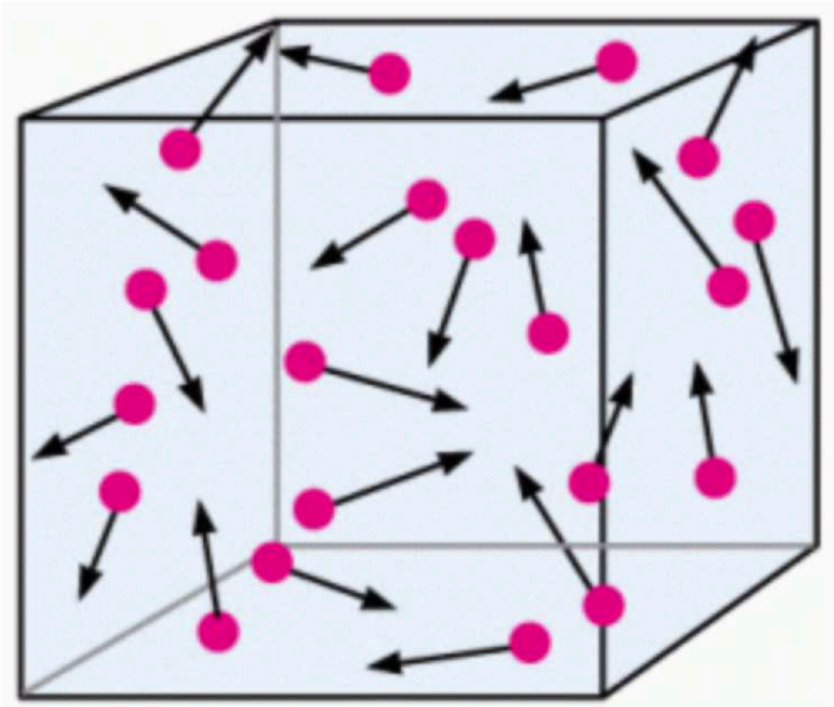
LECTURE 8

The Fokker-Planck Equation

Part II: Kinetic Theory



Part II: Kinetic Theory

The Smoluchowski Equation

Consider three arbitrary times: $t_3 > t_2 > t_1$ and let x(t) be a random process

We can alway write that

$$P_2(x_3, t_3|x_1, t_1) = \int P_3(x_3, t_3|x_2, t_2; x_1, t_1)\, P_2(x_2, t_2|x_1, t_1)\, \mathrm{d}x_2$$

which simply states that as x transitions from x_1 at t_1 to x_3 at t_3, it must pass through some x_2 at t_2

Iff the random process is Markovian and stationary, we have that

$$P_3(x_3, t_3|x_2, t_2; x_1, t_1) = P_2(x_3, t_3|x_2, t_2) = P_2(x_3, t_3 - t_2|x_2)$$

In this case the expression simplifies to

$$\boxed{P_2(x_3, t_3|x_1) = \int P_2(x_3, t_3 - t_2|x_2)\, P_2(x_2, t_2|x_1)\, \mathrm{d}x_2}$$

known as the Smoluchowski equation (or the Chapman-Kolmogorov equation)

From Smoluchowski Equation to Fokker-Planck

$$P_2(x_3, t_3|x_1) = \int P_2(x_3, t_3 - t_2|x_2)\, P_2(x_2, t_2|x_1)\, \mathrm{d}x_2$$

$t_3 \to t + \Delta t$
$x_3 \to x$
$x_2 \to x - \Delta x$
$x_1 \to x_0$

$$\boxed{P_2(x, t + \Delta t) = \int \Psi(\Delta x, \Delta t|x - \Delta x, t)\, P_2(x - \Delta x, t|x_0)\mathrm{d}(\Delta x)}$$

only valid for stationary, Markovian random process

where $\Psi(\Delta x, \Delta t|x - \Delta x, t) = P_2(x, t + \Delta t|x - \Delta x, t)$ is the transition probability that starting from $x - \Delta x$ at t the random variable undergoes a change Δx in timestep Δt

Using a Taylor series expansion for the integrant in the above expression, we obtain

$$\begin{aligned}
P_2(x, t + \Delta t) &= \int \mathrm{d}(\Delta x) \sum_{n=0}^{\infty} \frac{(-\Delta x)^n}{n!} \frac{\partial^n}{\partial x^n} \left[\Psi(\Delta x, \Delta t|x - \Delta x, t)\, P_2(x - \Delta x, t|x_0)\right]_{x - \Delta x = x} \\
&= \int \mathrm{d}(\Delta x) \sum_{n=0}^{\infty} \frac{(-\Delta x)^n}{n!} \frac{\partial^n}{\partial x^n} \left[\Psi(\Delta x, \Delta t|x, t)\, P_2(x, t|x_0)\right] \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{\partial^n}{\partial x^n} \left[P_2(x, t|x_0) \int \mathrm{d}(\Delta x)\, (\Delta x)^n\, \Psi(\Delta x, \Delta t|x - \Delta x, t)\right] \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{\partial^n}{\partial x^n} \left[\langle(\Delta x)^n\rangle\, P_2(x, t|x_0)\right]
\end{aligned}$$

