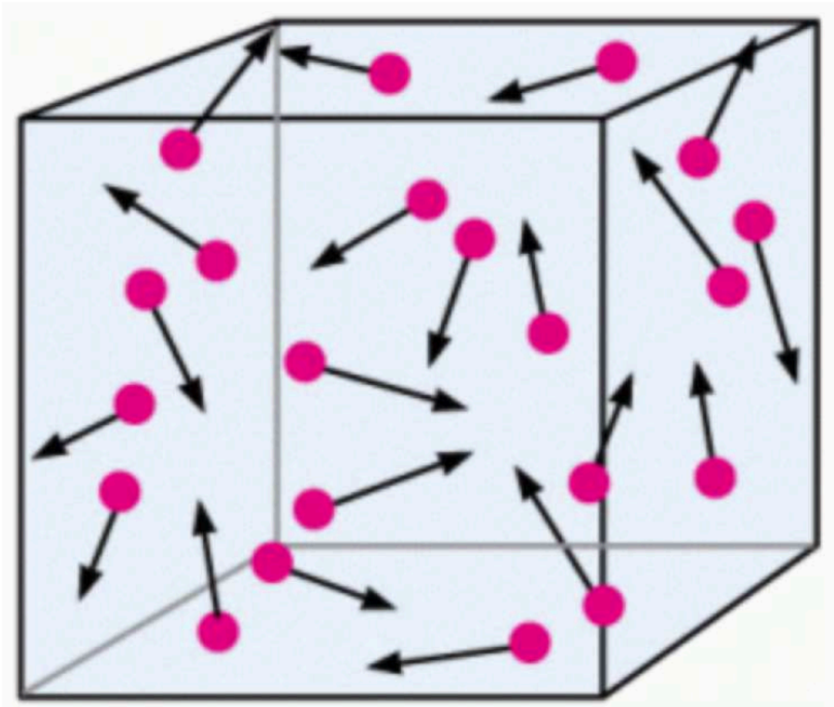
LECTURE 6

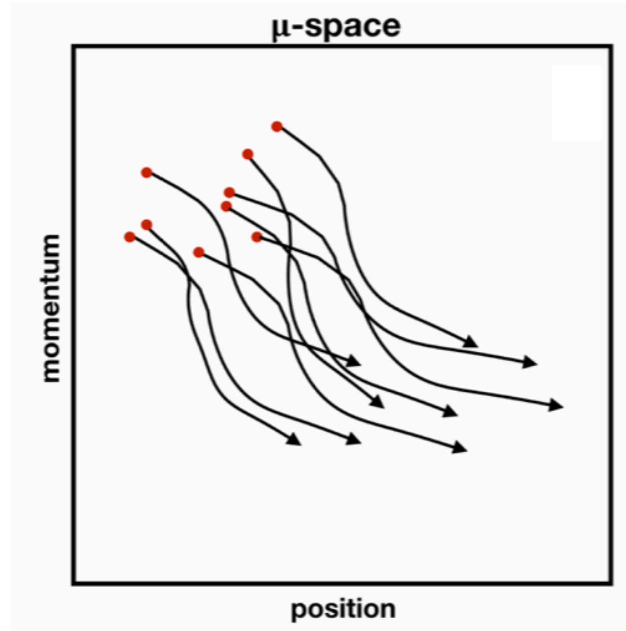
Part II: Kinetic Theory



Part II: Kinetic Theory

N-particle Phase-Space (Γ-space):

the 6N-dimensional phase-space of a dynamical system is a space in which all possible states of a system are represented, which each possible space corresponding to one unique point in that phase-space.



1-particle Phase-Space (μ-space):

the 6-dimensional phase-space of a dynamical system is a space covering all possible phase-space coordinates of individual particles. Each particles corresponding to one point in that phase-space.

Note: unlike in Γ-space, in which two trajectories can *never* intersect one-another, in μ-space the trajectories (of individual particles) *can* cross one another.

