LECTURE 4

Recap: Hamiltonian Dynamics

action

$$S = \int_{t_1}^{t_2} \mathcal{L}(\vec{q}, \dot{\vec{q}}, t)\, \mathrm{d}t$$

least action principle

$$\delta S = 0 \rightarrow$$

Euler-Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0\,, \qquad (i = 1, 2, ..., n)$$

n second-order differential equations

conjugate momentum

$$p_i \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

Hamiltonian

$$\mathcal{H}(\vec{q}, \vec{p}, t) = \dot{\vec{q}} \cdot \vec{p} - \mathcal{L}(\vec{q}, \dot{\vec{q}}, t) \rightarrow$$

Legendre transformation

Hamiltonian equations of motion

$$\dot{\vec{q}} = \frac{\partial \mathcal{H}}{\partial \vec{p}}\,, \qquad \dot{\vec{p}} = -\frac{\partial \mathcal{H}}{\partial \vec{q}}$$

$2n$ first-order differential equations

$$\{A, B\} = \frac{\partial A}{\partial \vec{q}} \cdot \frac{\partial B}{\partial \vec{p}} - \frac{\partial A}{\partial \vec{p}} \cdot \frac{\partial B}{\partial \vec{q}}$$

Poisson brackets

$$\vec{w} = (\vec{q}, \vec{p}) \qquad \dot{\vec{w}} = \{\vec{w}, \mathcal{H}\}$$

$$\{q_i, q_j\} = 0, \qquad \{p_i, p_j\} = 0, \qquad \{q_i, p_j\} = \delta_{ij}$$

canonical commutation relations

For any $f(\boldsymbol{q}, \boldsymbol{p}, t)$

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \{f, \mathcal{H}\} + \frac{\partial f}{\partial t}$$

Let F_2 be the generator of a canonical transformation ($\boldsymbol{q},\boldsymbol{p}$) → ($\boldsymbol{Q},\boldsymbol{P}$) for which all Q are cyclic.

Hamilton-Jacobi equation

$$\mathcal{H}\left(q_i, \frac{\partial F_2}{\partial q_i}\right) = E$$

n-dimensional PDE

Hamilton-Jacobi equation

If we can solve the Hamilton-Jacobi equation, we have a complete, trivial solution for the full dynamics of the Hamiltonian system:

$$Q_i(t) = \omega_i t + Q_i(0)\,, \qquad P_i(t) = P_i(0) \qquad \text{with } \omega_i \equiv \partial\mathcal{H}'/\partial P_i$$

Unfortunately, solving an n-dimensional PDE is extremely hard…

However, if Hamilton's characteristic function is separable, which means can be written as

$$W(\vec{q}, \vec{P}) = \sum_{i=1}^{n} W_i(q_i, \vec{P})$$

then the Hamilton-Jacobi equation reduces to a set of n first-order ODEs, which are easily solved by quadrature (i.e., can be written as n integral equations).

If the Hamilton-Jacobi equation is separable, we say that the Hamiltonian is integrable

Liouville's Theorem of Integrable Systems

If a system with n degrees of freedom has n mutually Poisson commuting integrals of motion $I_1, I_2, \ldots, I_n$ then the system is integrable.

Which of the following Hamiltonian systems is integrable and why?

[1] a single particle with one degree of freedom

$$\mathcal{H}(q,p) = \frac{p^2}{2m} + V(q)$$

ANSWER: yes, no explicit time dependence of Hamiltonian → energy is integral of motion
→ integrable according to Liouville's theorem

Every time-independent Hamiltonian with one dof is integrable

[2] two uncoupled Hamiltonians with one degree of freedom each

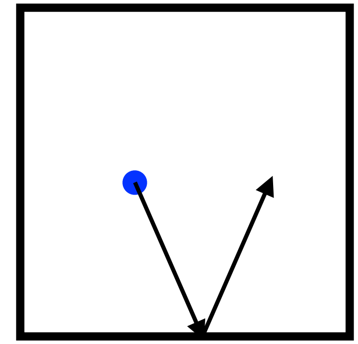
$$\mathcal{H}(q_1, q_2, p_1, p_2) = \mathcal{H}_1(q_1, p_1) + \mathcal{H}_2(q_2, p_2)$$

ANSWER: yes, both $\mathcal{H}_1$ and $\mathcal{H}_2$ are uncoupled and time-independent → both are integrals of motion
$\{\mathcal{H}_1,\mathcal{H}_2\}=0$ → integrable according to Liouville's theorem

any Hamiltonian $\mathcal{H}(q_1, ..., q_n, p_1, ..., p_n) = \sum_{i=1}^{n} \mathcal{H}_i(q_i, p_i)$, representing a sum of n uncoupled, time-independent Hamiltonians of 1 dof, is integrable

Which of the following Hamiltonian systems is integrable and why?