From Smoluchowski Equation to Fokker-Planck

Using that the n=0 term in the Taylor series expansion is nothing but $P_2(x,t|x_0)$ we obtain

$$\begin{aligned}\frac{\partial P_2(x,t|x_0)}{\partial t} &= \lim_{\Delta t\to 0}\frac{P_2(x,t+\Delta t|x_0)-P_2(x,t|x_0)}{\Delta t}\\ &= \sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\frac{\partial^n}{\partial x^n}\left[D^{(n)}(x,t)\,P_2(x,t|x_0)\right]\end{aligned}$$

where we have defined $$D^{(n)}(x,t)\equiv\lim_{\Delta t\to 0}\frac{\langle(\Delta x)^n\rangle}{\Delta t}$$

If we only keep the first two terms in the Taylor series expansion (i.e., we assume that Δx is small enough such that the higher order terms can be ignored), then we obtain the

Fokker-Planck equation $$\frac{\partial P_2}{\partial t} = -\frac{\partial}{\partial x}[D^{(1)}P_2]+\frac{1}{2}\frac{\partial^2}{\partial x^2}[D^{(2)}P_2]$$

This is a generalized diffusion equation for the evolution of P_2

The Fokker-Planck Equation

Validity: when is the FP equation valid?

Assumptions made: random process is stationary & Markovian
diffusive limit (Δx small)

A stochastic force is Markovian if `sampled' on time scale $\Delta t > \tau_{\rm coll}$

As discussed in the lecture notes, these assumptions are valid for describing the collisional evolution of a globular cluster, or the Brownian motion of a pollen floating in the air (if pollen is massive compared to air molecules).

However, the FP equation can not be used to describe the diffusion of individual air molecules.

$$\boxed{\frac{\partial P_2}{\partial t} = -\frac{\partial}{\partial x}[D^{(1)}P_2] + \frac{1}{2}\frac{\partial^2}{\partial x^2}[D^{(2)}P_2]}$$

$$\boxed{D^{(n)}(x,t) \equiv \lim_{\Delta t \to 0} \frac{\langle (\Delta x)^n \rangle}{\Delta t}}$$

Diffusion in velocity space: $\Delta v = v(\Delta t) - v$

$$D^{(1)} = \lim_{\Delta t \to 0} \frac{\langle \Delta v \rangle}{\Delta t} = \lim_{\Delta t \to 0} \frac{1}{\Delta t}[\langle v(\Delta t)\rangle - v]$$

$$D^{(2)} = \lim_{\Delta t \to 0} \frac{\langle (\Delta v)^2 \rangle}{\Delta t} = \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left[\langle v(\Delta t) v(\Delta t)\rangle - 2v\langle v(\Delta t)\rangle + v^2\right]$$

For white noise we have that $\langle v(t)\rangle = v\,\mathrm{e}^{-\gamma t/m}$

$$\langle v(t)v(t)\rangle = \langle v(t)\rangle^2 + \frac{D\gamma}{m}\left(1 - \mathrm{e}^{-2\gamma t/m}\right)$$

Substitution in the expressions for $D^{(1)}$ and $D^{(2)}$ and using Taylor series expansion for exponential:

$$D^{(1)} = -v\frac{\gamma}{m}, \qquad D^{(2)} = 2D\frac{\gamma^2}{m^2}$$

$$\boxed{\frac{\partial P(v,t)}{\partial t} = \frac{1}{m}\left[\gamma \frac{\partial\, vP(v,t)}{\partial v} + \frac{D\gamma^2}{m}\frac{\partial^2 P(v,t)}{\partial v^2}\right] = \frac{1}{\tau_{\mathrm{d}}}\frac{\partial}{\partial v}\left[vP(v,t) + \frac{D}{\tau_{\mathrm{d}}}\frac{\partial P(v,t)}{\partial v}\right]}$$

Recall: dissipation time $\tau_{\mathrm{d}} = m/\gamma$

$$\frac{\partial P(v,t)}{\partial t} = \frac{1}{m}\left[\gamma \frac{\partial\, vP(v,t)}{\partial v} + \frac{D\gamma^2}{m}\frac{\partial^2 P(v,t)}{\partial v^2}\right] = \frac{1}{\tau_{\rm d}}\frac{\partial}{\partial v}\left[vP(v,t) + \frac{D}{\tau_{\rm d}}\frac{\partial P(v,t)}{\partial v}\right]$$

Let's see what this does:

Let's assume that $P(v,0)$ is a narrow Gaussian centered on $v_0 > 0$

The FP expression has three terms:

- $P(v,t)$ which is positive for all v
- $v\, \partial P/\partial v$ which is positive for $0 < v < v_0$, negative otherwise
- $\partial^2 P/\partial v^2$ which is negative for $|v - v_0| < \sigma$, positive otherwise

The second term causes the PDF to move towards v=0 (friction)

The third term causes the PDF to broaden (diffusion)

At equilibrium $\partial P(v,t)/\partial t = 0$. This requires that $\frac{\partial P}{\partial v} = -\frac{v\tau_{\rm d}}{D}P \;\Rightarrow\; P \propto \exp\left(-\frac{mv^2}{2D\gamma}\right)$

Using the Einstein-Smoluchowski equation, according to which $D\gamma = k_{\rm B}T$ and requiring normalization, we obtain the Maxwell-Boltzmann distribution

$$P_{\rm eq}(v) = \left(\frac{m}{2\pi k_{\rm B}T}\right)^{1/2} \exp\left(-\frac{mv^2}{2k_{\rm B}T}\right)$$

Extension to higher dimensions:

The FP equation we derived thus far is valid for one-dimensional Markov processes x(t).