The BBGKY Hierarchy

$$\frac{\mathrm{d}f^{(N)}}{\mathrm{d}t} = \frac{\partial f^{(N)}}{\partial t} + \{f^N, \mathcal{H}\} = 0$$

Liouville Theorem

$$\frac{\partial f^{(k)}}{\partial t} = \{\mathcal{H}^{(k)}, f^{(k)}\} + \sum_{i=1}^{k} \int \mathrm{d}^3\vec{q}_{k+1}\, \mathrm{d}^3\vec{p}_{k+1}\, \frac{\partial U(|\vec{q}_i - \vec{q}_{k+1}|)}{\partial \vec{q}_i} \cdot \frac{\partial f^{(k+1)}}{\partial \vec{p}_i}$$

$$\frac{\partial f^{(1)}}{\partial t} = \{\mathcal{H}^{(1)}, f^{(1)}\} + \int \mathrm{d}^3\vec{q}_2\, \mathrm{d}^3\vec{p}_2\, \frac{\partial U(|\vec{q}_1 - \vec{q}_2|)}{\partial \vec{q}_1} \cdot \frac{\partial f^{(2)}}{\partial \vec{p}_1}$$

Hamiltonian $\mathcal{H}(\vec{q}_i, \vec{p}_i) = \sum_{i=1}^{N} \frac{\vec{p}_i^{\,2}}{2m} + \sum_{i=1}^{N} V(\vec{q}_i) + \frac{1}{2} \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} U(|\vec{q}_i - \vec{q}_j|)$

external potential

2-body interaction potential

reduced k-particle DF $f^{(k)}(\vec{w}_1, \vec{w}_2, ..., \vec{w}_k, t) \equiv \frac{N!}{(N-k)!} \int \prod_{i=k+1}^{N} \mathrm{d}^6\vec{w}_i\, f^{(N)}(\vec{w}_1, \vec{w}_2, ..., \vec{w}_N, t)$

one-particle DF $f^{(1)}(\vec{w}_1, t) \equiv N \int \prod_{i=2}^{N} \mathrm{d}^6\vec{w}_i\, f^{(N)}(\vec{w}_1, \vec{w}_2, ..., \vec{w}_N, t)$ → $f^{(1)}(\vec{q}, \vec{p}, t) = \mathrm{d}N/\mathrm{d}^3\vec{q}\,\mathrm{d}^3\vec{p}$

The Mayer Cluster Expansion

$$\boxed{f^{(2)}(\vec{q}_1, \vec{q}_2, \vec{p}_1, \vec{p}_2) = f^{(1)}(\vec{q}_1, \vec{p}_1)\, f^{(1)}(\vec{q}_2, \vec{p}_2) + g(\vec{q}_1, \vec{q}_2, \vec{p}_1, \vec{p}_2)}$$

↓ short-hand notation

$$f^{(2)}(1,2) = f^{(1)}(1)\, f^{(1)}(2) + g(1,2)$$

2-point correlation function

$$f^{(3)}(1,2,3) = f(1)\, f(2)\, f(3) + f(1)\, g(2,3) + f(2)\, g(1,3) + f(3)\, g(1,2) + h(1,2,3)$$

3-point correlation function

etc.

Correlations are induced by collisions (interactions) among the particles

[A] Collisionless System → $g(1,2)=0$ → Collisionless Boltzmann Equation (CBE)

$$\frac{\mathrm{d}f^{(1)}}{\mathrm{d}t} = \frac{\partial f^{(1)}}{\partial t} + \{f^{(1)}, \mathcal{H}^{(1)}\} = \frac{\partial f^{(1)}}{\partial t} + \vec{v} \cdot \frac{\partial f^{(1)}}{\partial \vec{x}} - \nabla\Phi \cdot \frac{\partial f^{(1)}}{\partial \vec{v}} = 0$$

[B] System with short-range collisions (neutral gas or liquid)

assumption of molecular chaos; $f^{(2)}(\vec{q}, \vec{q}, \vec{p}_1, \vec{p}_2) = f^{(1)}(\vec{q}, \vec{p}_1)\, f^{(1)}(\vec{q}, \vec{p}_2)$

→ Boltzmann Equation

$$\frac{\mathrm{d}f^{(1)}}{\mathrm{d}t} = \frac{\partial f^{(1)}}{\partial t} + \{f^{(1)}, \mathcal{H}^{(1)}\} = \frac{\partial f^{(1)}}{\partial t} + \vec{v} \cdot \frac{\partial f^{(1)}}{\partial \vec{x}} - \nabla\Phi \cdot \frac{\partial f^{(1)}}{\partial \vec{v}} = I[f^{(1)}]$$