[3] a free particle with two degrees of freedom moving inside a box



$$\mathcal{H}(x, y, p_x, p_y) = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + V(x, y)$$

$V(x,y) = 0$ inside the box and ∞ outside the box (reflective boundary conditions)

For this Hamiltonian to be integrable, we need two integrals of motion in involution

Neither p_x nor p_y are IoM, but p_x^2 and p_y^2 are

$$\{p_x^2, p_y^2\} = \{p_x p_x, p_y p_y\} = p_x\{p_x, p_y p_y\} + p_x\{p_x, p_y p_y\} = -2p_x\{p_y p_y, p_x\} = -2p_x\left[p_y\{p_y, p_x\} + p_y\{p_y, p_x\}\right] = 4p_x p_y\{p_x, p_y\} = 0$$

Last step follows from fact that p_x and p_y are canonical momenta

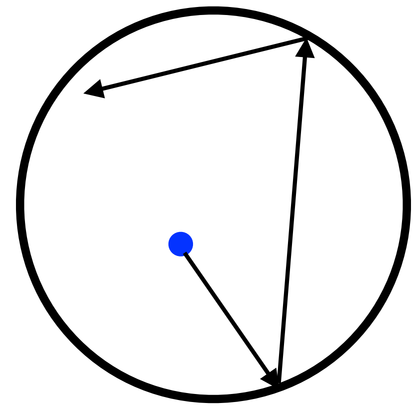
ANSWER: yes, this Hamiltonian is integrable

NOTE: Hamiltonian itself is also an IoM, and it is easy to show that it Poisson commutes with p_x^2 and p_y^2. However, since it is NOT independent of p_x^2 and p_y^2 it is not considered an independent third IoM.

Which of the following Hamiltonian systems is integrable and why?

[4] a free particle with two degrees of freedom moving inside a circular stadium

$$\mathcal{H}(x,y,p_x,p_y) = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + V(x,y)$$



$V(x,y) = 0$ inside the circle and ∞ outside the box

Upon inspection, it is clear that neither p_x and p_y nor p_x^2 and p_y^2 are integrals of motion

The Hamiltonian (total energy), though, is an integral of motion (no explicit time-dependence of $\mathcal{H}$)

For system to be integrable, we need a second IoM that is in involution with the Hamiltonian

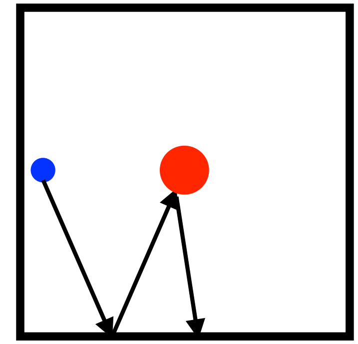
Rotational symmetry → Noether's theorem demands conservation of angular momentum

$L_z = x\,p_y - y\,p_x$ is a second integral of motion, and it is easy to show that $\{L_z,\mathcal{H}\} = 0$

ANSWER: yes, this Hamiltonian is integrable

Which of the following Hamiltonian systems is integrable and why?

[5] a free particle with two degrees of freedom moving inside a square stadium with circular object in center



$$\mathcal{H}(x, y, p_x, p_y) = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + V(x, y)$$

$V(x,y) = 0$ outside the red circle and inside the box, ∞ everywhere else

Upon inspection, it is clear that neither p_x and p_y nor p_x^2 and p_y^2 are integrals of motion

The Hamiltonian (total energy), though, is an integral of motion (no explicit time-dependence of $\mathcal{H}$)

For system to be integrable, we need a second IoM that is in involution with the Hamiltonian

This time there is no rotational symmetry; angular momemta not conserved

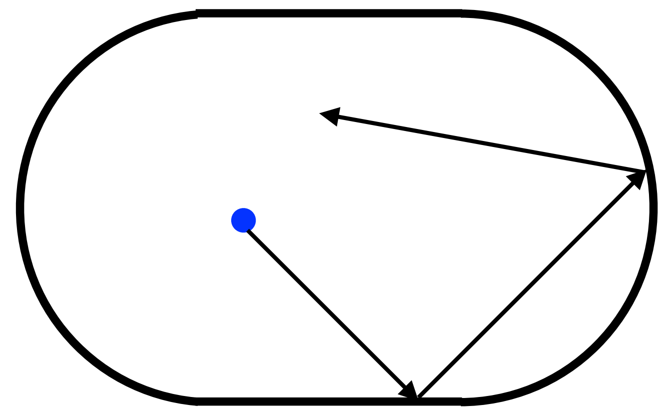
ANSWER: no, this Hamiltonian is NOT integrable

