{N} V(\vec{q}_i) + \frac{1}{2} \sum_{i=k+1}^{N} \sum_{\substack{j=k+1 \\ j \neq i}}^{N} U_{ij}$$

To see this, consider the U_{ij} term, for which we can write

$$\begin{aligned} \sum_{i=1}^{N} \sum_{\substack{j=i \\ j \neq i}}^{N} U_{ij} &= \sum_{i=1}^{k} \sum_{\substack{j=1 \\ j \neq i}}^{N} U_{ij} + \sum_{i=k+1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} U_{ij} \\ &= \sum_{i=1}^{k} \sum_{\substack{j=1 \\ j \neq i}}^{k} U_{ij} + \sum_{i=1}^{k} \sum_{j=k+1}^{N} U_{ij} + \sum_{i=k+1}^{N} \sum_{j=1}^{k} U_{ij} + \sum_{i=k+1}^{N} \sum_{\substack{j=k+1 \\ j \neq i}}^{N} U_{ij} \end{aligned}$$

The second and third terms are identical (since $U_{ij} = U_{ji}$) so that upon substitution we obtain the above relation between $\mathcal{H}^{(N)}$ and $\mathcal{H}^{(k)}$.

Now let's take the **Liouville equation** and integrate it over the entire phase-space of particles $k+1$ to N:

$$\int \prod_{n=k+1}^{N} \mathrm{d}^3\vec{q}_n \, \mathrm{d}^3\vec{p}_n \, \frac{\partial f^{(N)}}{\partial t} = \int \prod_{n=k+1}^{N} \mathrm{d}^3\vec{q}_n \, \mathrm{d}^3\vec{p}_n \, \{\mathcal{H}^{(N)}, f^{(N)}\}$$

First the LHS: using that the integration is independent of time, we take the time derivative outside of the integral, yielding

$$\frac{\partial}{\partial t} \int \prod_{n=k+1}^{N} \mathrm{d}^3\vec{q}_n \, \mathrm{d}^3\vec{p}_n \, f^{(N)} = \frac{(N-k)!}{N!} \, \frac{\partial f^{(k)}}{\partial t}$$

where we have made use of the definition of the **reduced k-particle distribution function** (see Chapter 6). Writing out the Poisson brackets in the RHS, and splitting

the summation over i in two parts, we can write the RHS as the sum of two integrals, $\mathcal{I}_1$ plus $\mathcal{I}_2$, where

$$\mathcal{I}_1 = \int \prod_{n=k+1}^{N} \mathrm{d}^3\vec{q}_n \, \mathrm{d}^3\vec{p}_n \sum_{i=1}^{k} \left(\frac{\partial \mathcal{H}^{(N)}}{\partial \vec{q}_i} \cdot \frac{\partial f^{(N)}}{\partial \vec{p}_i} - \frac{\partial \mathcal{H}^{(N)}}{\partial \vec{p}_i} \cdot \frac{\partial f^{(N)}}{\partial \vec{q}_i} \right)$$

and

$$\mathcal{I}_2 = \int \prod_{n=k+1}^{N} \mathrm{d}^3\vec{q}_n \, \mathrm{d}^3\vec{p}_n \sum_{i=k+1}^{N} \left(\frac{\partial \mathcal{H}^{(N)}}{\partial \vec{q}_i} \cdot \frac{\partial f^{(N)}}{\partial \vec{p}_i} - \frac{\partial \mathcal{H}^{(N)}}{\partial \vec{p}_i} \cdot \frac{\partial f^{(N)}}{\partial \vec{q}_i} \right)$$

Integral $\mathcal{I}_2$ vanishes. To see this, realize that $\partial \mathcal{H}^{(N)}/\partial \vec{q}_i$ is independent of $\vec{p}_i$ and $\partial \mathcal{H}^{(N)}/\partial \vec{p}_i$ is independent of $\vec{q}_i$ (this follows from the definition of the Hamiltonian). Because of this, each terms in $\mathcal{I}_2$ can be cast in the form

$$\int_{-\infty}^{+\infty} \mathrm{d}x \int_{-\infty}^{+\infty} \mathrm{d}y \, g(x) \, \frac{\partial f(x,y)}{\partial y} = \int_{-\infty}^{+\infty} \mathrm{d}x \, g(x) \, \left[f(x,y) \right]_{y=-\infty}^{y=+\infty}$$

i.e., these turn into surface integrals, and since $f^{(N)}(\vec{q}_i, \vec{p}_i) = 0$ in the limits $|\vec{q}_i| \to \infty$ (systems are of finite extent) and $|\vec{p}_i| \to \infty$ (particles have finite speed), we see that $\mathcal{I}_2$ must vanish.

In order to compute $\mathcal{I}_1$, we first write $\mathcal{H}^{(N)}$ in terms of $\mathcal{H}^{(k)}$ and $\mathcal{H}^{(k,N)}$ as indicated above. This allows us split the result in three terms, $\mathcal{I}_{1a}$, $\mathcal{I}_{1b}$ and $\mathcal{I}_{1c}$ given by

$$\mathcal{I}_{1a} = \int \prod_{n=k+1}^{N} \mathrm{d}^3\vec{q}_n \, \mathrm{d}^3\vec{p}_n \sum_{i=1}^{k} \left(\frac{\partial \mathcal{H}^{(k)}}{\partial \vec{q}_i} \cdot \frac{\partial f^{(N)}}{\partial \vec{p}_i} - \frac{\partial \mathcal{H}^{(k)}}{\partial \vec{p}_i} \cdot \frac{\partial f^{(N)}}{\partial \vec{q}_i} \right),$$

$$\mathcal{I}_{1b} = \int \prod_{n=k+1}^{N} \mathrm{d}^3\vec{q}_n \, \mathrm{d}^3\vec{p}_n \sum_{i=1}^{k} \left(\frac{\partial \mathcal{H}^{(k,N)}}{\partial \vec{q}_i} \cdot \frac{\partial f^{(N)}}{\partial \vec{p}_i} - \frac{\partial \mathcal{H}^{(k,N)}}{\partial \vec{p}_i} \cdot \frac{\partial f^{(N)}}{\partial \vec{q}_i} \right),$$

and

$$\mathcal{I}_{1c} = \int \prod_{n=k+1}^{N} \mathrm{d}^3\vec{q}_n \, \mathrm{d}^3\vec{p}_n \sum_{i=1}^{k} \left(\frac{\partial}{\partial \vec{q}_i} \left[\sum_{l=1}^{k} \sum_{j=k+1}^{N} U_{kl} \right] \cdot \frac{\partial f^{(N)}}{\partial \vec{p}_i} - \frac{\partial}{\partial \vec{p}_i} \left[\sum_{l=1}^{k} \sum_{j=k+1}^{N} U_{kl} \right] \cdot \frac{\partial f^{(N)}}{\partial \vec{q}_i} \right)$$

We now examine each of these in turn. Starting with $\mathcal{I}_{1a}$, for which we realize that the operator for $f^{(N)}$ is independent of the integration variables, such that we can take it outside of the integral. Hence, we have that

$$\mathcal{I}_{1a} = \sum_{i=1}^{k} \left(\frac{\partial \mathcal{H}^{(k)}}{\partial \vec{q}_i} \cdot \frac{\partial}{\partial \vec{p}_i} - \frac{\partial \mathcal{H}^{(k)}}{\partial \vec{p}_i} \cdot \frac{\partial}{\partial \vec{q}_i} \right) \int \prod_{n=k+1}^{N} \mathrm{d}^3 \vec{q}_n \, \mathrm{d}^3 \vec{p}_n \, f^{(N)}$$

Using the definition of the reduced k-particle distribution function, this can be written as

$$\mathcal{I}_{1a} = \frac{(N-k)!}{N!} \sum_{i=1}^{k} \left(\frac{\partial \mathcal{H}^{(k)}}{\partial \vec{q}_i} \cdot \frac{\partial f^{(k)}}{\partial \vec{p}_i} - \frac{\partial \mathcal{H}^{(k)}}{\partial \vec{p}_i} \cdot \frac{\partial f^{(k)}}{\partial \vec{q}_i} \right) = \frac{(N-k)!}{N!} \, \{\mathcal{H}^{(k)}, f^{(k)}\}$$

where we have made use of the definition of the Poisson brackets (see Chapter 4).