[C] System with long-range collisions (collisional Plasma or low-N gravitational system)

assumptions; $h(1,2,3)=0$ + fluid is homogeneous + $g(1,2)$ relaxes faster than $f^{(1)}$

→ Lenard-Balescu equation

$$\frac{\partial f(\vec{v},t)}{\partial t} = -\frac{8\pi^4 n_\mathrm{e}}{m_\mathrm{e}^2} \frac{\partial}{\partial \vec{v}} \int \mathrm{d}\vec{k}\, \mathrm{d}\vec{v}'\, \vec{k}\vec{k} \cdot \frac{\phi^2(k)}{|\varepsilon(\vec{k}, \vec{k}\cdot\vec{v})|^2}\, \delta[\vec{k}\cdot(\vec{v}-\vec{v}')] \left[f(\vec{v}) \frac{\partial f}{\partial \vec{v}'} - f(\vec{v}') \frac{\partial f}{\partial \vec{v}} \right]$$

see chapter 27

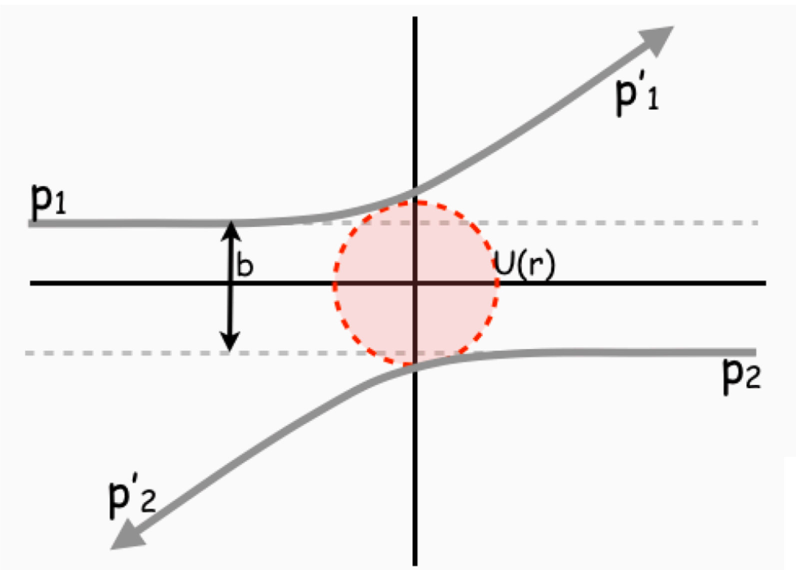
here $f = f^{(1)}$

From Boltzmann to Navier-Stokes

In what follows we focus on the 1-particle distribution function f, dropping the $^{(1)}$-superscript

We consider collision integral term in the Boltzmann equation:

$$I[f] = (\partial f/\partial t)_{\rm coll} = \int \mathrm{d}^3\vec{q}_2\, \mathrm{d}^3\vec{p}_2\, \frac{\partial U(|\vec{q}_1 - \vec{q}_2|)}{\partial \vec{q}_1} \cdot \frac{\partial f^{(2)}}{\partial \vec{p}_1}$$



consider the following elastic two-particle collision

$$\vec{p}_1 + \vec{p}_2 \to \vec{p}_1{}' + \vec{p}_2{}'$$

these collisions obey:

momentum conservation: $\vec{p}_1 + \vec{p}_2 = \vec{p}_1{}' + \vec{p}_2{}'$

energy conservation: $|\vec{p}_1|^2 + |\vec{p}_2|^2 = |\vec{p}_1{}'|^2 + |\vec{p}_2{}'|^2$

Write the rate at which particles of momentum $\boldsymbol{p}_1$ at $\boldsymbol{x}$ experience collisions $\vec{p}_1 + \vec{p}_2 \to \vec{p}_1{}' + \vec{p}_2{}'$ as:

$$\mathcal{R} = \omega(\vec{p}_1, \vec{p}_2 | \vec{p}_1{}', \vec{p}_2{}')\, f^{(2)}(\vec{x}, \vec{x}, \vec{p}_1, \vec{p}_2)\, \mathrm{d}^3\vec{p}_2\, \mathrm{d}^3\vec{p}_1{}'\, \mathrm{d}^3\vec{p}_2{}'$$