NOTE: it is easy to see that system is subject to chaos (extreme sensitivity to initial conditions)

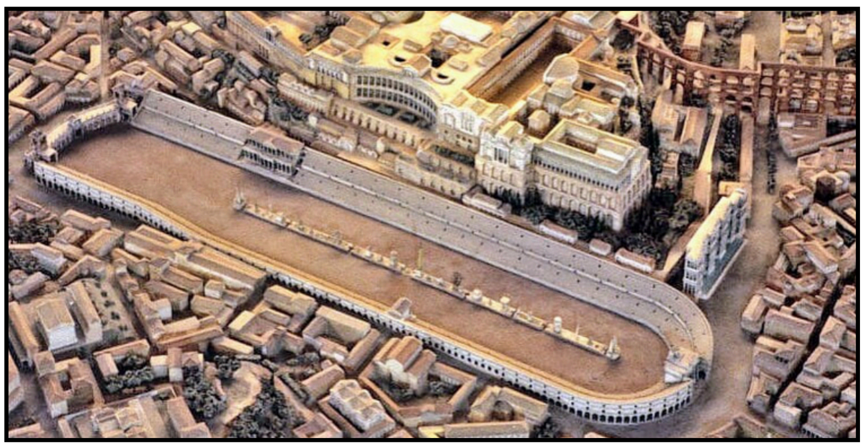
Which of the following Hamiltonian systems is integrable and why?

[6] a free particle with two degrees of freedom moving inside Bunimovich stadium

$$\mathcal{H}(x,y,p_x,p_y) = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + V(x,y)$$



$V(x,y) = 0$ inside the stadium, ∞ everywhere else



Bunimovich stadium is constructed by cutting a circle in two halves, and connecting part by straight lines. Example is Circus Maximus in ancient Rome

Because of lack of symmetry, no other IoM other than Hamiltonian (energy)

ANSWER: no, this Hamiltonian is NOT integrable

even systems with only 2 degrees of freedom can produce chaos

Which of the following Hamiltonian systems is integrable and why?

[7] a particle in a central potential in 3D

$$\mathcal{H}(\vec{q},\vec{p}) = \mathcal{H}(\vec{r},\vec{p}) = \frac{\vec{p}^2}{2m} + V(r)$$

For system to be integrable, we need (at least) three independent IoM in involution

Hamiltonian itself is an IoM and the system has spherical symmetry → $\boldsymbol{L} = (L_x, L_y, L_z)$ is conserved

Hence, each L_i is an integral of motion, but they are NOT in involution: $\{L_i, L_j\} \neq 0$

But, we have that $\{\boldsymbol{L}^2, L_i\} = 0$, and thus we have 3 independent IoM in involution: $\mathcal{H}$, $\boldsymbol{L}^2$, L_z

ANSWER: yes, this Hamiltonian is integrable

Every central force problem is integrable in 3D, independent of $V(r)$

Which of the following Hamiltonian systems is integrable and why?

[8] a two-particle system in 3D

$$\mathcal{H}(\vec{r}_1, \vec{r}_2, \vec{p}_1, \vec{p}_2) = \frac{\vec{p}_1^2}{2m_1} + \frac{\vec{p}_2^2}{2m_2} + V(\vec{r}_1, \vec{r}_2)$$

For system to be integrable, we need (at least) six independent IoM in involution

Hamiltonian itself is an IoM but system has no additional symmetries…

Not even the total linear momentum is conserved, because $V = V(\vec{r}_1, \vec{r}_2)$

ANSWER: no, this Hamiltonian is NOT integrable

Which of the following Hamiltonian systems is integrable and why?

[9] a two-particle system in 3D with $V = V(|\mathbf{r}_1 - \mathbf{r}_2|)$

$$\mathcal{H}(\vec{r}_1, \vec{r}_2, \vec{p}_1, \vec{p}_2) = \frac{\vec{p}_1^{\,2}}{2m_1} + \frac{\vec{p}_2^{\,2}}{2m_2} + V(|\vec{r}_1 - \vec{r}_2|)$$

For system to be integrable, we need (at least) six independent IoM in involution