Next up is $\mathcal{I}_{1b}$. It is clear that this integral must vanish, since both $\partial \mathcal{H}^{(k,N)}/\partial \vec{q}_i$ and $\partial \mathcal{H}^{(k,N)}/\partial \vec{p}_i$ are equal to zero. After all, the index i runs from 1 to k, and the phase-space coordinates of those particles do not appear in $\mathcal{H}^{(k,N)}$. This leaves $\mathcal{I}_{1c}$; since U_{kl} is independent of momentum, the second term within the brackets vanishes, leaving only

$$\mathcal{I}_{1c} = \int \prod_{n=k+1}^{N} \mathrm{d}^3 \vec{q}_n \, \mathrm{d}^3 \vec{p}_n \sum_{i=1}^{k} \left(\sum_{j=k+1}^{N} \frac{\partial U_{ij}}{\partial \vec{q}_i} \cdot \frac{\partial f^{(N)}}{\partial \vec{p}_i} \right)$$

Upon inspection, you can see that each term of the j-summation is equal (this follows from the fact that we integrate over all of $\vec{q}_j$ for each $j = k+1, ..., N$). Hence, since there are $N-k$ such terms we have that

$$
\begin{aligned}
\mathcal{I}_{1c} &= (N-k)\sum_{i=1}^{k}\int\prod_{n=k+1}^{N}\mathrm{d}^3\vec{q}_n\,\mathrm{d}^3\vec{p}_n\left(\frac{\partial U_{i,k+1}}{\partial\vec{q}_i}\cdot\frac{\partial f^{(N)}}{\partial\vec{p}_i}\right)\\
&= (N-k)\sum_{i=1}^{k}\int\mathrm{d}^3\vec{q}_{k+1}\,\mathrm{d}^3\vec{p}_{k+1}\left(\frac{\partial U_{i,k+1}}{\partial\vec{q}_i}\cdot\frac{\partial}{\partial\vec{p}_i}\right)\int\prod_{n=k+2}^{N}\mathrm{d}^3\vec{q}_n\,\mathrm{d}^3\vec{p}_n\,f^{(N)}\\
&= (N-k)\,\frac{(N-k-1)!}{N!}\sum_{i=1}^{k}\int\mathrm{d}^3\vec{q}_{k+1}\,\mathrm{d}^3\vec{p}_{k+1}\left(\frac{\partial U_{i,k+1}}{\partial\vec{q}_i}\cdot\frac{\partial f^{(k+1)}}{\partial\vec{p}_i}\right)\\
&= \frac{(N-k)!}{N!}\sum_{i=1}^{k}\int\mathrm{d}^3\vec{q}_{k+1}\,\mathrm{d}^3\vec{p}_{k+1}\left(\frac{\partial U_{i,k+1}}{\partial\vec{q}_i}\cdot\frac{\partial f^{(k+1)}}{\partial\vec{p}_i}\right)
\end{aligned}
$$

where as before we have taken the operator outside of the integral, and we have used the definition of the reduced distribution functions.

Combining everything, we obtain our final expression for the evolution of the reduced k-particle distribution function

$$
\boxed{\frac{\partial f^{(k)}}{\partial t} = \{\mathcal{H}^{(k)}, f^{(k)}\} + \sum_{i=1}^{k}\int\mathrm{d}^3\vec{q}_{k+1}\,\mathrm{d}^3\vec{p}_{k+1}\left(\frac{\partial U_{i,k+1}}{\partial\vec{q}_i}\cdot\frac{\partial f^{(k+1)}}{\partial\vec{p}_i}\right)}
$$

Note that the evolution of $f^{(k)}$ thus depends on $f^{(k+1)}$, such that the above expression represents a set of N coupled equations, known as the **BBGKY hierarchy**. Note also that this derivation is completely general; the ONLY assumption we have made along the way is that the dynamics are Hamiltonian!

For the **1-particle distribution function** the above expression reduces to

$$
\frac{\partial f^{(1)}}{\partial t} = \{\mathcal{H}^{(1)}, f^{(1)}\} + \int\mathrm{d}^3\vec{q}_2\,\mathrm{d}^3\vec{p}_2\left(\frac{\partial U_{12}}{\partial\vec{q}_1}\cdot\frac{\partial f^{(2)}}{\partial\vec{p}_1}\right)
$$

with $\mathcal{H}^{(1)} = \frac{p^2}{2m} + V(\vec{q})$ the 1-particle Hamiltonian and $f^{(2)} = f^{(2)}(\vec{q}_1, \vec{q}_2, \vec{p}_1, \vec{p}_2, t)$ the **2-particle distribution function**. This equation forms the basis for the **Boltzmann equation**, as discussed in Chapters 6 and 7.

Appendix I

Equations of State

Equation of State (EoS): a thermodynamic equation describing the state of matter under a given set of physical conditions. In what follows we will always write our EoS in the form $P = P(\rho, T)$. Other commonly used forms are $P = P(\rho, \varepsilon)$ or $P = P(\rho, S)$.

Ideal Gas: a hypothetical gas that consists of identical point particles (i.e. of zero volume) that undergo perfectly elastic collisions and for which interparticle forces can be neglected.

An ideal gas obeys the **ideal gas law:** $P\,V = N\,k_{\rm B}\,T$.

Here N is the total number of particles, $k_{\rm B}$ is Boltzmann's constant, and V is the volume occupied by the fluid. Using that $\rho = N\,\mu m_{\rm p}/V$, where μ is the **mean molecular weight** in units of the proton mass $m_{\rm p}$, we have that the **EoS for an ideal gas** is given by

$$P = \frac{k_{\rm B}\,T}{\mu\,m_{\rm p}}\,\rho$$

`NOTE:` astrophysical gases are often well described by the ideal gas law. Even for a fully ionized gas, the interparticle forces (Coulomb force) can typically be neglected (i.e., the potential energies involved are typically $< 10\%$ of the kinetic energies). Ideal gas law breaks down for dense, and cool gases, such as those present in gaseous planets.

Maxwell-Boltzmann Distribution: the distribution of particle momenta, $\vec{p} = m\vec{v}$, of an ideal gas follows the Maxwell-Boltzmann distribution.

$$\mathcal{P}(\vec{p})\,\mathrm{d}^3\vec{p} = \left(\frac{1}{2\pi m k_\mathrm{B} T}\right)^{3/2} \exp\left(-\frac{p^2}{2m k_\mathrm{B} T}\right)\,\mathrm{d}^3\vec{p}$$

where $p^2 = \vec{p}\cdot\vec{p}$. This distribution follows from maximizing entropy under the following assumptions:

1. all magnitudes of velocity are *a priori* equally likely
2. all directions are equally likely (isotropy)
3. total energy is constrained at a fixed value
4. total number of particles is constrained at a fixed value

Using that $E = p^2/2m$ we thus see that $\mathcal{P}(\vec{p}) \propto e^{-E/k_\mathrm{B}T}$.

Pressure: pressure arises from (elastic) collisions of particles. A particle hitting a wall head on with momentum $p = mv$ results in a transfer of momentum to the wall of $2mv$. Using this concept, and assuming isotropy for the particle momenta, it can be shown that

$$\boxed{P = \zeta\, n\, \langle E\rangle}$$

where $\zeta = 2/3$ ($\zeta = 1/3$) in the case of a non-relativistic (relativistic) fluid, and

$$\langle E\rangle = \int_0^\infty E\,\mathcal{P}(E)\,\mathrm{d}E$$

is the average, translational energy of the particles. In the case of our ideal (non-relativistic) fluid,

$$\langle E\rangle = \left\langle \frac{p^2}{2m}\right\rangle = \int_0^\infty \frac{p^2}{2m}\,\mathcal{P}(p)\,\mathrm{d}p = \frac{3}{2}k_\mathrm{B}T$$

Hence, we find that the **EoS for an ideal gas** is indeed given by

$$\boxed{P = \frac{2}{3}\, n\, \langle E \rangle = n\, k_{\rm B}\, T = \frac{k_{\rm B} T}{\mu m_{\rm p}}\, \rho}$$

Specific Internal Energy: the internal energy per unit mass for an ideal gas is

$$\varepsilon = \frac{\langle E \rangle}{\mu m_{\rm p}} = \frac{3}{2} \frac{k_{\rm B} T}{\mu m_{\rm p}}$$

Actually, the above derivation is only valid for a true 'ideal gas', in which the particles are point particles. More generally,

$$\boxed{\varepsilon = \frac{1}{\gamma - 1} \frac{k_{\rm B} T}{\mu m_{\rm p}}}$$

where γ is the **adiabatic index**, which for an ideal gas is equal to $\gamma = (q+5)/(q+3)$, with q the internal degrees of freedom of the fluid particles: $q = 0$ for point particles (resulting in $\gamma = 5/3$), while diatomic particles have $q = 2$ (at sufficiently low temperatures, such that they only have rotational, and no vibrational degrees of freedom). The fact that $q = 2$ in that case arises from the fact that a diatomic molecule only has two relevant rotation axes; the third axis is the symmetry axis of the molecule, along which the molecule has negligible (zero in case of point particles) moment of inertia. Consequently, rotation around this symmetry axis carries no energy.

Photon gas: Having discussed the EoS of an ideal gas, we now focus on a gas of photons. Photons have energy $E = h\nu$ and momentum $p = E/c = h\nu/c$, with h the Planck constant.

Black Body: an idealized physical body that absorbs all incident radiation. A black body (BB) in thermal equilibrium emits electro-magnetic radiation called **black body radiation**.