The function $\omega(\vec{p}_1, \vec{p}_2 | \vec{p}_1{}', \vec{p}_2{}')$ depends on the interaction potential $U(\mathbf{r})$ and can be calculated (in principle) via differential cross sections

Momentum & energy conservation → $\omega(\vec{p}_1, \vec{p}_2 | \vec{p}_1{}', \vec{p}_2{}') \propto \delta^3(\vec{P} - \vec{P}')\, \delta(E - E')$

with $\vec{P} = \vec{p}_1 + \vec{p}_2$ and $\vec{P}' = \vec{p}_1{}' + \vec{p}_2{}'$

Time reversibility → $\omega(\vec{p}_1, \vec{p}_2 | \vec{p}_1{}', \vec{p}_2{}') = \omega(\vec{p}_1{}' \vec{p}_2{}' | \vec{p}_1, \vec{p}_2)$

Using principle of molecular chaos → $f^{(2)}(\vec{x}, \vec{x}, \vec{p}_1, \vec{p}_2) = f^{(1)}(\vec{x}, \vec{p}_1)\, f^{(1)}(\vec{x}, \vec{p}_2)$

$$\Rightarrow \boxed{I[f] = \int \mathrm{d}^3\vec{p}_2\, \mathrm{d}^3\vec{p}_1{}'\, \mathrm{d}^3\vec{p}_2{}'\, \omega(\vec{p}_1{}', \vec{p}_2{}' | \vec{p}_1, \vec{p}_2)\, [f(\vec{p}_1{}')\, f(\vec{p}_2{}') - f(\vec{p}_1)\, f(\vec{p}_2)]}$$

replenishing collisions — depleting collisions

for brevity we no longer write out the explicit *x*-dependence of the DF

What can we learn about the equilibrium distribution function, $f_{\rm eq}(\boldsymbol{x},\boldsymbol{p})$?

Equilibrium → $\partial f_{\rm eq}/\partial t = 0$

Ignore external potential & spatial homogeneity → $\{\mathcal{H}, f_{\rm eq}\} = 0$

$\Big\}$ $I[f] = 0$

$$I[f] = \int \mathrm{d}^3\vec{p}_2\, \mathrm{d}^3\vec{p}_1{}'\, \mathrm{d}^3\vec{p}_2{}'\, \omega(\vec{p}_1{}', \vec{p}_2{}' | \vec{p}_1, \vec{p}_2)\, \left[f(\vec{p}_1{}')\, f(\vec{p}_2{}') - f(\vec{p}_1)\, f(\vec{p}_2)\right] = 0$$

Detailed balance → $f(\vec{x}, \vec{p}_1{}')\, f(\vec{x}, \vec{p}_2{}') - f(\vec{x}, \vec{p}_1)\, f(\vec{x}, \vec{p}_2) = 0$

→ $\log[f(\vec{p}_1)] + \log[f(\vec{p}_2)] = \log[f(\vec{p}_1{}')] + \log[f(\vec{p}_2{}')]$

This has form of a conservation law, and suggests that log[*f*] must be equal to sum of conserved quantities, *A*(***p***), that obey $A(\vec{p}_1) + A(\vec{p}_2) = A(\vec{p}_1{}') + A(\vec{p}_2{}')$

We have the following collisional invariants:

$A = 1$	**particle number conservations**
$A = \vec{p}$	**momentum conservation**
$A = \vec{p}^{\,2}/(2m)$	**energy conservation**

This therefore suggests that $\log[f_{\rm eq}(\vec{p})] \propto a_1 + a_2\, \vec{p} + a_3\, |\vec{p}|^2$

This therefore suggests that $\log[f_{\rm eq}(\vec{p})] \propto a_1 + a_2\,\vec{p} + a_3\,|\vec{p}|^2$

It can be shown that this implies the Maxwell-Boltzmann distribution

$$f_{\rm eq}(p) = \frac{n}{(2\pi m k_{\rm B} T)^{3/2}} \exp\left[-\frac{p^2}{2 m k_{\rm B} T}\right]$$

In other words: the MB-distribution is the equilibrium solution of the Boltzmann equation

It can be shown that $$\boxed{\int \mathrm{d}^3\vec{p}\, A(\vec{p}) \left(\frac{\partial f}{\partial t}\right)_{\mathrm{coll}} = 0}$$