Convert to center-of-mass coordinates:

$$\vec{r} = \vec{r}_1 - \vec{r}_2 \qquad \vec{R} = (m_1\vec{r}_1 + m_2\vec{r}_2)/M$$
$$\vec{p} = \mu(\vec{v}_1 - \vec{v}_2) \qquad \vec{P} = \vec{p}_1 + \vec{p}_2$$
$$\mu = m_1 m_2/M \qquad M = m_1 + m_2$$

$$\longrightarrow \quad \mathcal{H}(\vec{r}, \vec{R}, \vec{p}, \vec{P}) = \frac{\vec{P}^2}{2M} + \frac{\vec{p}^2}{2\mu} + V(r)$$

Hamiltonian itself (total energy) is an integral of motion

$\boldsymbol{R}$ is cyclic, and thus $\boldsymbol{P}$ is an integral of motion (3 components)

Spherical symmetry → angular momentum conserved → L_z and $\boldsymbol{L}^2$ are also integrals of motion

As you can easily verify, all of these are in involution with each other

ANSWER: yes, this Hamiltonian is integrable

A two-body system with a central force is integrable

Which of the following Hamiltonian systems is integrable and why?

[10] an n-particle system in 3D with central forces

$$\mathcal{H}(\vec{q}_1, \vec{q}_2, ..., \vec{q}_n, \vec{p}_1, \vec{p}_2, ..., \vec{p}_n) = \sum_{i=1}^{n} \frac{\vec{p}_i^{\,2}}{2m_i} + \sum_{i \neq j} V(|\vec{r}_i - \vec{r}_j|)$$

For system to be integrable, we need (at least) $3n$ independent IoM in involution

After converting to the center-of-mass frame we have the following constants of motion:

- Hamiltonian (or total energy), because $\mathcal{H} \neq \mathcal{H}(t)$
- The total angular momentum vector $\vec{L}_{\rm tot} = \sum_i \vec{r}_i \times \vec{p}_i$ (spherical symmetry)
- The total linear momentum vector $\vec{P}_{\rm tot} = \sum_i \vec{p}_i$ (no external forces)
- The initial position vector of center of mass $\vec{R}_0 = \vec{R}(t) - (\vec{P}_{\rm tot}/M)t$

These 10 constants of motion are known as the 10 Galilean invariants

However, these are not all in involution: For angular momentum, only 2 of 3 are independent; L_z and $\boldsymbol{L}^2$
It is also clear that $\vec{R}_0$ and $\vec{P}_{\rm tot}$ are not independent

ANSWER: no, this Hamiltonian is NOT integrable unless $n \leq 2$

An N-body system in 3D is NOT integrable for $N > 2$, even when all the forces are central

Which of the following Hamiltonian systems is integrable and why?

[11] an n-particle system in 3D with central forces **in the limit $n\rightarrow\infty$**

$$\mathcal{H}(\vec{q}_1, \vec{q}_2, ..., \vec{q}_n, \vec{p}_1, \vec{p}_2, ..., \vec{p}_n) = \sum_{i=1}^{n} \frac{\vec{p}_i^2}{2m_i} + \sum_{i \neq j} V(|\vec{r}_i - \vec{r}_j|)$$

As indicated on previous slide, this system is NOT integrable for n>2

However, in the limit $n\rightarrow\infty$ the system becomes collisionless, and we can write

$$V(\vec{q}_1, \vec{q}_2, ..., \vec{q}_N) = \sum_i V_{\rm ext}(\vec{q}_i)$$

which implies that we can write the Hamiltonian as

$$\mathcal{H}(\vec{q}_1, \vec{q}_2, ..., \vec{q}_N, \vec{p}_1, \vec{p}_2, ..., \vec{p}_N) = \sum_{i=1}^{N} \left[\frac{\vec{p}_i^2}{2m} + V_{\rm ext}(\vec{q}_i)\right] = \sum_{i=1}^{N} \mathcal{H}_i(\vec{q}_i, \vec{p}_i)$$

If the system as a whole has spherical symmetry, such that $V_{\rm ext}(\vec{q}) = V_{\rm ext}(r)$, then this Hamiltonian is simply the sum of n independent central force problems which **is** integrable (see **[7]**)

The $3n$ integrals of motion in involution are: E_i, $L_{z,i}$ and $\boldsymbol{L}^2{}_i$ for all i=1,2,…n particles

If $V_{\rm ext}(\vec{q}) = V_{\rm ext}(R, z)$, such that the system is axisymmetric, then each particle has two integrals of motion (E and L_z). Integrability requires a third, and only a small subset of axisymmetric systems obey such a third integral of motion (called I_3) and are thus integrable.