The **spectral number density distribution** of BB photons is given by

$$n_\gamma(\nu, T) = \frac{8\pi\nu^2}{c^3} \frac{1}{e^{h\nu/k_{\rm B}T} - 1}$$

which implies a **spectral energy distribution**

$$u(\nu, T) = n_\gamma(\nu, T)\, h\nu = \frac{8\pi h\nu^3}{c^3}\, \frac{1}{e^{h\nu/k_{\rm B}T} - 1}$$

and thus an **energy density** of

$$u(T) = \int_0^\infty u(\nu, T)\, {\rm d}\nu = \frac{4\sigma_{\rm SB}}{c}\, T^4 \equiv a_{\rm r}\, T^4$$

where

$$\sigma_{\rm SB} = \frac{2\pi^5 k_{\rm B}^4}{15 h^3 c^2}$$

is the **Stefan-Boltzmann constant** and $a_{\rm r} \simeq 7.6 \times 10^{-15} {\rm erg\, cm^{-3}\, K^{-4}}$ is called the **radiation constant**.

Radiation Pressure: when the photons are reflected off a wall, or when they are absorbed and subsequently re-emitted by that wall, they transfer twice their momentum in the normal direction to that wall. Since photons are relativistic, we have that **the EoS for a photon gas** is given by

$$\boxed{P = \frac{1}{3}\, n\, \langle E \rangle = \frac{1}{3}\, n_\gamma\, \langle h\nu \rangle = \frac{1}{3} u(T) = \frac{aT^4}{3}}$$

where we have used that $u(T) = n_\gamma \langle E \rangle$.

Quantum Statistics: according to quantum statistics, a collection of many indistinguishable elementary particles in **thermal equilibrium** has a momentum distribution given by

$$f(\vec{p})\, {\rm d}^3\vec{p} = \frac{g}{h^3} \left[\exp\left(\frac{E(p) - \mu}{k_{\rm B}T} \right) \pm 1 \right]^{-1} {\rm d}^3\vec{p}$$

where the signature $\pm$ takes the positive sign for fermions (which have half-integer spin), in which case the distribution is called the **Fermi-Dirac distribution**, and the negative sign for bosons (particles with zero or integer spin), in which case the distribution is called the **Bose-Einstein distribution**. The factor g is the **spin degeneracy factor**, which expresses the number of spin states the particles can have ($g = 1$ for neutrinos, $g = 2$ for photons and charged leptons, and $g = 6$

for quarks). Finally, μ is called the **chemical potential**, and is a form of potential energy that is related (in a complicated way) to the number density and temperature of the particles (see Appendix K).

Classical limit: In the limit where the mean interparticle separation is much larger than the de Broglie wavelength of the particles, so that quantum effects (e.g., Heisenberg's uncertainty principle) can be ignored, the above distribution function of momenta can be accurately approximated by the **Maxwell-Boltzmann distribution**.

Heisenberg's Uncertainty Principle: $\Delta x\ \Delta p_x > h$ (where $h = 6.63{\times}10^{-27}\mathrm{g\,cm^2\,s^{-1}}$ is Planck's constant). One interpretation of this quantum principle is that phase-space is quantized; no particle can be localized in a phase-space element smaller than the fundamental element

$$\Delta x\ \Delta y\ \Delta z\ \Delta p_x\ \Delta p_y\ \Delta p_z = h^3$$

Pauli Exclusion Principle: no more than one fermion of a given spin state can occupy a given phase-space element h^3. Hence, for electrons, which have $g = 2$, the maximum phase-space density is $2/h^3$.

Degeneracy: When compressing and/or cooling a fermionic gas, at some point all possible low momentum states are occupied. Any further compression therefore results in particles occupying high (but the lowest available) momentum states. Since particle momentum is ultimately responsible for pressure, this degeneracy manifests itself as an extremely high pressure, known as **degeneracy pressure**.

Fermi Momentum: Consider a **fully degenerate** gas of electrons of electron density n_e. It will have fully occupied the part of phase-space with momenta $p \leq p_{\rm F}$. Here $p_{\rm F}$ is the maximum momentum of the particles, and is called the **Fermi momentum**. The energy corresponding to the Fermi momentum is called the **Fermi energy**, $E_{\rm F}$ and is equal to $p_{\rm F}^2/2m$ in the case of a non-relativistic gas, and $p_{\rm F}c$ in the case of a relativistic gas.

Let V_x be the volume occupied in configuration space, and $V_p = \frac{4}{3}\pi p_{\rm F}^3$ the volume occupied in momentum space. If the total number of particles is N, and the gas is

fully degenerate, then

$$V_x\, V_p = \frac{N}{2} h^3$$

Using that $n_e = N/V_x$, we find that

$$p_{\rm F} = \left(\frac{3}{8\pi} n_e\right)^{1/3} h$$

EoS of Non-Relativistic, Degenerate Gas: Using the information above, it is relatively straightforward (see Problem Set 4) to compute the EoS for a fully degenerate gas. Using that for a non-relativistic fluid $E = p^2/2m$ and $P = \frac{2}{3}\, n\, \langle E \rangle$, while degeneracy implies that

$$\langle E \rangle = \frac{1}{N} \int_0^{E_{\rm f}} E\, N(E)\, {\rm d}E = \frac{1}{N} \int_0^{p_{\rm F}} \frac{p^2}{2m}\, \frac{2}{h^3}\, V_x\, 4\pi p^2\, {\rm d}p = \frac{3}{5}\, \frac{p_{\rm F}^2}{2m}$$

we obtain that

$$P = \frac{1}{20} \left(\frac{3}{\pi}\right)^{2/3} \frac{h^2}{m^{8/3}}\, \rho^{5/3}$$

EoS of Relativistic, Degenerate Gas: In the case of a relativistic, degenerate gas, we use the same procedure as above. However, this time we have that $P = \frac{1}{3}\, n\, \langle E \rangle$ while $E = p\, c$, which results in

$$P = \frac{1}{8} \left(\frac{3}{\pi}\right)^{1/3} \frac{c\, h}{m^{4/3}}\, \rho^{4/3}$$

White Dwarfs and the Chandrasekhar limit: White dwarfs are the end-states of stars with mass low enough that they don't form a neutron star. When the pressure support from nuclear fusion in a star comes to a halt, the core will start to contract until degeneracy pressure kicks in. The star consists of a fully ionized plasma. Assume for simplicity that the plasma consists purely of hydrogen, so that the number density of protons is equal to that of electrons: $n_{\rm p} = n_{\rm e}$. Because of **equipartition**

$$\frac{p_{\rm p}^2}{2m_{\rm p}} = \frac{p_{\rm e}^2}{2m_{\rm e}}$$

Since $m_{\rm p} >> m_{\rm e}$ we have also that $p_{\rm p} >> p_{\rm e}$ (in fact $p_{\rm p}/p_{\rm e} = \sqrt{m_{\rm p}/m_{\rm e}} \simeq 43$). Consequently, when cooling or compressing the core of a star, the electrons will become degenerate well before the protons do. Hence, white dwarfs are held up against collapse by the **degeneracy pressure from electrons**. Since the electrons are typically non-relativistic, the EoS of the white dwarf is: $P \propto \rho^{5/3}$. If the white dwarf becomes more and more massive (i.e., because it is accreting mass from a companion star), the Pauli-exclusion principle causes the Fermi momentum, $p_{\rm F}$, to increase to relativistic values. This **softens** the EoS towards $P \propto \rho^{4/3}$. Such an equation of state is too soft to stabilize the white dwarf against gravitational collapse; the white dwarf collapses until it becomes a **neutron star**, at which stage it is supported against further collapse by the degeneracy pressure from neutrons. This happens when the mass of the white dwarf reaches $M_{\rm lim} \simeq 1.44 M_\odot$, the so-called **Chandrasekhar limit**.