It is straightforward to extent this to multi-dimensional Markov processes **x**(t)=[x_1(t),x_2(t),…x_n(t)]

$$\frac{\partial P_2}{\partial t} = -\frac{\partial}{\partial x}[D^{(1)}P_2] + \frac{1}{2}\frac{\partial^2}{\partial x^2}[D^{(2)}P_2] \quad \Rightarrow \quad \frac{\partial P}{\partial t} = -\frac{\partial}{\partial x_i}[D_i^{(1)}P_2] + \frac{1}{2}\frac{\partial^2}{\partial x_i \partial x_j}[D_{ij}^{(2)}P_2]$$

$$D_i^{(1)} = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\int \mathrm{d}^n(\Delta\vec{x})\,(\Delta x)_i\,\Psi(\Delta\vec{x},\Delta t|\vec{x}) = \lim_{\Delta t\to 0}\frac{\langle(\Delta x)_i\rangle}{\Delta t}$$

$$D_{ij}^{(2)} = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\int \mathrm{d}^n(\Delta\vec{x})\,(\Delta x)_i\,(\Delta x)_j\,\Psi(\Delta\vec{x},\Delta t|\vec{x}) = \lim_{\Delta t\to 0}\frac{\langle(\Delta x)_i\,(\Delta x)_j\rangle}{\Delta t}$$

Note that the first-order diffusion `coefficient’ is now a vector, and the second-order diffusion `coefficient; has become a tensor

The above multi-dimentional FP equation has many applications in physics, mathematics and beyond…

The collision integral as a Fokker-Planck equation:

Thus far we focussed on FP applications in which we want to describe the evolution of a PDF $P(x,t)$ that starts from some x_0 at $t=t_0$

Sometimes, though, we want to describe the evolution of an unconstrained PDF, $P(x,t)$, under the influence of some stochastic force.

A particular application of the latter is to describe the evolution of the 1-particle distribution function, $f=f^{(1)}$, under the influence of long-range forces (i.e., gravity) in cases where the collisionality of the system is not negligible.

This evolution is described by the Boltzmann equation

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \{f, \mathcal{H}\} = \left(\frac{\partial f}{\partial t}\right)_{\mathrm{c}}$$

with the collision integral given by

$$\left(\frac{\partial f}{\partial t}\right)_{\mathrm{c}} = \int \mathrm{d}^3\vec{q}_2\, \mathrm{d}^3\vec{p}_2\, \frac{\partial U(|\vec{q}_1 - \vec{q}_2|)}{\partial \vec{q}_1} \cdot \frac{\partial f^{(2)}}{\partial \vec{p}_1}$$

which is a complicated-to-solve integro-differential equation that depends on the 2-particle DF, which thus requires input from higher-order equations in the BBGKY hierarchy

The collision integral as a Fokker-Planck equation:

As an alternative approach, we now try to use the Fokker-Planck equation to solve this

Hence, we seek a FP equation that describes the evolution of $f(\boldsymbol{q},\boldsymbol{p},t)$, due to collisionality, rather than the evolution of $P_2(\boldsymbol{q},\boldsymbol{p},t|\boldsymbol{q}_0,\boldsymbol{p}_0,t)$

Our stochastic variable is $\mathbf{w}(t) = (\boldsymbol{q}(t), \boldsymbol{p}(t))$

Let $\Psi(\Delta\vec{w},\vec{w})\,\mathrm{d}^6(\Delta\vec{w})\,\Delta t$ be the transition probability that a particle with phase-space coordinates $\vec{w}$ is scattered to the phase-space volume $\mathrm{d}^6(\Delta\vec{w})$ centered on $\vec{w}+\Delta\vec{w}$ during Δt

Stochastic collisions cause the distribution function to change based on a competition between a gain term and a loss term:

$$\text{loss term}: \quad \left(\frac{\partial f(\vec{w})}{\partial t}\right)_{-} = -f(\vec{w})\int \mathrm{d}^6(\Delta\vec{w})\,\Psi(\Delta\vec{w},\vec{w})\,,$$

$$\text{gain term}: \quad \left(\frac{\partial f(\vec{w})}{\partial t}\right)_{+} = \int \mathrm{d}^6(\Delta\vec{w})\,\Psi(\Delta\vec{w},\vec{w}-\Delta\vec{w})\,f(\vec{w}-\Delta\vec{w})$$

Hence, we can write the collision integral as

$$\boxed{\left(\frac{\partial f}{\partial t}\right)_{\mathrm{c}} = \int \mathrm{d}^6(\Delta\vec{w})\,\left[\Psi(\Delta\vec{w},\vec{w}-\Delta\vec{w})\,f(\vec{w}-\Delta\vec{w}) - \Psi(\Delta\vec{w},\vec{w})\,f(\vec{w})\right]}$$