Proof:

define $$\mathcal{I}_1 = \int \mathrm{d}^3\vec{p}_1\, \mathrm{d}^3\vec{p}_2\, \mathrm{d}^3\vec{p}_1{}'\, \mathrm{d}^3\vec{p}_2{}'\, \omega(\vec{p}_1{}', \vec{p}_2{}' | \vec{p}_1, \vec{p}_2)\, A(\vec{p}_1)\, [f(\vec{p}_1{}')\, f(\vec{p}_2{}') - f(\vec{p}_1)\, f(\vec{p}_2)]$$

re-labelling $1 \leftrightarrow 2$ and re-ordering yields

$$\mathcal{I}_2 = \int \mathrm{d}^3\vec{p}_1\, \mathrm{d}^3\vec{p}_2\, \mathrm{d}^3\vec{p}_1{}'\, \mathrm{d}^3\vec{p}_2{}'\, \omega(\vec{p}_1{}', \vec{p}_2{}' | \vec{p}_1, \vec{p}_2)\, A(\vec{p}_2)\, [f(\vec{p}_1{}')\, f(\vec{p}_2{}') - f(\vec{p}_1)\, f(\vec{p}_2)]$$

Starting from the first expression and swapping $p_1 \leftrightarrow p'_1$ yields

$$\mathcal{I}_3 = -\int \mathrm{d}^3\vec{p}_1\, \mathrm{d}^3\vec{p}_2\, \mathrm{d}^3\vec{p}_1{}'\, \mathrm{d}^3\vec{p}_2{}'\, \omega(\vec{p}_1, \vec{p}_2 | \vec{p}_1{}', \vec{p}_2{}')\, A(\vec{p}_1{}')\, [f(\vec{p}_1{}')\, f(\vec{p}_2{}') - f(\vec{p}_1)\, f(\vec{p}_2)]$$

re-labelling $1 \leftrightarrow 2$ and re-ordering yields

$$\mathcal{I}_4 = -\int \mathrm{d}^3\vec{p}_1\, \mathrm{d}^3\vec{p}_2\, \mathrm{d}^3\vec{p}_1{}'\, \mathrm{d}^3\vec{p}_2{}'\, \omega(\vec{p}_1, \vec{p}_2 | \vec{p}_1{}', \vec{p}_2{}')\, A(\vec{p}_2{}')\, [f(\vec{p}_1{}')\, f(\vec{p}_2{}') - f(\vec{p}_1)\, f(\vec{p}_2)]$$

Time reversibility $\omega(\vec{p}_1{}', \vec{p}_2{}' | \vec{p}_1, \vec{p}_2) = \omega(\vec{p}_1, \vec{p}_2 | \vec{p}_1{}', \vec{p}_2{}')$ implies that $\mathcal{I}_4 = \mathcal{I}_3 = \mathcal{I}_2 = \mathcal{I}_1$

and thus $\mathcal{I}_1 = [\mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4]/4$

$$\mathcal{I}_1 = [\mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4]/4$$

substituting the expressions from the previous page:

$$\mathcal{I}_1 = \frac{1}{4}\int \mathrm{d}^3\vec{p}_1\, \mathrm{d}^3\vec{p}_2\, \mathrm{d}^3\vec{p}_1{}'\, \mathrm{d}^3\vec{p}_2{}'\, \omega(\vec{p}_1{}', \vec{p}_2{}'|\vec{p}_1, \vec{p}_2) \times \\ \{A(\vec{p}_1) + A(\vec{p}_2) - A(\vec{p}_1{}') - A(\vec{p}_2{}')\}\ [f(\vec{p}_1{}')\, f(\vec{p}_2{}') - f(\vec{p}_1)\, f(\vec{p}_2)]$$

$A(\boldsymbol{p})$ is a collisional invariant → $A(\vec{p}_1) + A(\vec{p}_2) - A(\vec{p}_1{}') - A(\vec{p}_2{}') = 0$ → $\mathcal{I}_1 = 0$

Q.E.D.