	Non-Relativistic	**Relativistic**
non-degenerate	$P \propto \rho T$	$P \propto T^4$
degenerate	$P \propto \rho^{5/3}$	$P \propto \rho^{4/3}$

Summary of equations of state for different kind of fluids

Appendix J

Derivation of the Energy Equation

The energy equation can be obtained from the **master moment equation**

$$\boxed{\frac{\partial}{\partial t} n\langle Q\rangle + \frac{\partial}{\partial x_i}\Big[n\langle Q v_i\rangle\Big] + \frac{\partial \Phi}{\partial x_i} n\left\langle \frac{\partial Q}{\partial v_i}\right\rangle = 0}$$

by substituting

$$Q = \frac{1}{2}mv^2 = \frac{m}{2}v_i v_i = \frac{m}{2}(u_i + w_i)(u_i + w_i) = \frac{m}{2}(u^2 + 2u_i w_i + w^2)$$

Hence, we have that $\langle Q\rangle = \frac{1}{2}mu^2 + \frac{1}{2}m\langle w^2\rangle$ where we have used that $\langle u\rangle = u$ and $\langle w\rangle = 0$. Using that $\rho = m\,n$, the first term in the master moment equation thus becomes

$$\frac{\partial}{\partial t}\left[n\langle Q\rangle\right] = \frac{\partial}{\partial t}\left[\rho\frac{u^2}{2} + \rho\varepsilon\right]$$

where we have used that the specific internal energy $\varepsilon = \frac{1}{2}\langle w^2\rangle$. For the second term, we use that

$$\begin{aligned}
n\langle v_k Q\rangle &= \frac{\rho}{2}\langle (u_k + w_k)(u^2 + 2u_i w_i + w^2)\rangle \\
&= \frac{\rho}{2}\langle u^2 u_k + 2u_i u_k w_i + w^2 u_k + u^2 w_k + 2u_i w_i w_k + w^2 w_k\rangle \\
&= \frac{\rho}{2}\left[u^2 u_k + u_k\langle w^2\rangle + 2u_i\langle w_i w_k\rangle + \langle w^2 w_k\rangle\right] \\
&= \rho\frac{u^2}{2}u_k + \rho\varepsilon u_k + \rho u_i\langle w_i w_k\rangle + \rho\langle w_k\frac{1}{2}w^2\rangle
\end{aligned}$$

Using that $\rho\langle w_i w_k\rangle = -\sigma_{ik}$, the second term of the master moment equation becomes

$$\frac{\partial}{\partial x_k}\left[n\langle v_k Q\rangle\right] = \frac{\partial}{\partial x_k}\left[\rho\left(\frac{u^2}{2} + \varepsilon\right)u_k - \sigma_{ik}u_i + \rho\langle w_k\frac{1}{2}w^2\rangle\right]$$

Finally, for the third term we use that

$$\frac{\partial Q}{\partial v_k} = \frac{m}{2}\frac{\partial v^2}{\partial v_k} = m v_k$$

To understand the last step, note that in Cartesian coordinates $v^2 = v_x^2 + v_y^2 + v_z^2$. Hence, we have that

$$n\frac{\partial \Phi}{\partial x_k}\left\langle \frac{\partial Q}{\partial v_k} \right\rangle = \rho \frac{\partial \Phi}{\partial x_k}\langle v_k \rangle = \rho \frac{\partial \Phi}{\partial x_k} u_k$$

Combining the three terms in the master moment equation, we finally obtain the following form of the **energy equation**:

$$\frac{\partial}{\partial t}\left[\rho\left(\frac{u^2}{2} + \varepsilon\right)\right] = -\frac{\partial}{\partial x_k}\left[\rho\left(\frac{u^2}{2} + \varepsilon\right) u_k - \sigma_{jk} u_j + \rho\langle w_k \frac{1}{2} w^2 \rangle\right] - \rho\, u_k \frac{\partial \Phi}{\partial x_k}$$

Appendix K

The Chemical Potential

Consider a system which can exchange energy **and particles** with a reservoir, and the volume of which can change. There are three ways for this system to increase its internal energy; heating, changing the system's volume (i.e., doing work on the system), or adding particles. Hence,

$$\mathrm{d}U = T\,\mathrm{d}S - P\,\mathrm{d}V + \mu\,\mathrm{d}N$$

Note that this is the first law of thermodynamics, but now with the added possibility of changing the number of particles of the system. The scalar quantity μ is called the **chemical potential**, and is defined by

$$\mu = \left(\frac{\partial U}{\partial N}\right)_{S,V}$$

This is not to be confused with the μ used to denote the mean weight per particle, which ALWAYS appears in combination with the proton mass, m_{p}. As is evident from the above expression, the chemical potential quantifies how the internal energy of the system changes if particles are added or removed, while keeping the entropy and volume of the system fixed. The chemical potential appears in the Fermi-Dirac distribution describing the momentum distribution of a gas of fermions or bosons.

Consider an ideal gas, of volume V, entropy S and with internal energy U. Now imagine adding a particle of zero energy ($\epsilon = 0$), while keeping the volume fixed. Since $\epsilon = 0$, we also have that $\mathrm{d}U = 0$. But what about the entropy? Well, we have increased the number of ways in which we can redistribute the energy U (a macrostate quantity) over the different particles (different microstates). Hence, by adding this particle we have increased the system's entropy. If we want to add a particle while keeping S fixed, we need to decrease U to offset the increase in the number of 'degrees of freedom' over which to distribute this energy. Hence, keeping S (and V) fixed, requires that the particle has negative energy, and we thus see that $\mu < 0$.

For a fully degenerate Fermi gas, we have that $T = 0$, and thus $S = 0$ (i.e., there is only one micro-state associated with this macrostate, and that is the fully degenerate one). If we now add a particle, and demand that we keep $S = 0$, then that particle must have the Fermi energy (see Chapter 13); $\epsilon = E_{\rm f}$. Hence, for a fully degenerate gas, $\mu = E_{\rm f}$.

Finally, consider a photon gas in thermal equilibrium inside a container. Contrary to an ideal gas, in a photon gas the number of particles (photons) cannot be arbitrary. The number of photons at given temperature, T, and thus at given U, is given by the Planck distribution and is constantly adjusted (through absorption and emission against the wall of the container) so that the photon gas remains in thermal equilibrium. In other words, N_γ is not a degree of freedom for the system, but it set by the volume and the temperature of the gas. Since we can't change N while maintaining S (or T) and V, we have that $\mu = 0$ for photons.

To end this discussion of the chemical potential, we address the origin of its name, which may, at first, seem weird. Let's start with the 'potential' part. The origin of this name is clear from the following. According to its definition (see above), the chemical potential is the 'internal energy' per unit amount (moles). Now consider the following correspondences:

Gravitational potential is the gravitational energy per unit mass:

$$W = \frac{G\,m_1\,m_2}{r} \qquad \Rightarrow \qquad \phi = \frac{G\,m}{r} \qquad \Rightarrow \qquad \phi = \frac{\partial W}{\partial m}$$

Similarly, electrical potential is the electrical energy per unit charge

$$V = \frac{1}{4\pi\varepsilon_0}\frac{q_1\,q_2}{r} \qquad \Rightarrow \qquad \phi = \frac{1}{4\pi\varepsilon_0}\frac{q}{r} \qquad \Rightarrow \qquad \phi = \frac{\partial V}{\partial q}$$

These examples make it clear why μ is considered a 'potential'. Finally, the word chemical arises from the fact that the μ plays an important role in chemistry (i.e., when considering systems in which chemical reactions take place, which change the particles). In this respect, it is important to be aware of the fact that μ is an additive quantity that is conserved in a chemical reaction. Hence, for a chemical

reaction $i + j \rightarrow k + l$ one has that $\mu_i + \mu_j = \mu_k + \mu_l$. As an example, consider the annihilation of an electron and a positron into two photons. Using that $\mu = 0$ for photons, we see that the chemical potential of elementary particles (i.e., electrons) must be opposite to that of their anti-particles (i.e., positrons).

Because of the additive nature of the chemical potential, we also have that the above equation for $\mathrm{d}U$ changes slightly whenever the gas consists of different particle species; it becomes

$$\mathrm{d}U = T\,\mathrm{d}S - P\,\mathrm{d}V + \sum_i \mu_i\,\mathrm{d}N_i$$

where the summation is over all species i. If the gas consists of equal numbers of elementary particles and anti-particles, then the total chemical potential of the system will be equal to zero. In fact, in many treatments of fluid dynamics it may be assumed that $\sum_i \mu_i\,\mathrm{d}N_i = 0$; in particular when the relevant reactions are 'frozen' (i.e., occur on a timescales τ_{react} that are much longer than the dynamical timescales τ_{dyn} of interest), so that $\mathrm{d}N_i = 0$, or if the reactions go so fast ($\tau_{\mathrm{react}} \ll \tau_{\mathrm{dyn}}$) that each reaction and its inverse are in local thermodynamic equilibrium, in which case $\sum_i \mu_i\,\mathrm{d}N_i = 0$ for those species involved in the reaction. Only in the rare, intermediate case when $\tau_{\mathrm{react}} \sim \tau_{\mathrm{dyn}}$ is it important to keep track of the relative abundances of the various chemical and/or nuclear species.

Appendix L

The Lighthill Equation

As discussed in Chapter 17, acoustic waves result from disturbances in a compressible fluid. These disturbances may arise from objects being moved through the fluid. However, sound waves can also be sourced by fluid motions themselves. A familiar example is the noise from jet-engines; the noise emenates from the turbulent wake created by engines. In astrophysics, turbulence will also typically create sound waves. In general these sound waves will not have an important impact on the physics. A potential exception is the heating of the ICM by sound waves created by turbulent wakes created by AGN feedback. We now derive the main equation that describes how fluid motion can source sound waves.