Note the subtle assumption that $f(\vec{w})$ and $\Psi(\Delta\vec{w},\vec{w})$ are statistically independent: molecular chaos

The collision integral as a Fokker-Planck equation:

Now let's restrict ourselves to weak encounters only (those for which $|\Delta\vec{w}|$ is small)

Then we can Taylor expand and truncate at second order (i.e., make FP ansatz)

$$\Psi(\Delta\vec{w},\vec{w}-\Delta\vec{w})\,f(\vec{w}-\Delta\vec{w}) = \Psi(\Delta\vec{w},\vec{w})\,f(\vec{w}) - \sum_{i=1}^{6}\Delta w_i \frac{\partial}{\partial w_i}\left[\Psi(\Delta\vec{w},\vec{w})\,f(\vec{w})\right] + \frac{1}{2}\sum_{i,j=1}^{6}\Delta w_i\,\Delta w_j \frac{\partial^2}{\partial w_i \partial w_j}\left[\Psi(\Delta\vec{w},\vec{w})\,f(\vec{w})\right]$$

Substituting in our expression for the collision integral then yields

$$\left(\frac{\partial f}{\partial t}\right)_{\rm c} = -\sum_{i=1}^{6}\frac{\partial}{\partial w_i}\left\{D[\Delta w_i]\,f(\vec{w})\right\} + \frac{1}{2}\sum_{i,j=1}^{6}\frac{\partial^2}{\partial w_i \partial w_j}\left\{D[\Delta w_i \Delta w_j]\,f(\vec{w})\right\}$$

with the following diffusion coefficients:

[.] not 'is a function of', but 'average of'

$$D[\Delta w_i] \equiv \int {\rm d}^6(\Delta\vec{w})\,\Delta w_i\,\Psi(\Delta\vec{w},\vec{w})$$

$$D[\Delta w_i \Delta w_j] \equiv \int {\rm d}^6(\Delta\vec{w})\,\Delta w_i \Delta w_j\,\Psi(\Delta\vec{w},\vec{w})$$

These express the expectation values for changes in Δw_i and $\Delta w_i \Delta w_j$ per unit time interval

The collision integral as a Fokker-Planck equation:

Substituting this expression for the collision integral (which is a Fokker-Planck equation) in the Boltzmann equation yields

$$\frac{\mathrm{d}f}{\mathrm{d}t} = -\sum_{i=1}^{6} \frac{\partial}{\partial w_i} \left\{ D[\Delta w_i]\, f(\vec{w}) \right\} + \frac{1}{2} \sum_{i,j=1}^{6} \frac{\partial^2}{\partial w_i \partial w_j} \left\{ D[\Delta w_i \Delta w_j]\, f(\vec{w}) \right\}$$

Kramer's equation of Schwarzschild equation

NOTE: sometimes the above equation is simply referred to as the Fokker-Planck equation

The above equation describes the Lagrangian evolution of the distribution function due to long-range collisions (for which, as we will see, the assumption of weak collisisions is justified).

Note that it is a differential equation, rather than an integro-differential equation, which is much easier to solve.

Key is to compute the first and second order diffusion coefficients…

This Kramer's equation is the primary tool we have (in addition to N-body simulations) to describe the evolution of a gravitational system under the influence of collisions

Weak vs. Strong Encounters

The Fokker-Planck equation, and thus Kramer's equation, is only valid for weak encounters

But among the numerous collisions in a gravitational system, there will always be some some encounters that are strong (cause a large $|\Delta\vec{w}|$)...

We now demonstrate that their impact is negligible and can thus be ignored

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The velocity impuls of a subject mass due to a high-speed encounter with a field particle of mass m with impact parameter b and velocity v is given by

$$\Delta v_\perp \simeq \frac{2G\,m}{b\,v}$$

(you will derive this in one of the problem sets)

Let's define strong collisions as those for which the impact parameter $b < b_{90}$ where b_{90} is the impact parameter for which $\Delta v_\perp = v$, i.e., for which the deflection is 90 degrees

$$b_{90} = \frac{2Gm}{v^2}$$

The surface density of field particles in a system of size R is roughly $N/(\pi R^2)$

Hence, when a subject mass crosses the system once it has $\frac{\mathrm{d}N}{\mathrm{d}b}\,\mathrm{d}b = \frac{N}{\pi R^2}\cdot 2\pi\,b\,\mathrm{d}b = \frac{2N}{R^2}\,b\,\mathrm{d}b$ encounters with impact parameters in the range b, $b+\mathrm{d}b$

Weak vs. Strong Encounters

If we assume that the system is homogeneous, then $\langle \Delta v_\perp \rangle = 0$ (average out)

However, $$\overline{\Delta v^2} = \int_{b_{\rm min}}^{b_{\rm max}} (\Delta v_\perp)^2(b) \frac{{\rm d}N}{{\rm d}b}\, {\rm d}b = 8\,N \left(\frac{Gm}{Rv}\right)^2 \int_{b_{\rm min}}^{b_{\rm max}} \frac{{\rm d}b}{b} = 8\,N \left(\frac{Gm}{Rv}\right)^2 \ln\Lambda$$

where we have defined the Coulomb logarithm $$\ln\Lambda = \ln\left(\frac{b_{\rm max}}{b_{\rm min}}\right)$$