As long as $A(\boldsymbol{p})$ is a collisional invariant we have that

$$\boxed{\int \mathrm{d}^3\vec{p}\, A(\vec{p}) \left(\frac{\partial f}{\partial t}\right)_{\mathrm{coll}} = 0}$$

we will use this shortly to obtain the Navier-Stokes equations from the Boltzmann equation

Solving the Boltzmann equation

$$\boxed{\frac{\mathrm{d}f^{(1)}}{\mathrm{d}t} = \frac{\partial f^{(1)}}{\partial t} + \{f^{(1)}, \mathcal{H}^{(1)}\} = \frac{\partial f^{(1)}}{\partial t} + \vec{v}\cdot\frac{\partial f^{(1)}}{\partial \vec{x}} - \nabla\Phi\cdot\frac{\partial f^{(1)}}{\partial \vec{v}} = I[f^{(1)}]}$$

for the 7-dimensional DF $f\,(\boldsymbol{x},\boldsymbol{v},t)$ is a non-trivial task

Rather, we are going to solve moment equations of the Boltzmann equation

Consider a scalar function $Q(\boldsymbol{v})$. The expectation value for Q at location $\boldsymbol{x}$ at time t is given by

$$\langle Q\rangle = \langle Q\rangle(\vec{x},t) = \frac{\int Q(\vec{v})\, f(\vec{x},\vec{v},t)\,\mathrm{d}^3\vec{v}}{\int f(\vec{x},\vec{v},t)\,\mathrm{d}^3\vec{v}}$$

using that

$$n = n(\vec{x},t) = \int f(\vec{x},\vec{v},t)\,\mathrm{d}^3\vec{v}$$

we see that

$$\boxed{\int Q(\vec{v})\, f(\vec{x},\vec{v},t)\,\mathrm{d}^3\vec{v} = n\,\langle Q\rangle}$$

Define $g(\vec{x},t) = \int Q(\vec{v})\, f(\vec{x},\vec{v},t)\, \mathrm{d}^3\vec{v}$

$Q(\vec{v}) = 1$	$\Rightarrow$	$g(\vec{x},t) = n(\vec{x},t)$	number density
$Q(\vec{v}) = m$	$\Rightarrow$	$g(\vec{x},t) = \rho(\vec{x},t)$	mass density
$Q(\vec{v}) = m\,\vec{v}$	$\Rightarrow$	$g(\vec{x},t) = \rho(\vec{x},t)\,\vec{u}(\vec{x},t)$	momentum flux density
$Q(\vec{v}) = \frac{1}{2}m(\vec{v}-\vec{u})^2$	$\Rightarrow$	$g(\vec{x},t) = \rho(\vec{x},t)\,\varepsilon(\vec{x},t)$	specific energy density

these are our macroscopic quantities of interest…

NOTE: ***u***(***x***,*t*) = <***v***(***x***,*t*)> is the mean velocity of all particles at location ***x*** at time *t*

Rather than solving the Boltzmann equation $\frac{\partial f}{\partial t} + \vec{v} \cdot \nabla f - \nabla \Phi \cdot \frac{\partial f}{\partial \vec{v}} = \left(\frac{\partial f}{\partial t}\right)_{\rm coll}$

we seek to solve the following moment equations:

$$\int Q(\vec{v}) \left[\frac{\partial f}{\partial t} + \vec{v} \cdot \nabla f - \nabla \Phi \cdot \frac{\partial f}{\partial \vec{v}}\right] {\rm d}^3\vec{v} = \int Q(\vec{v}) \left(\frac{\partial f}{\partial t}\right)_{\rm coll} {\rm d}^3\vec{v}$$

In particular, we will focus on Q(***v***) that are collisional invariants for which the rhs vanishes

$$\int Q(\vec{v}) \frac{\partial f}{\partial t} {\rm d}^3\vec{v} + \int Q(\vec{v})\, \vec{v} \cdot \nabla f \, {\rm d}^3\vec{v} - \int Q(\vec{v})\, \nabla \Phi \cdot \frac{\partial f}{\partial \vec{v}} {\rm d}^3\vec{v} = 0$$

NOTE: if we had started with the CBE, rather than the Boltzmann equation, we would have obtained the same moment equations.