In the linear perturbation theory used in Chapter 17, we neglected the inertial acceleration term $\vec{u} \cdot \nabla \vec{u}$ since it is quadratic in the (assumed small) velocity. When developing a theory in which the sound waves are sourced by fluid, the velocities are not necessarily small, and we cannot neglect the inertial acceleration term. To proceed, it is advantageous to start from the Euler equation in flux-conservative form (see Chapter 12)

$$\frac{\partial \rho u_i}{\partial t} + \frac{\partial \Pi_{ij}}{\partial x_j} = 0$$

Here Π_{ij} is the **momentum flux density tensor** which, for an inviscid fluid, is given by

$$\Pi_{ij} = P\delta_{ij} + \rho u_i u_j$$

We now set $\rho = \rho_0 + \rho_1$ and $P = P_0 + P_1$, where the subscript '0' indicates the unperturbed equilibrium flow, and '1' refers to the perturbed quantities. Using that, $\partial \rho_0 / \partial t = 0$, the continuity equation reduces to

$$\frac{\partial \rho_1}{\partial t} + \frac{\partial \rho u_i}{\partial x_i} = 0$$

In addition, using the fact that we can ignore P_0, since the pressure only enters the Euler equation via its gradient, and $\nabla P_0 = 0$ (i.e., the unperturbed medium is in pressure equilibrium), we can write the momentum flux density tensor as

$$\Pi_{ij} = \rho u_i u_j + P_1 \delta_{ij} + c_{\rm s}^2 \rho_1 \delta_{ij} - c_{\rm s}^2 \rho_1 \delta_{ij} = c_{\rm s}^2 \rho_1 \delta_{ij} + Q_{ij}$$

where we have introduced the tensor

$$\boxed{Q_{ij} \equiv \rho u_i u_j + (P_1 - c_{\rm s}^2 \rho_1)\delta_{ij}}$$

which describes the departure of Π_{ij} from linear theory. To see this, recall that in linear theory any term that is quadratic in velocity is ignored, and that in linear theory the perturbed pressure and density are related according to $P_1 = c_{\rm s}^2 \rho_1$. Hence, in linear theory $Q_{ij} = 0$.

Substituting the above expression for Π_{ij} in the Euler equation in flux-conservative form yields

$$\frac{\partial \rho u_i}{\partial t} + c_{\rm s}^2 \frac{\partial \rho_1}{\partial x_i} = -\frac{\partial Q_{ij}}{\partial x_j}$$

Next we take the time-derivative of the continuity equation to obtain

$$\frac{\partial^2 \rho_1}{\partial t^2} + \frac{\partial}{\partial x_i}\left[\frac{\partial \rho u_i}{\partial t}\right] = 0$$

Substituting the Euler equation finally yields the **inhomogeneous wave equation**

$$\boxed{\frac{\partial^2 \rho_1}{\partial t^2} - c_{\rm s}^2 \frac{\partial^2 \rho_1}{\partial x_i^2} = \frac{\partial^2 Q_{ij}}{\partial x_i \partial x_j}}$$

which is known as the **Lighthill equation**, after M.J. Lighthill who first derived it in 1952. It is an example of an **inhomogeneous wave equation**; the term on the rhs is a **source term**, and its presence makes the PDE inhomogeneous.

Inhomogeneous wave equations are well known in Electrodynamics. In particular, introducing the (scalar) electric potential, ϕ, and the (vector) magnetic potential $\vec{A}$, defined from the $\vec{E}$ and $\vec{B}$ fields by

$$\vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t} \qquad \vec{B} = \nabla \times \vec{A}$$

and adopting the **Lorenz gauge condition**

$$\frac{1}{c^2}\frac{\partial \phi}{\partial t} + \nabla \cdot \vec{A} = 0$$

the four **Maxwell equations** in a vacuum with charge ρ and current $\vec{J}$ reduce to two uncoupled, inhomogeneous wave equations that are symmetric in the potentials:

$$\begin{aligned} \nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} &= -\frac{\rho}{\varepsilon_0} \\ \nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} &= -\mu_0\,\vec{J}. \end{aligned}$$

As discussed in many standard textbooks on electromagnetism, the general solution of the inhomogeneous wave equation

$$\frac{\partial^2\rho_1}{\partial t^2} - c_{\rm s}^2\frac{\partial^2\rho_1}{\partial x_i^2} = \mathcal{G}(\vec{x},t)$$

is given by

$$\rho_1(\vec{x},t) = \frac{1}{4\pi c_{\rm s}^2}\int \frac{\mathcal{G}(\vec{x}',t-|\vec{x}-\vec{x}'|/c_{\rm s})}{|\vec{x}-\vec{x}'|}\,{\rm d}V'$$

This represent a superposition of spherical acoustic waves traveling outward from their sources located at $\vec{x}'$.

Thus, we have seen that keeping the higher-order velocity terms yields a source term of acoustic waves, given by

$$\mathcal{G}(\vec{x},t) = \frac{\partial^2}{\partial x_i \partial x_j}\left[\rho u_i u_j + (P_1 - c_{\rm s}^2\rho_1)\delta_{ij}\right]$$

Note that this is a scalar quantity (Einstein summation). Although this equation gives some insight as to how fluid motion can spawn sound waves, actually solving the Lighthill equation for a turbulent velocity field $\vec{u}(\vec{x},t)$ is obviously horrendously difficult.

We end by considering the impact of **viscosity** and **conductivity**. As discussed in Chapter 17, these transport mechanisms can be ignored as long as the wavelength of the sound wave is much larger than the mean-free path of the fluid particles. However, in the long run viscosity and conductivity will cause some level of momentum dissipation and energy diffusion, which will cause the sound waves to die out. In fact, for a non-ideal fluid the **momentum flux density tensor** is given by

$$\Pi_{ij} \equiv \rho\langle v_i v_j\rangle = \rho u_i u_j + P\delta_{ij} - \tau_{ij}$$

Hence, the **viscous stress tensor**, τ_{ij}, enters the tensor Q_{ij} that describes the departure of Π_{ij} from linear theory. Hence, viscosity can simply be included in the source (or sink) term of the Lighthill equation. Note that in its derivation we did NOT linearize; we did NOT make the assumptions that $\rho_1 \ll \rho_0$ or $P_1 \ll P_0$. The Lighthill equation is therefore valid *in general* as long as ρ_1 and P_1 are perturbations away from an equilibrium solution with $\partial\rho_0/\partial t = 0$ and $\nabla P_0 = 0$.

Appendix M

The Forced, Damped Oscillator

In order to understand how a many-body system responds to a perturbing (oscillating) force, it is often insightful to first consider one of the simplest examples: the **forced damped oscillator** which you have undoubtedly encountered as part of your undergraduate physics courses. A solid understanding of this system is extremely beneficial for understanding, for example, how galaxies react to external forcing. After all, you can think of a galaxy as a huge collection of harmonic oscillators. As we have seen in Part I, in an integrable Hamiltonian system the motion of the stars is quasi-periodic, being made up of three oscillations (i.e., round-and-round, in-and-out, up-and-down), each with their own frequencies. Each of these oscillations is like a harmonic oscillator, and will respond to the forcing as such.

Let us start with a simple, unforced, damped oscillator consisting of a mass on a spring bobbing up and down in a vessel with a liquid that causes a friction. The equation of motion is

$$m\ddot{x} = -k\,x - b\,\dot{x}$$

where dots indicate time-derivatives, k is the spring constant, and b is the friction term. Dividing by mass yields

$$\ddot{x} + \gamma\,\dot{x} + \omega_0^2\,x = 0$$

Here $\omega_0 \equiv \sqrt{k/m}$ is the natural, angular frequency of the system and $\gamma \equiv b/m$. It is common practice to also define the **quality parameter** $Q \equiv \omega_0/\gamma$. In the absence of friction ($\gamma = 0$) the solution to the above diffential equation is simply

$$x(t) = A\cos(\omega_0 t + \alpha)$$

where A and α are two constants whose values derive from the initial conditions (ICs). We can find the solution to the above differential equation in the case where $\gamma \neq 0$ by substituting a trial solution $x(t) \propto \mathrm{e}^{-i\omega t}$. This yields that

$$\omega^2 + i\gamma\omega - \omega_0^2 = 0$$

which has solutions

$$\omega = -i\frac{\gamma}{2} \pm \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}\,, \qquad \omega_0^2 > \gamma^2/4$$

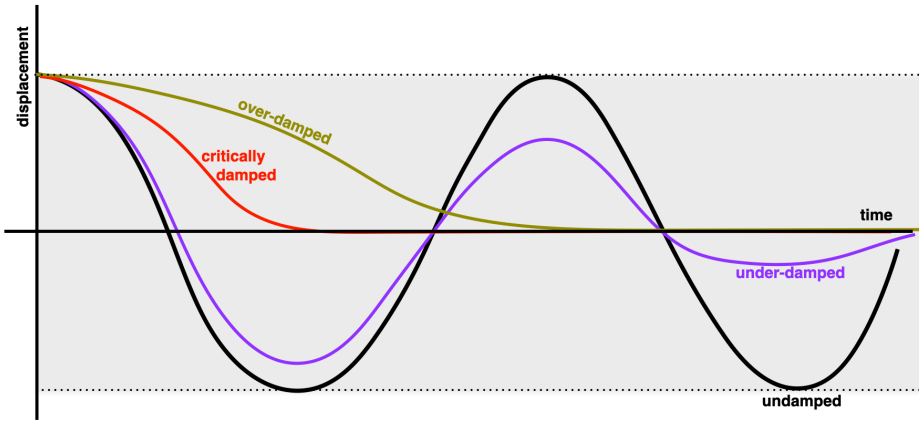

Figure 36: *The various responses of a damped, harmonic oscillator. See text for details.*