Triaxial systems have only one `classical' integral of motion (E); integrability is even less likely

The Structure of Integrable Hamiltonian Systems

If a Hamiltonian system is integrable, the solution to the equations of motion can be written as

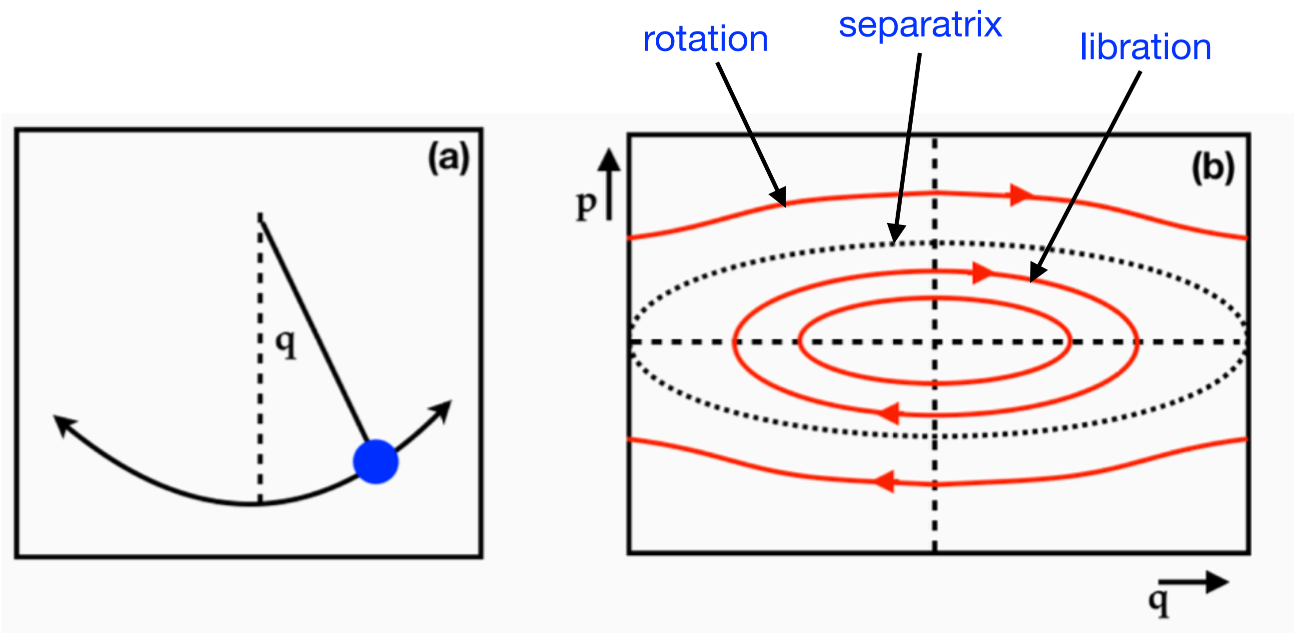
$$Q_i(t) = \omega_i t + Q_i(0)\,, \qquad P_i(t) = P_i(0) \quad \text{with} \quad \omega_i = \partial \mathcal{H}' / \partial P_i$$

The evolution of each Q_i is cyclic with frequency $\boldsymbol{\omega}_i$

We distinguish two different kinds of periodic motion:

Libration: motion between states in which the generalized momentum vanishes

Rotation: motion for which the generalized momentum remains always non-zero



canonical transformation to cyclic Q

Action-Angle Variables

Consider an integrable Hamiltonian with *n* degrees of freedom and with $(I_1, I_2, \ldots, I_n)$ a set of *n* integrals of motion in involution.

Define: $I_a = (I_1 + I_2) / 2$ and $I_b = (I_1 - I_2) / 2$

By construction I_a and I_b are also integrals of motion, and it is trivial to show that $\{I_a, I_b\} = 0$

Hence, $(I_a, I_b, I_3, \ldots, I_n)$ is also a set of *n* integrals of motion in involution.

QUESTION: is there an *optimal* set of integrals of motion to use?

ANSWER: yes, the set of action-angle variables.

The actions, I_i, take the role of the generalized momenta and are defined as

$$I_i = \oint p_i \, \mathrm{d}q_i$$

No Einstein summation!

The angles, θ_i, are the corresponding generalized coordinates, and are proper angles.

Action-Angle Variables

What is so special about ($\boldsymbol{I}$,$\boldsymbol{\Theta}$)?

If the Hamiltonian is integrable, Hamilton's characteristic function is separable:

$$W(\vec{q},\vec{P}) = \sum_i W_i(q_i, P_1, P_2, ..., P_n) = \sum_i W_i(q_i, \vec{P})$$

According to the corresponding transformation rules

$$p_i = \frac{\partial W}{\partial q_i} = \frac{\partial W_i}{\partial q_i} = p_i(q_i, \vec{P})$$

Hence, we have that

$$I_i = \oint p_i \, \mathrm{d}q_i = \oint p_i(q_i, \vec{P}) \, \mathrm{d}q_i = I_i(\vec{P}) \quad \longrightarrow \quad P_i = P_i(\vec{I})$$

and since the P_i are integrals of motion, so are the actions. After all,

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \sum_i \frac{\partial I}{\partial P_i} \dot{P}_i = 0$$

Action-Angle Variables

What is so special about ($\boldsymbol{I},\boldsymbol{\Theta}$)?