For weak encounters we can set $b_{min} = b_{90}$ and $b_{max} = R$

Substituting the expression for b_{90} we then have that $\ln \Lambda \simeq \ln N$

Using that the typical velocity $v \sim \sqrt{\frac{GM}{R}} = \sqrt{\frac{GNm}{R}}$ we obtain that $\overline{\Delta v^2} = \frac{8}{N} \ln N v^2$

Thus, it takes of the order of $N/(8 \ln N)$ crossings for the net effect of weak encounters to be such that $(\Delta v_\perp)^2 \sim v^2$

This is called the two-body relaxation time $$\tau_{\rm relax} = \frac{N}{8\,\ln N} \tau_{\rm cross}$$

Weak vs. Strong Encounters

For comparison, we now compute how long it takes for strong collisions to have a comparable effect

Since a single strong collision already causes $(\Delta v_{\perp})^2 \geq v^2$ we just need to calculate the collision time for strong collisions

$$\tau_{\rm coll} = (n\sigma v)^{-1}$$

$$n = 3N/4\pi R^3$$

$$\sigma = \pi(2b_{90})^2 = 4\pi b_{90}^2$$

→

$$\tau_{\rm coll}^{\rm strong} = \frac{1}{3N}\left(\frac{R}{b_{90}}\right)^2 \tau_{\rm cross}$$

$$b_{90} = \frac{2Gm}{v^2}$$

$$v^2 = \frac{GNm}{R}$$

→

$$\tau_{\rm coll}^{\rm strong} = \frac{N}{12}\tau_{\rm cross} = \frac{2}{3}\ln N\,\tau_{\rm relax}$$

The net impact of **weak encounters** is of order $\ln N$ times as important as that of **strong encounters**

The Diffusion Coefficients:

Solving the Fokker-Planck equation basically boils down to computing the diffusion coefficients $D[\Delta w_i]$ and $D[\Delta w_i \Delta w_j]$ with w_i being a 6D phase-space vector

In general, doing this in 6D phase-space is extremely complicated. However, we can simplify things

From the expression for $\overline{\Delta v^2}$ we see that each octave in impact parameter makes an equal contribution

Let us use this to derive the impact parameter $b_{1/2}$ such that encounters with $b < b_{1/2}$ contribute 50 percent to the total. This requires solving

$$\ln(b_{1/2}) - \ln(b_{\rm min}) = \ln(b_{\rm max}) - \ln(b_{\rm min})$$

$$b_{\rm max} = R$$

$$b_{\rm max}/b_{\rm min} = \Lambda \sim N$$

$$\Rightarrow \quad b_{1/2} = \frac{R}{\sqrt{N}}$$

More than 50 percent of the total impact of collisions is due to those with an impact parameter that is significantly smaller than the mean particle separation $\lambda_{\rm int} = R/N^{1/3}$

⇒ local approximation is justified (unless resonance effects are important…)

$|\Delta x| \ll |\Delta v|$ this is true because encounter time b/v is much smaller than orbital time

(you will derive this in one of the problem sets)

The Diffusion Coefficients:

Hence, if we pick $\vec{w} = (\vec{x}, \vec{v})$ then we are justified in setting $D[\Delta x_i] = D[\Delta x_i \Delta x_j] = D[\Delta x_i \Delta v_j] = 0$ and the Fokker-Planck equation simplifies to

$$\left(\frac{\partial f}{\partial t}\right)_{\rm c} = -\sum_{i=1}^{3} \frac{\partial}{\partial v_i} \left\{ D[\Delta v_i]\, f(\vec{w}) \right\} + \frac{1}{2} \sum_{i,j=1}^{3} \frac{\partial^2}{\partial v_i \partial v_j} \left\{ D[\Delta v_i \Delta v_j]\, f(\vec{w}) \right\}$$

and we are left with the task to compute $D[\Delta v_i]$ and $D[\Delta v_i \Delta v_j]$

Working out how encounters with impact parameter *b* between a field particle and a subject mass impact the velocity of the latter and computing the expectation values $\langle \Delta v_i \rangle$ and $\langle \Delta v_i \, \Delta v_j \rangle$ by integrating over *b* and the velocity distribution $f(v)$ of the field particles, yields

$$D[\Delta v_i] = \langle \Delta v_i \rangle = 4\pi G^2 m_{\rm a} (m + m_{\rm a}) \ln \Lambda \, \frac{\partial}{\partial v_i} h(\vec{v})$$

$$D[\Delta v_i \Delta v_j] = \langle \Delta v_i \Delta v_j \rangle = 4\pi G^2 m_{\rm a}^2 \ln \Lambda \, \frac{\partial^2}{\partial v_i \partial v_j} g(\vec{v})$$

with $h(v)$ and $g(v)$ known as the Rosenbluth potentials, given by

$$h(\vec{v}) \equiv \int {\rm d}^3 \vec{v}_{\rm a} \frac{f_{\rm a}(\vec{v}_{\rm a})}{|\vec{v} - \vec{v}_{\rm a}|}$$

$$g(\vec{v}) \equiv \int {\rm d}^3 \vec{v}_{\rm a} f_{\rm a}(\vec{v}_{\rm a}) \, |\vec{v} - \vec{v}_{\rm a}|$$