Taking the real part of our trial solution with our expression for ω ultimately yields the solution

$$\boxed{x(t) = A\,\mathrm{e}^{-\frac{\gamma}{2}t}\,\cos(\omega t + \alpha)}$$

where, as before, A and α are dictated by the ICs, and

$$\boxed{\omega = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}} = \omega_0\sqrt{1 - \frac{1}{4Q^2}}\,.}$$

This is the solution for an **underdamped oscillator**. If $\gamma^2 > 4\omega_0^2$ then the system is said to be **overdamped**: the friction is so strong that the response is damped without any oscillations. When $\gamma = 2\omega_0$ the system is **critically damped**, which is the solution for which the system returns to its equilibrium situation the fastest, without overshooting (see Fig. 36).

Now let's start forcing this damped oscillator with an oscillating force $F(t) = F_0\cos(\omega t)$. The equation of motion now becomes

$$\ddot{x} + \gamma\,\dot{x} + \omega_0^2\,x = \frac{F_0}{m}\cos(\omega t)$$

This is an inhomogeneous differential equation, whose solution is the sum of two parts: the *complementary solution* (aka transient solution), which is the solution to the homogeneous equation (i.e., with $F_0 = 0$, which produces the solution of the damped oscillator discussed above), plus a *particular solution*, (aka steady-state solution) which describes the **steady-state** solution which is the solution to which

the system asymptotes at late times. When forcing an oscillator with a frequency ω, the steady-state solution will be an oscillation with the same frequency as the forcing. Hence, we consider a trial solution (for the *particular solution* only) of the form $A_{\rm s}{\rm e}^{i\omega t+\phi_{\rm s}}$. Here ω is the forcing frequency, and $\phi_{\rm s}$ is an angle that describes the phase-difference between the driving and the response. Taking the above expression to the complex plane (i.e., $\cos(\omega t) \to e^{i\omega t}$) and substituting this trial solution gives

$$-\omega^2 - i\gamma\omega + \omega_0^2 = \frac{F_0}{A_{\rm s}\, m} e^{i\phi_{\rm s}} = \frac{F_0}{A_{\rm s}\, m}(\cos\phi_{\rm s} + i\sin\phi_{\rm s})$$

Seperately equating the real and imaginary parts yields the following two equations:

$$\begin{aligned} -\omega^2 + \omega_0^2 &= \frac{F_0}{A_{\rm s}\, m}\cos\phi_{\rm s} \\ -\omega\,\gamma &= \frac{F_0}{A_{\rm s}\, m}\sin\phi_{\rm s}\,. \end{aligned}$$

Adding these two expressions in quadrature gives the amplitude of our trial solution

$$\boxed{A_{\rm s} = \frac{F_0/m}{\sqrt{(\omega_0^2-\omega^2)^2 + (\omega\gamma)^2}}}$$

while dividing both equations gives a solution for the phase-difference

$$\boxed{\tan\phi_{\rm s} = \frac{\omega\,\gamma}{\omega_0^2 - \omega^2}}$$

Note that there are no longer any free parameters. The steady-state solution is independent of the ICs. By the time the system settles in this steady-state solution, it has lost all its memory of the ICs!

Hence, the full solution (complementary + particular) for the forced, damped oscillator is given by

$$x(t) = A\,{\rm e}^{-\frac{\gamma}{2}t}\,\cos(\omega_1 t + \alpha) + A_{\rm s}\cos(\omega t + \phi_{\rm s})$$

with $\omega_1 = \omega_0\sqrt{1-(2Q)^{-1}}$, and A and α two constants that depend on the ICs. At early times, before the first term has damped away, the system is beating with frequencies ω_1 and the driving frequency ω, which can cause erratic behavior, knowns as **transient behavior**. At sufficiently late times, the complementary solution has damped away, and the system settles in its steady-state solution, oscillating with the

same frequency as the forcing, but with an amplitude and phase-difference given by the expressions for $A_{\rm s}$ and $\phi_{\rm s}$ given above.

It is useful to examine these expressions in some detail. When $\omega \ll \omega_0$ then $A = F_0/(m\,\omega_0^2) = F_0/k$. Hence, the amplitude is governed by the amplitude of the forcing plus the spring constant. In addition, as intuitively clear, the response is exactly in phase with the perturber ($\phi_{\rm s} = 0$). When $\omega \gg \omega_0$ the system is unable to keep up with the forcing, and the response dies away ($A_{\rm s} \rightarrow 0$). Interestingly, the phase-difference $\phi_{\rm s} \rightarrow \pi$, indicating that the response is exactly out-of-phase with the perturber. Finally, when $\omega = \omega_0$ the system is **in resonance**, causing a large response given by $A_{\rm s} = F_0/(m\omega_0\gamma) = Q\,(F_0/k)$. Note that $A_{\rm s} \rightarrow \infty$ in the absence of any damping ($\gamma = 0$). A key insight from this treatment of the forced, damped oscillator is that when one crosses the resonance, the sign of the response changes with respect to the forcing (i.e., goes from in-phase to out-of-phase, or vice versa).

Appendix N

The Dielectric Function

In electromagnetism, a dielectric (or dielectric medium) is an electrical insulator that can be polarised by an applied electric field. When a dielectric material is placed in an electric field, electric charges do not flow through the material as in an electrical conductor, because they have no free, electrons that may drift through the material, but instead they shift, only slightly, from their average equilibrium positions, causing dielectric polarisation. Because of dielectric polarisation, positive charges are displaced in the direction of the field and negative charges shift in the direction opposite to the field. This creates an internal electric field that reduces the overall field within the dielectric itself.

Hence, an electric field applied to a dielectric introduces an electric displacement field, $\vec{D}$, as well as a polarization field, $\vec{P}$, inside the dielectric. These fields are related to the applied electric field according to

$$\vec{D} = \varepsilon \, \vec{E} \qquad\qquad \vec{P} = \chi \, \vec{E}$$

Here ε is the **relative permittivity** of the medium, which is also known as the **dielectric constant**, and χ is the **electric susceptibility**. Note that

$$\varepsilon = 1 + 4\pi\chi$$

If the applied electric field is static, and the dielectric material is not anisotropic, then ε and χ are scalars. If the material is anisotropic, though, both are tensors.

Now consider applying a electric field that varies spatially and temporally according to, $\vec{E} \propto \exp[i(\vec{k} \cdot \vec{x} + \omega t)]$. Now the electric displacement, $\vec{D}$, and polarization, $\vec{P}$, also depend on location and time, and, in Fourier space, the dielectric constant now becomes a **dielectric function** $\varepsilon(\vec{k}, \omega)$. Hence, the dielectric function is basically a **linear response function**, describing the correlation between the imposed electric field $\vec{E}(\vec{r}, t)$ and the resulting electric displacement, $\vec{D}(\vec{r}, t)$ (i.e., the response) in the medium.

The **dielectric function** depends on the dielectric. A plasma is basically a conductor, as it has free electrons, and it indeed acts like a conductor when a static electric field is applied to it. However, when applying an electric field of the form

$\vec{E} \propto \exp[i(\vec{k}\cdot\vec{x}+\omega t)]$ it responds in a complicated way, as described by the dielectric function. In particular, the dielectric function of a plasma describes how it responds to EM waves. Note that a dielectric function is typically a complex function. If it has an imaginary part then it implies that the response is unstable, either showing (exponential) damping or growth.

Relation to dispersion relation: Important insights come from considering the zeros of the dielectric function; the relation between ω and $\vec{k}$ for which $\varepsilon(\vec{k},\omega) = 0$ is called the **dispersion relation** $\omega(k)$, which describes the **normal modes** of the plasma.

Normal Modes: a normal model of a dynamical system is a pattern of motion that can be sustained without external forcing and in which all parts of the system move sinusoidally with the same frequency and with a fixed phase relation. In other words, it is a mode of the form $\exp[i(\vec{k}\cdot\vec{x}+\omega t)]$ that is a solution to the perturbation analysis of the dynamical system. The frequencies of normal modes are known as its **natural frequencies** or **resonant frequencies** (driving a system at its natural frequencies causes resonance, i.e., a large response). The most general motion of a (linear) system is a superposition of its normal modes. Normal modes are orthogonal to each other, which implies that exciting one normal mode can never cause excitation of another mode.