Since $\mathcal{H}=\mathcal{H}(\boldsymbol{P})$ we have that $\mathcal{H}=\mathcal{H}(\boldsymbol{I})$, which indicates that the angles, θ_i, are all cyclic

From Hamilton's equations of motion we have that

$$\omega_i \equiv \dot{\theta}_i = \frac{\partial \mathcal{H}}{\partial I_i} = \omega_i(I_1, ..., I_n)$$

and since the actions are integrals of motion, the frequencies are constant as well

Hence, we have that the standard solution for an integrable Hamiltonian can be written as

$$\theta_i(t) = \omega_i t + \theta_i(0)\,, \qquad I_i(t) = I_i(0)$$

so far, nothing special here, since the same holds for ($\boldsymbol{Q},\boldsymbol{P}$)

Action-Angle Variables

What is so special about (***I***,***Θ***)?

What makes the action-angle variables so special is the following:

Let us compute by how much the angle changes during one period of its libration/rotation

$$\Delta\theta_i = \oint \frac{\partial\theta_i}{\partial q_i}\mathrm{d}q_i = \oint \frac{\partial^2 W}{\partial I_i\,\partial q_i}\mathrm{d}q_i = \frac{\partial}{\partial I_i}\oint \frac{\partial W}{\partial q_i}\mathrm{d}q_i = \frac{\partial}{\partial I_i}\oint p_i\,\mathrm{d}q_i = \frac{\partial I_i}{\partial I_i} = 1$$

Since $\theta_i(t) = \omega_i t + \theta_i(0)$, we also have that $\Delta\theta_i = \omega_i T$, where T is the period

We thus see that

$$\boxed{\omega_i = \dot{\theta}_i = \frac{1}{T}}$$

The time-derivative of the angle-variable, θ_i, is the frequency of motion in the `direction' associated with the i^{th} degree of freedom.

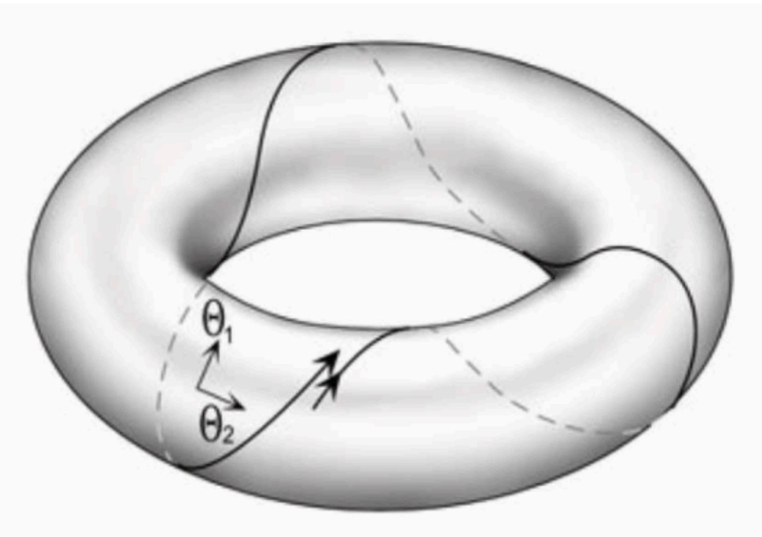
Action-Angle Variables

What is so special about ($\boldsymbol{I},\boldsymbol{\Theta}$)?

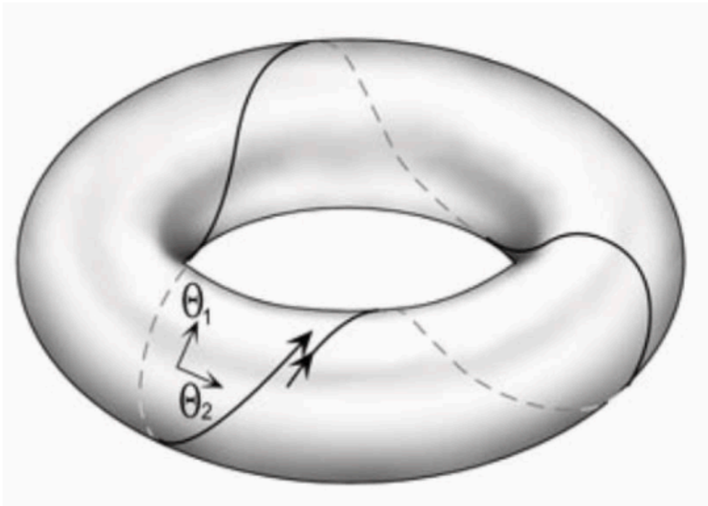
The action-angle formalism allows you to determine the frequencies of periodic motion without having to calculate the exact trajectories for the motion