(see Binney & Tremaine for detailed derivation)

The Diffusion Coefficients:

If the velocity distribution of field particles is isotropic, this simplifies to

$$\begin{aligned} D[\Delta v_{\|}] &= -16\pi^2 G^2 m_{\rm a}(m+m_{\rm a}) \ln\Lambda\, \mathcal{E}_2(v) \\ D[(\Delta v_{\|})^2] &= \frac{32}{3}\pi^2 G^2 m_{\rm a}^2 \ln\Lambda\, v\, [\mathcal{E}_4(v) + \mathcal{F}_1(v)] \\ D[(\Delta v_{\perp})^2] &= \frac{32}{3}\pi^2 G^2 m_{\rm a}^2 \ln\Lambda\, v\, [3\mathcal{E}_2(v) - \mathcal{E}_4(v) + 2\mathcal{F}_1(v)] \end{aligned}$$

where

$$\begin{aligned} \mathcal{E}_n(v) &= \int_0^v \left(\frac{v_{\rm a}}{v}\right)^n f_{\rm a}(v_{\rm a}) {\rm d}v_{\rm a} \\ \mathcal{F}_n(v) &= \int_v^\infty \left(\frac{v_{\rm a}}{v}\right)^n f_{\rm a}(v_{\rm a}) {\rm d}v_{\rm a} \end{aligned}$$

It is straightforward to compute related diffusion coefficients, i.e.,

$$\begin{aligned} D[\Delta E] &= \frac{1}{2}\langle(\vec{v}+\Delta\vec{v})^2 - \vec{v}^2\rangle = \langle \Delta v \cdot \vec{v}\rangle + \langle \Delta\vec{v}\cdot\Delta\vec{v}\rangle \\ &= v\, D[(\Delta v)_{\|}] + \frac{1}{2}D[((\Delta\vec{v})_{\|})^2] + \frac{1}{2}D[((\Delta\vec{v})_{\perp})^2] \end{aligned}$$

$$D[(\Delta E)^2] = v^2 D[(\Delta\vec{v})_{\|})^2]$$

The Diffusion Coefficients:

Now that we can compute the diffusion coefficient due to weak gravitational encounters we can compute a more accurate (and more local) estimate of the two-body relaxation time

Since the two-body relaxation time is defined as the time scale on which the cumulative effect of two-body collisions becomes significant, we have that

$$\tau_{\rm relax} = \frac{v_{\rm rms}^2}{D[(\Delta v_{\parallel})^2]}$$

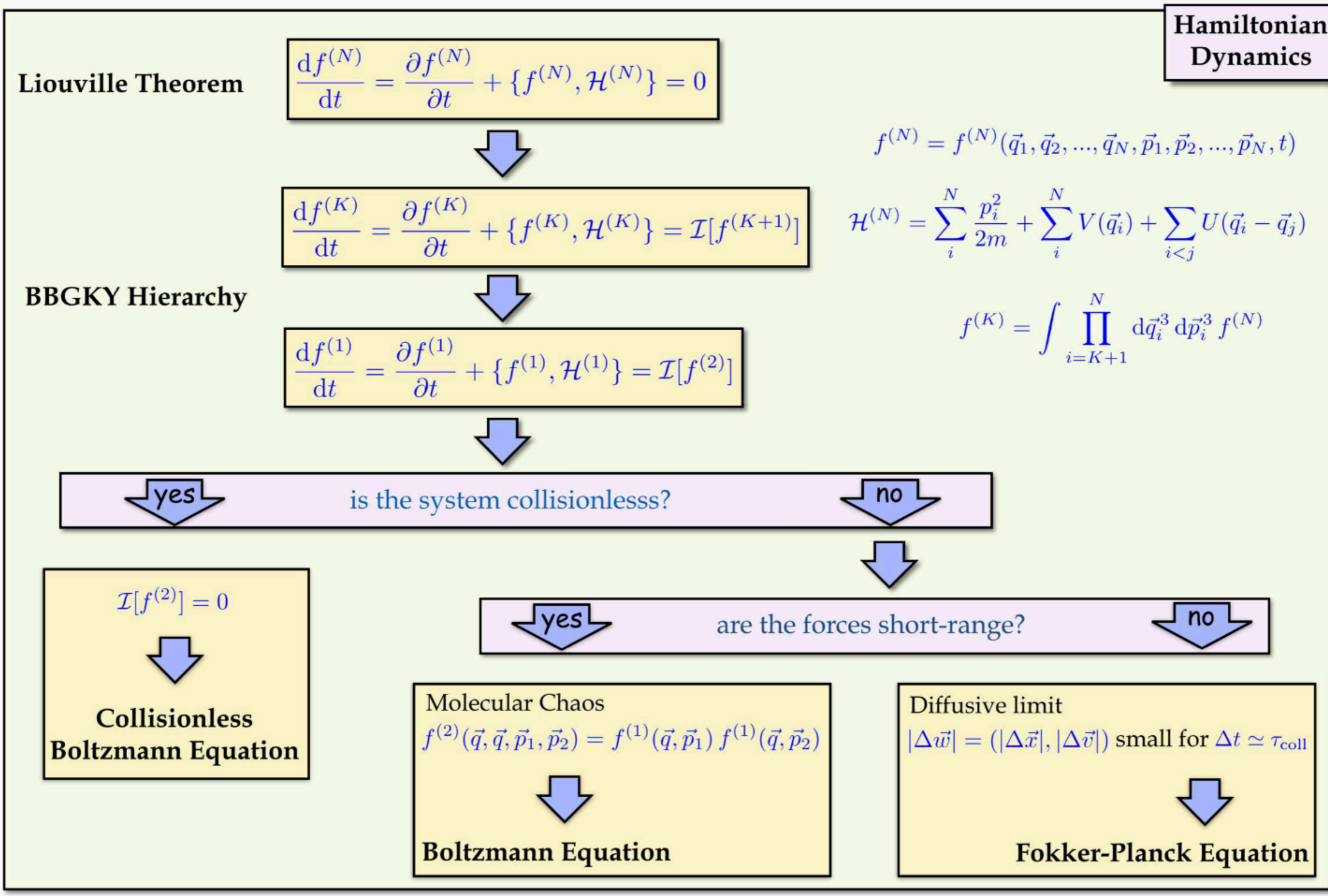
Assuming that the velocity distribution of field particles is isotropic and Maxwellian, that $v^2_{\rm rms} = \sigma^2$, and that the typical speed of a particle is equal to $v = \sqrt{3}\,\sigma$, one obtains