The **refractive index**, n, of a medium is the ratio of the speed of light in vacuum and the **phase velocity** in the medium: $n = c/v_\phi$. Hence, since plasmas typically have $0 < \varepsilon < 1$, and $n = \sqrt{\varepsilon}$, we have that the group velocity of EM waves in a plasma $v_\phi > c$. note that this is NOT a violation of special relativity, since information is propagated by the group velocity $v_\mathrm{g} = \partial\omega/\partial k$, and not by the **phase velocity** $v_\phi \equiv \omega/k$.

An example of a plasma with an index of refraction less than unity is Earth's ionosphere. This causes EM waves propagating through the ionosphere to be bent "away from the normal" allowing the radio wave to be refracted back toward earth, thus enabling long-distance radio communications

Recall, the refractive index may depend on wavelength, which is called **dispersion**, as for example in a prism. Since the dielectric function of a plasma in general depends on wavelength, most plasma's are dispersive

WORKSHEET 1

Gauging Prior Knowledge

The next page has a a list of concepts that will be discussed in ASTR 501. In order to gauge your level of knowledge at the beginning of this course, please indicate (using a cross) your level of familiarity with each topic:

Use 'Familiar' if you understand this topic well, to the point that you can explain it to you peers.

Use 'Somewhat' if you have heared of it, but are not comfortable with it.

Use 'Not at all' if you have never heared of it, or heared of it without knowing really what it is.

SPRING 2022

Term	Familiar	Somewhat	Not at all
Euler-Lagrange equations	6	1	
Hamilton-Jacobi equation	4	3	
Canonical transformation	3	4	
Noether's theorem	2	4	1
Poisson brackets	3	2	2
Action-Angle variables	2	1	4
Integrable Hamiltonian	2	4	1
Liouville Theorem	2	3	2
Molecular chaos	1	3	3
BBGKY hierarchy		3	4
Markovian process	3	2	2
Fluctuation-Dissipation Theorem		1	6
Langevin equation		1	6
Smoluchowski equation		1	6
Fokker-Planck equation	3	3	1
Lagrangian vs. Eulerian derivatives	5	2	
Navier-Stokes equation	5	2	
Baroclinicity	1	3	3
Kelvin's circulation theorem	2	1	4
Crocco's theorem			7
Reynold's number	2	5	
Reynold stress	1	3	3
Ranking-Hugoniot jump conditions	1	4	2
Rayleigh-Taylor instability	4	2	1
Kelvin-Helmholtz instability	5	2	
Thermal instability	4	3	
Jeans equations	3	4	
Jeans' theorem	4	3	
Impulse approximation	3	1	3
Gravothermal Catastrophe	2	2	3
Plasma frequency		5	2
Lenard-Balescu equation		2	5
Alfvén velocity		7	
Landau damping	1	4	2

FALL 2023

Term	**Familiar**	**Somewhat**	**Not at all**
Euler-Lagrange equations	8	2	
Hamilton-Jacobi equation	4	4	2
Canonical transformation	4	3	3
Noether's theorem	6	2	2
Poisson brackets	6	2	2
Action-Angle variables	2	4	4
Integrable Hamiltonian	1	6	3
Liouville Theorem	3	2	5
Molecular chaos	1	1	8
BBGKY hierarchy	2		8
Markovian process	2	8	
Fluctuation-Dissipation Theorem	1		9
Langevin equation	1	1	8
Smoluchowski equation		1	9
Fokker-Planck equation	1	6	3
Lagrangian vs. Eulerian derivatives	6	4	
Navier-Stokes equation	3	6	1
Baroclinicity		1	9
Kelvin's circulation theorem	2	1	7
Crocco's theorem		2	8
Reynold's number	7	3	
Reynold stress		4	6
Ranking-Hugoniot jump conditions	1	4	5
Rayleigh-Taylor instability	3	3	4
Kelvin-Helmholtz instability	3	5	2
Thermal instability	2	4	4
Jeans equations	3	5	2
Jeans' theorem	3	4	3
Impulse approximation	1	5	4
Gravothermal Catastrophe	1	4	5
Plasma frequency	2	2	6
Lenard-Balescu equation		2	8
Alfvén velocity	1	3	6
Landau damping	1	6	3

WORKSHEET 2

Integrable or Not Integrable?

Which of the following Hamilitonian systems is **integrable** and why?

Example 1: one-degree of freedom, single particle

The Hamiltonian of a $n = 1$ system is

$$\mathcal{H}(q,p) = \frac{p^2}{2m} + V(q)$$

According to Liouville's theorem of integrable systems (Chapter 4), a system of n degrees of freedom (hereafter dof) is integrable if it has (at least) n integrals of motion (hereafter IoM), I_i, in involution ($\{I_i, I_j\} = 0$).

Since $\mathcal{H} \neq \mathcal{H}(t)$ (no explicit time-dependence), we have that $\mathrm{d}\mathcal{H}/\mathrm{d}t = 0$ and the Hamiltonian itself is thus a constant of motion ($\mathcal{H} = E$, reflecting conservation of energy, consistent with symmetry under time-translation and Noether's theorem). Hence, the system is integrable.

Every time-independent Hamiltonian with one dof is integrable

Example 2: two uncoupled Hamiltonians of 1 degree of freedom

The hamiltonian of this $n = 2$ system is

$$\mathcal{H}(q_1, q_2, p_1, p_2) = \mathcal{H}_1(q_1, p_1) + \mathcal{H}_2(q_2, p_2)$$

In order for this system to be integrable it needs two IoM in involution. Since the two Hamiltonians are uncoupled and time-independent, both $\mathcal{H}_1$ and $\mathcal{H}_2$ are constants of motion. Furthermore, they are in involution, which follows directly from

$$\{\mathcal{H}_1, \mathcal{H}_2\} = \frac{\partial \mathcal{H}_1}{\partial q_i}\frac{\partial \mathcal{H}_2}{\partial p_i} - \frac{\partial \mathcal{H}_1}{\partial p_i}\frac{\partial \mathcal{H}_2}{\partial q_i} = 0$$

(i.e., one of the factors in each term is always zero. Hence, this system is integrable.

any Hamiltonian $\mathcal{H}(q_1, ..., q_n, p_1, ..., p_n) = \sum_{i=1}^{n} \mathcal{H}_i(q_i, p_i)$, representing a sum of n uncoupled, time-independent Hamiltonians of 1 dof, is integrable

Example 3: free particle ($n = 2$) moving inside a box

Consider a free particle (no force and thus no potential) moving in a $2D$ (x, y) system constrained by $0 \leq x \leq L$ and $0 \leq y \leq L$. These constraints are NOT holonomic, and therefore do not reduce the number of degrees of freedom. Rather, they specify what in dynamics is called the **stadium** (see first panel of Fig. 37).

The Hamiltonian of this system is given by

$$\mathcal{H}(x, y, p_x, p_y) = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + V(x, y)$$

The potential only specifies the stadium and is such that $V(x, y) = 0$ inside the stadium, and ∞ outside (indicating a reflective boundary). Although this makes the Hamiltonian not differentiable, this is only a problem at the boundary, not in the stadium, and therefore of no concern.

For the system to be integrable it needs $n = 2$ IoM in involution. So let's think about what are constants of motion. First of all, since $\mathcal{H} \neq \mathcal{H}(t)$, we have that the Hamiltonian itself is an integral of motion. Note that p_x and/or p_y are NOT IoM; after all, when you bounce against the wall at $x = L$ then p_x changes sign. What is conserved, though, is the square of the momentum; hence, both p_x^2 and p_y^2 are IoM. It is easy to see that these are in involution:

$$\{p_x^2, p_y^2\} = \{p_x p_x, p_y p_y\} = p_x\{p_x, p_y p_y\} + p_x\{p_x, p_y p_y\} = -2p_x\{p_y p_y, p_x\} = \\ -2p_x\left[p_y\{p_y, p_x\} + p_y\{p_y, p_x\}\right] = 4p_x p_y\{p_x, p_y\} = 0$$

The last step follows from the fact that p_x and p_y are canonical momenta.

So we have two IoM in involution, and therefore the system is integrable. Note that we do NOT have three independent IoM in involution; although the Hamiltonian is another IoM, and it is in involution with both p_x^2 and p_y^2 (which is easy to see), it is not independent; after all, inside the stadium $\mathcal{H} = (p_x^2/2m) + (p_y^2/2m)$.