- Calculate $I_i = \oint p_i \, \mathrm{d}q_i$
- Express original Hamiltonian as a function of the actions: $\mathcal{H}=\mathcal{H}(I_1, I_2, \ldots, I_n)$
- Compute the frequencies using $\omega_i = \partial\mathcal{H}/\partial I_i$

Also, the action-angle variables are the natural coordinates to describe the orbital structure of the Hamiltonian system. Holding the actions fixed, the corresponding angles trace out an n-torus in phase-space



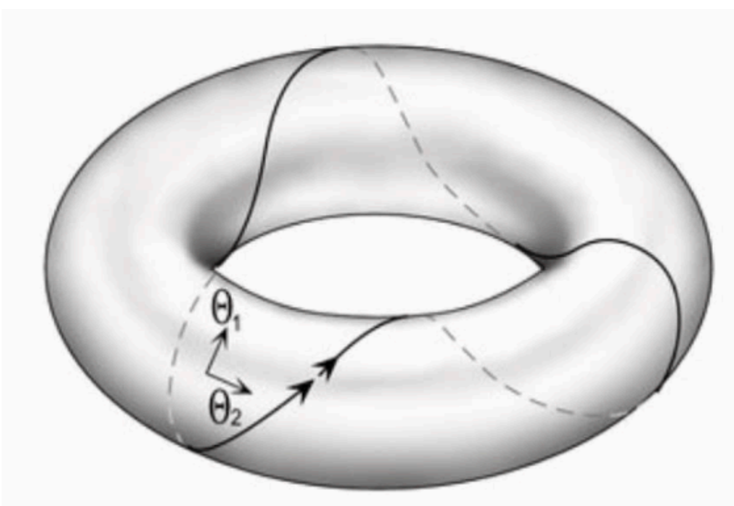
The Structure of Integrable Hamiltonian Systems



Non-Resonant

If the frequencies are incommensurable, then over time the phase-space trajectory will densely cover the entire surface of the n-torus.

We say that the orbit is non-resonant



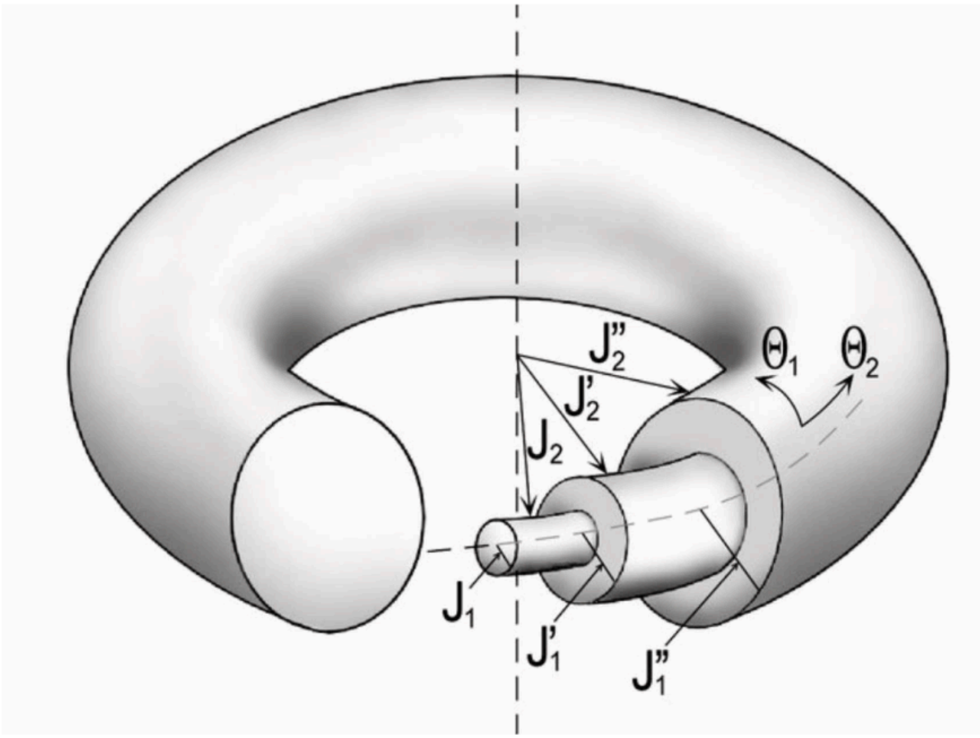
Resonant

If two or more of the frequencies are commensurable, (i.e., ω_i/ω_j is rational) then the phase-space trajectory is closed, and the trajectory is a n-1 dimensional manifold on the surface of the n-torus.