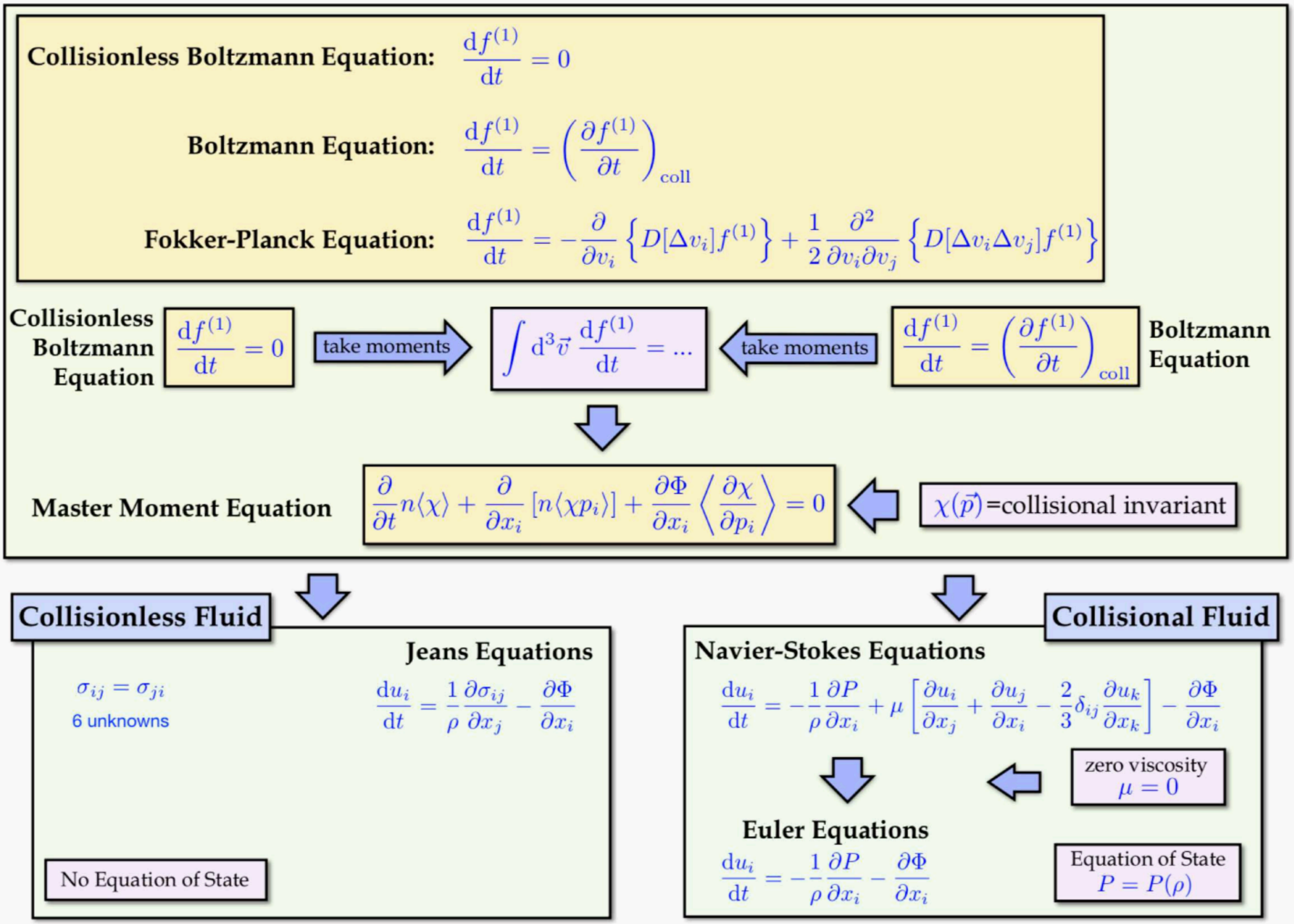
$$\boxed{\tau_{\rm relax} = 0.34 \frac{\sigma^3}{G^2\, m\, \rho\, \ln\Lambda}}$$