Finally, you may wonder what happens to a particle that I start from the center $(x, y) = (L/2, L/2)$ and shoot under an angle of 45° at one of the corners. How do you apply the law of reflection in this case. This case cannot be treated, and because

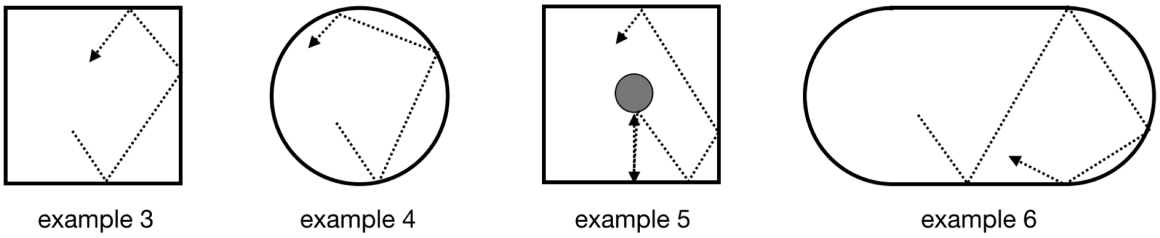

Figure 37: *The stadiums discussed in examples 3 to 6.*

of this Hamiltonian system is actually **pseudo-integrable**: there is a subset of initial conditions of measure zero for which you can not write down what the solution is.

Example 4: free particle ($n = 2$) moving inside a circular stadium

Same situation as in example 3, except that this time the stadium is circular with a radius R (see second panel of Fig. 37).

The Hamiltonian of this system is given by

$$\mathcal{H}(x, y, p_x, p_y) = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + V(x, y)$$

where this time $V(x, y) = 0$ if $r = \sqrt{x^2 + y^2} < R$ and infinity otherwise. Upon inspection it is clear that neither p_x or p_y, nor p_x^2 or p_y^2 are IoM. The Hamiltonian itself, though, still is one.

In this case, the system has circular symmetry. I can rotate the stadium around the z-axis and everything remains the same. Hence, the Hamiltonian obeys a continuous symmetry, and Noether's theorem tells us that there must be an associated constant of the motion. It is easy to see that this is the angular momentum $L_z = xp_y - yp_x$. We leave it as an exercise for the reader to confirm that $\{\mathcal{H}, L_z\} = 0$, and thus that we have two independent IoM in involution. Yes, this system too is integrable.

Example 5: free particle ($n = 2$) moving inside a square stadium with circular object in center

In this example we place a circular object at the center of a square stadium, as depicted in the third panel of Fig. 37. Note that this time there is no circular symmetry, and thus no conservation of angular momentum. Neither are p_x, p_y, p_x^2 or p_y^2 conserved. The only constant of motion is the actual Hamiltonian, and this is

thus a non-integrable system! You can easily see that it is subject to chaos (extreme sensitivity to initial conditions), by considering the two trajectories shown in Fig. 37.

Example 6: free particle ($n = 2$) moving inside a Bunimovich stadium

The Bunimovich stadium is depicted in the rightmost panel of Fig. 37. It is constructed by cutting a circle in two halves, and connecting them with straight lines, to make a shape similar to that of the Circus Maximus in ancient Rome. Similar to example 5, because of the lack of symmetry, there is no IoM other than $\mathcal{H}$, and this is another example of a non-integrable Hamiltonian system. The last two examples show that

even systems with only 2 degrees of freedom can produce chaos

Example 7: particle in a central potential in 3D

The Hamiltonian for this example is

$$\mathcal{H}(\vec{q},\vec{p}) = \mathcal{H}(\vec{r},\vec{p}) = \frac{\vec{p}^2}{2m} + V(r)$$

For this Hamiltonian with 3 degrees of freedom to be integrable, it need to have 3 independent IoM in involution. As always, because the Hamiltonian is autonomous, $\mathcal{H}$ is an IoM. Also, because the potential is central, we have spherical symmetry, and thus $\vec{L}$ is an integral of motion. Now, $\vec{L}$ has three components, so these are really three IoM. However, they are NOT in involution with each other; $\{L_i, L_j\} \neq 0$ when $i \neq j$ (think of quantum mechanics, in which $\hat{L}_x$ and $\hat{L}_y$ do not commute). Hence, we can only pick one of the three components, and we shall pick L_z. It is not difficult to check that L_i Poisson commutes with $\mathcal{H}$. Thus, thus far we have two IoM in involution. As it turns out, one can take $\vec{L}^2$ as a third one, since $\{L_z, \vec{L}^2\} = 0$. Hence, this system is integrable!

Every central force problem is integrable in 3D, independent of $V(r)$

Example 8: a two-particle system in 3D

For this case we have that

$$\mathcal{H}(\vec{r}_1,\vec{r}_2,\vec{p}_1,\vec{p}_2) = \frac{\vec{p}_1^2}{2m_1} + \frac{\vec{p}_2^2}{2m_2} + V(\vec{r}_1,\vec{r}_2)$$

For this system to be integrable, we need a total of 6 independent IoM in involution (there are 6 degrees of freedom). However, the system doesn't have any symmetry, and no IoM other than $\mathcal{H}$ is obvious. In order to get some insight, it is useful to go to the center of mass coordinate system. Let $\vec{r} = \vec{r}_1 - \vec{r}_2$, $\vec{R} = (m_1\vec{r}_1 + m_2\vec{r}_2)/M$, $M = m_1 + m_2$, $\mu = m_1 m_2/M$, $\vec{P} = \vec{p}_1 + \vec{p}_2$ and $\vec{p} = \mu(\vec{v}_1 - \vec{v}_2)$. In these new coordinates, the Hamiltonian is given by

$$\mathcal{H}(\vec{r}, \vec{R}, \vec{p}, \vec{P}) = \frac{\vec{P}^2}{2M} + \frac{\vec{p}^2}{2\mu} + V(\vec{r}, \vec{R})$$

Note that this does not really simplify the situation. As it turns out, this general two-particle system has no IoM other than $\mathcal{H}$. Note that not even $\vec{P}$ is conserved, which is obvious from the fact that the potential depends on $\vec{R}$. Clearly, then, this system is not integrable.

Example 9: a two-particle system in 3D for which $V = V(|\vec{r}_1 - \vec{r}_2|)$.

For this case we have that in the center-of-mass frame, the Hamiltonian is given by

$$\mathcal{H}(\vec{r}, \vec{R}, \vec{p}, \vec{P}) = \frac{\vec{P}^2}{2M} + \frac{\vec{p}^2}{2\mu} + V(r)$$

Hence, we see that $\vec{R}$ is cyclic, and thus that the conjugate momenta $\vec{P} = (P_x, P_y, P_z)$ are conserved. In fact, upon closed inspection one notices that the above Hamiltonian can be split in two: $\mathcal{H}(\vec{r}, \vec{p}, \vec{P}) = \mathcal{H}_1(\vec{P}) + \mathcal{H}_2(\vec{r}, \vec{p})$. The first term corresponds to a free system (the center of mass), which is integrable, and the second corresponds to a 3D central force system, which, as we have seen in example 7 is also integrable. Hence, the total system must be integrable as well, and indeed, with respect to the center of mass the angular momentum vector is conserved (spherical symmetry $\rightarrow$ Noether's theorem), and we can add L_z and $\vec{L}^2$ to our list of four IoM for a grand total of 6, as required.

A two-body system with a central force is integrable

Example 10: an n-particle system in 3D with central forces

Consider the following Hamiltonian

$$\mathcal{H}(\vec{q}_1, \vec{q}_2, ..., \vec{q}_n, \vec{p}_1, \vec{p}_2, ..., \vec{p}_n) = \sum_{i=1}^{n} \frac{\vec{p}_i^2}{2m_i} + \sum_{i \neq j} V(|\vec{r}_i - \vec{r}_j|)$$

As we have seen, this is integrable for $n = 2$; what about $n > 2$? Converting again to the center of mass frame, one infers the following integrals of motion (we'll worry about involution later):

- the Hamiltonian $\mathcal{H}$ itself (because $\mathcal{H} \neq \mathcal{H}(t)$)
- the total angular momentum vector $\vec{L}_{\rm tot} = \sum_i \vec{r}_i \times \vec{p}_i$
- the total momentum vector $\vec{P}_{\rm tot} = \sum_i \vec{p}_i$ (since there are no external forces, the total momentum is conserved; see Chapter 3)
- the initial position vector, $\vec{R}_0$, of the center of mass. Since $\vec{R}(t) = \vec{R}_0 + (\vec{P}_{\rm tot}/M)t$, we have that $R_0 = R(t) - (\vec{P}_{\rm tot}/M)t$ is an integral of motion

No matter how hard you try, you will not find another independent integral of motion. Hence, we have a grand total of only 10 integrals of motion (these are known as the 10 **Galilean invariants**). For an n particle system in 3D we have $3n$ degrees of freedom, and we thus need $3n$ IoM in involution. Hence, we immediately see that the system will NOT be integrable for $n \geq 4$. For the case $n = 3$ it comes down to whether or not the the Galilean invariants are in involution or not. We already know that $\vec{L}_{\rm tot}$ only yields 2, rather than 3 independent IoM in involution ($L_{{\rm tot},z}$ and $\vec{L}^2_{\rm tot}$). We leave it as an exercise for the student to show that not all of the remaining IoM are in involution, and we thus conclude that

An N-body system in 3D is NOT integrable for $N > 2$, even when all the forces are central