HENCE: what follows is valid for both collisional (short-range collisions) and collisionless systems

$$\int Q(\vec{v})\,\frac{\partial f}{\partial t}\,\mathrm{d}^3\vec{v} + \int Q(\vec{v})\,\vec{v}\cdot\nabla f\,\mathrm{d}^3\vec{v} - \int Q(\vec{v})\,\nabla\Phi\cdot\frac{\partial f}{\partial \vec{v}}\,\mathrm{d}^3\vec{v} = 0$$

I + **II** - **III** $= 0$

I	$\int Q(\vec{v})\,\frac{\partial f}{\partial t}\,\mathrm{d}^3\vec{v}$
II	$\int Q(\vec{v})\,v_i\frac{\partial f}{\partial x_i}\,\mathrm{d}^3\vec{v}$
III	$\int Q(\vec{v})\,\frac{\partial \Phi}{\partial x_i}\,\frac{\partial f}{\partial v_i}\,\mathrm{d}^3\vec{v}$

Integral I

$$\int Q(\vec{v})\,\frac{\partial f}{\partial t}\,\mathrm{d}^3\vec{v} = \int \frac{\partial Q f}{\partial t}\,\mathrm{d}^3\vec{v} = \frac{\partial}{\partial t}\int Q f\,\mathrm{d}^3\vec{v} = \frac{\partial}{\partial t} n\langle Q\rangle$$

Integral II

$$\int Q(\vec{v})\,v_i\frac{\partial f}{\partial x_i}\,\mathrm{d}^3\vec{v} = \int \frac{\partial Q\,v_i\,f}{\partial x_i}\,\mathrm{d}^3\vec{v} = \frac{\partial}{\partial x_i}\int Q\,v_i\,f\,\mathrm{d}^3\vec{v} = \frac{\partial}{\partial x_i}\Big[n\,\langle Q\,v_i\rangle\Big]$$

where we have used that

$$Q\,v_i\,\frac{\partial f}{\partial x_i} = \frac{\partial (Q\,v_i\,f)}{\partial x_i} - f\,\frac{\partial Q\,v_i}{\partial x_i} = \frac{\partial (Q\,v_i\,f)}{\partial x_i}$$

Integral III

$$\begin{aligned}\int Q\,\vec{F}\cdot\nabla_v f\,\mathrm{d}^3\vec{v} &= \int \nabla_v\cdot(Qf\vec{F})\mathrm{d}^3\vec{v} - \int f\,\nabla_v\cdot(Q\vec{F})\,\mathrm{d}^3\vec{v}\\ &= \int Qf\vec{F}\mathrm{d}^2S_v - \int f\,\frac{\partial QF_i}{\partial v_i}\,\mathrm{d}^3\vec{v}\\ &= -\int fQ\frac{\partial F_i}{\partial v_i}\,\mathrm{d}^3\vec{v} - \int fF_i\,\frac{\partial Q}{\partial v_i}\,\mathrm{d}^3\vec{v}\\ &= -\int f\frac{\partial \Phi}{\partial x_i}\,\frac{\partial Q}{\partial v_i}\,\mathrm{d}^3\vec{v} = -\frac{\partial \Phi}{\partial x_i}n\left\langle\frac{\partial Q}{\partial v_i}\right\rangle\end{aligned}$$

I + II - III $= 0$ ➡ $$\boxed{\frac{\partial}{\partial t}n\langle Q\rangle + \frac{\partial}{\partial x_i}\left[n\langle Qv_i\rangle\right] + \frac{\partial \Phi}{\partial x_i}n\left\langle\frac{\partial Q}{\partial v_i}\right\rangle = 0}$$

Master Moment equation

This master moment equation holds for any collisional invariant, Q(***v***), and for both collisionless and collisional (short-range forces only) systems

Let's consider Q=m (mass conservation) and substitute this in the Master Moment Equation

using that $\langle m \rangle = m$, that $mn = \rho$, and that $\langle m v_i \rangle = m \langle v_i \rangle = m u_i$ we obtain that

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_i}{\partial x_i} = 0}$$

which is the continuity equation (in index-form)

Let's consider Q=mv_j (momentum conservation) and substitute this in the Master Moment Equation

using that $n\langle mv_j v_i\rangle = \rho\langle v_i v_j\rangle$, and that $\frac{\partial\Phi}{\partial x_i} n \left\langle \frac{\partial m v_j}{\partial v_i} \right\rangle = \frac{\partial\Phi}{\partial x_i}\rho \left\langle \frac{\partial v_j}{\partial v_i} \right\rangle = \frac{\partial\Phi}{\partial x_i}\rho\delta_{ij} = \rho\frac{\partial\Phi}{\partial x_j}$

one finds that

$$\frac{\partial \rho u_j}{\partial t} + \frac{\partial \rho\langle v_i v_j\rangle}{\partial x_i} + \rho\frac{\partial\Phi}{\partial x_j} = 0$$