We say that the orbit is resonant

The Structure of Integrable Hamiltonian Systems

If a Hamiltonian is integrable, the entire phase-space is foliated with nested n-tori



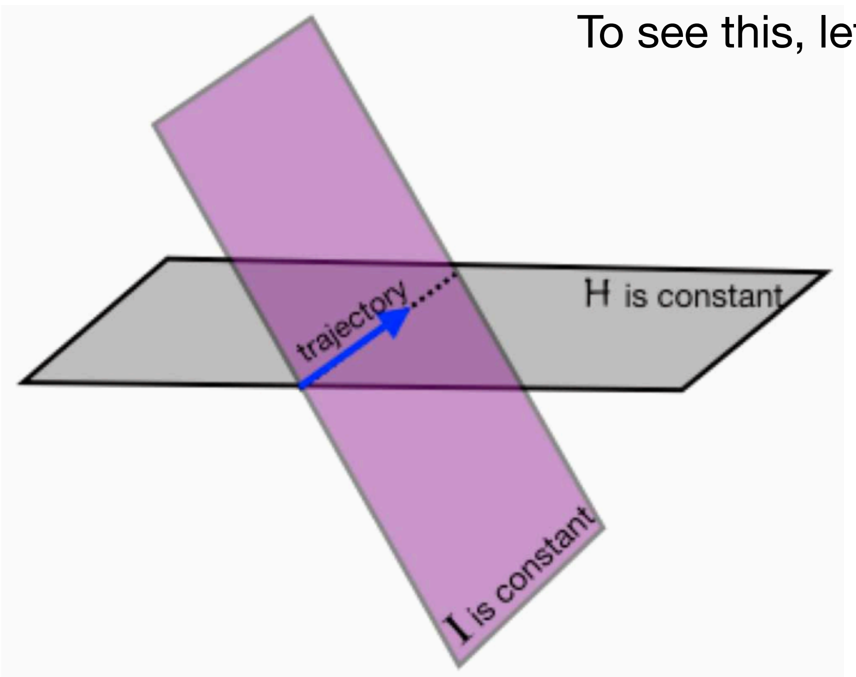
The surface of each torus is characterized (labeled) by the corresponding actions (here Ji rather than I_i), and all phase-space trajectories are restricted to move on the surface of their n-dimensional torus.

Hence, each trajectory is restricted to move on an n-dimensional manifold in $2n$-dimensional phase space. **!**

The Structure of Integrable Hamiltonian Systems

This is related to an important geometric property of Hamiltonian systems:

For each integral of motion in involution the dimensionality of the manifold traced out by the particle is reduced by one **!**



To see this, let $I = I(\boldsymbol{q},\boldsymbol{p})$ be an integral of motion, then

$$\nabla I \cdot (\dot{\vec{q}}, \dot{\vec{p}}) = \left(\frac{\partial I}{\partial \vec{q}}, \frac{\partial I}{\partial \vec{p}}\right) \cdot (\dot{\vec{q}}, \dot{\vec{p}})$$
$$= \frac{\partial I}{\partial q_i}\frac{\partial \mathcal{H}}{\partial p_i} - \frac{\partial I}{\partial p_i}\frac{\partial \mathcal{H}}{\partial q_i} = \{I, \mathcal{H}\} = 0$$

Let $\mathcal{H} = \mathcal{H}(\boldsymbol{q},\boldsymbol{p})$ be another integral of motion, then

$$\nabla \mathcal{H} \cdot (\dot{\vec{q}}, \dot{\vec{p}}) = \left(\frac{\partial \mathcal{H}}{\partial \vec{q}}, \frac{\partial \mathcal{H}}{\partial \vec{p}}\right) \cdot (\dot{\vec{q}}, \dot{\vec{p}}) = -\dot{\vec{p}} \cdot \dot{\vec{q}} + \dot{\vec{q}} \cdot \dot{\vec{p}} = 0$$

We see that the trajectory $(\dot{\boldsymbol{q}},\dot{\boldsymbol{p}})$ is limited to the intersection of I=cst and $\mathcal{H}$ =cst