Unlike the more common $\tau_{\rm relax} = N/(8\ln N)\,\tau_{\rm cross}$ this expression is based on local quantities

Solar neighborhood: $(\sigma = 30\ {\rm km\,s^{-1}},\ \rho = 0.04\ {\rm M_\odot\ pc^{-3}},\ m = 1\ {\rm M_\odot})$

$$\tau_{\rm relax} = 6 \times 10^{14}\ {\rm yr}(\ln\Lambda/18.5)^{-1}$$


Hamiltonian Dynamics
Liouville Theorem
$\frac{\mathrm{d}f^{(N)}}{\mathrm{d}t} = \frac{\partial f^{(N)}}{\partial t} + \{f^{(N)}, \mathcal{H}^{(N)}\} = 0$
$f^{(N)} = f^{(N)}(\vec{q}_1, \vec{q}_2, ..., \vec{q}_N, \vec{p}_1, \vec{p}_2, ..., \vec{p}_N, t)$
$\frac{\mathrm{d}f^{(K)}}{\mathrm{d}t} = \frac{\partial f^{(K)}}{\partial t} + \{f^{(K)}, \mathcal{H}^{(K)}\} = \mathcal{I}[f^{(K+1)}]$
$\mathcal{H}^{(N)} = \sum_i^N \frac{p_i^2}{2m} + \sum_i^N V(\vec{q}_i) + \sum_{i<j} U(\vec{q}_i - \vec{q}_j)$
BBGKY Hierarchy
$f^{(K)} = \int \prod_{i=K+1}^{N} \mathrm{d}\vec{q}_i^{\,3}\, \mathrm{d}\vec{p}_i^{\,3}\, f^{(N)}$
$\frac{\mathrm{d}f^{(1)}}{\mathrm{d}t} = \frac{\partial f^{(1)}}{\partial t} + \{f^{(1)}, \mathcal{H}^{(1)}\} = \mathcal{I}[f^{(2)}]$
yes
is the system collisionlesss?
no
$\mathcal{I}[f^{(2)}] = 0$
Collisionless Boltzmann Equation
yes
are the forces short-range?
no
Molecular Chaos
$f^{(2)}(\vec{q}, \vec{q}, \vec{p}_1, \vec{p}_2) = f^{(1)}(\vec{q}, \vec{p}_1)\, f^{(1)}(\vec{q}, \vec{p}_2)$
Boltzmann Equation
Diffusive limit
$|\Delta\vec{w}| = (|\Delta\vec{x}|, |\Delta\vec{v}|)$ small for $\Delta t \simeq \tau_{\rm coll}$
Fokker-Planck Equation



Collisionless Boltzmann Equation: $\frac{df^{(1)}}{dt} = 0$
Boltzmann Equation: $\frac{df^{(1)}}{dt} = \left(\frac{\partial f^{(1)}}{\partial t}\right)_{coll}$
Fokker-Planck Equation: $\frac{df^{(1)}}{dt} = -\frac{\partial}{\partial v_i}\left\{D[\Delta v_i] f^{(1)}\right\} + \frac{1}{2}\frac{\partial^2}{\partial v_i \partial v_j}\left\{D[\Delta v_i \Delta v_j] f^{(1)}\right\}$
Collisionless Boltzmann Equation
$\frac{df^{(1)}}{dt} = 0$
take moments
$\int d^3\vec{v}\, \frac{df^{(1)}}{dt} = ...$
take moments
$\frac{df^{(1)}}{dt} = \left(\frac{\partial f^{(1)}}{\partial t}\right)_{coll}$
Boltzmann Equation
Master Moment Equation
$\frac{\partial}{\partial t} n\langle\chi\rangle + \frac{\partial}{\partial x_i}\left[n\langle\chi p_i\rangle\right] + \frac{\partial\Phi}{\partial x_i}\left\langle\frac{\partial\chi}{\partial p_i}\right\rangle = 0$
$\chi(\vec{p})$=collisional invariant
Collisionless Fluid
Jeans Equations
$\sigma_{ij} = \sigma_{ji}$
6 unknowns
$\frac{du_i}{dt} = \frac{1}{\rho}\frac{\partial\sigma_{ij}}{\partial x_j} - \frac{\partial\Phi}{\partial x_i}$
No Equation of State
Collisional Fluid
Navier-Stokes Equations
$\frac{du_i}{dt} = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} + \mu\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\frac{\partial u_k}{\partial x_k}\right] - \frac{\partial\Phi}{\partial x_i}$
zero viscosity
$\mu = 0$
Euler Equations
$\frac{du_i}{dt} = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} - \frac{\partial\Phi}{\partial x_i}$
Equation of State
$P = P(\rho)$