As detailed in the lecture notes, this can be manipulated to yield the momentum equations

$$\boxed{\rho\frac{\partial u_j}{\partial t} + \rho u_i \frac{\partial u_j}{\partial x_i} + \frac{\partial\left[\rho\langle v_i v_j\rangle - \rho u_i u_j\right]}{\partial x_i} + \rho\frac{\partial\Phi}{\partial x_j} = 0}$$

In order to make sense of the $\rho\langle v_i v_j\rangle - \rho u_i u_j$ term, it is useful to write the microscopic particle velocity, v, as the sum of a streaming motion, u, and a random velocity, w

$$\boxed{\vec{v} = \vec{u} + \vec{w}}$$

we have that $\langle\vec{w}\rangle = 0$ and $\langle\vec{v}\rangle = \vec{u}$

$\langle\cdot\rangle$ indicates an average over nearby particles (sometimes called a fluid element)

It is convenient to introduce the stress tensor $$\boxed{\sigma_{ij} \equiv -\rho\langle w_i w_j\rangle = -\rho\langle v_i v_j\rangle + \rho u_i u_j}$$

Substituting this in the momentum equations we obtain the more common form:

$$\boxed{\frac{\partial u_j}{\partial t} + u_i \frac{\partial u_j}{\partial x_i} = \frac{1}{\rho}\frac{\partial \sigma_{ij}}{\partial x_i} - \frac{\partial \Phi}{\partial x_j}}$$

Momentum equations

Recall that these apply to both collisional (short-range force) and collisionless systems….

QUESTION: where has the impact of collisions (recall we started with a collision integral) gone??

ANSWER: it is `hidden' in the detailed expression for the stress tensor

Momentum equations
$$\boxed{\frac{\partial u_j}{\partial t} + u_i \frac{\partial u_j}{\partial x_i} = \frac{1}{\rho}\frac{\partial \sigma_{ij}}{\partial x_i} - \frac{\partial \Phi}{\partial x_j}}$$

Collisionless Fluid:

No collisions, there is nothing special we can say about the stress tensor.

Momentum equations take on form above and are called the Jeans equations

Collisional Fluid:

As we will see, as long as the fluid is Newtonian, we have that $\sigma_{ij} = -P\delta_{ij} + \tau_{ij}$

with P the hydrodynamic pressure, and

$$\tau_{ij} = \mu \left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\frac{\partial u_k}{\partial x_k}\right] + \eta\,\delta_{ij}\frac{\partial u_k}{\partial x_k}$$

the deviatoric stress tensor with μ and η the shear viscosity and bulk viscosity, respectively.

Substituting this in the momentum equations yields the Navier-Stokes equations

Setting $\mu = \eta = 0$ (assuming an ideal fluid), these become the Euler equations

Let's consider Q=$mv^2/2$ (energy conservation) and substitute this in the Master Moment Equation

As detailed in Appendix J of the lecture notes, working out the various terms ultimately yields

$$\boxed{\frac{\partial}{\partial t}\left[\rho\left(\frac{u^2}{2}+\varepsilon\right)\right] = -\frac{\partial}{\partial x_k}\left[\rho\left(\frac{u^2}{2}+\varepsilon\right) u_k - \sigma_{jk}u_j + \rho\langle w_k \frac{1}{2} w^2\rangle\right] - \rho u_k \frac{\partial \Phi}{\partial x_k}}$$

which is known as the energy equation (in Lagrangian index-form).

Here $\varepsilon = \frac{1}{2}\langle w^2\rangle$ is the specific internal energy.

As we will see in Part III of the lecture notes, the energy equation (for collisional fluids) can be recast as

$$\rho \frac{\mathrm{d}\varepsilon}{\mathrm{d}t} = -P\frac{\partial u_k}{\partial x_k} + \mathcal{V} - \frac{\partial F_{\mathrm{cond},k}}{\partial x_k}$$

with $\mathcal{V}$ the rate of viscous dissipation, and $\boldsymbol{F}_{\mathrm{cond}}$ the conductive heat flux.