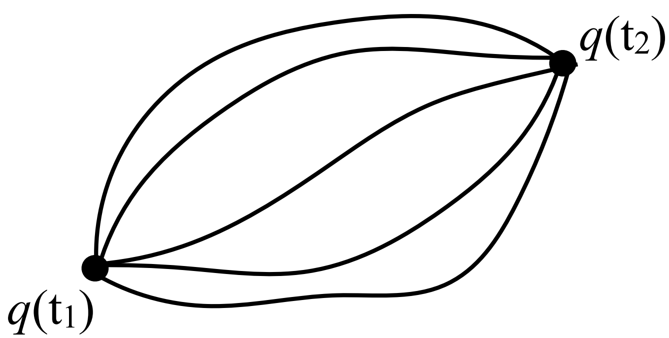
LECTURE 3

Classical Dynamics: a primer

Typically, a system of *N* particles has n=3*N* degrees of freedom (unless there are holonomic constraints, which is rarely the case in astrophysical fluids)

Let $\vec{q} = (q_1, q_2, ..., q_n)$ be the vector of generalized coordinates that specifies the positions of all *N* particles in configuration space

The action is defined as $$S = \int_{t_1}^{t_2} \mathcal{L}(\vec{q}, \dot{\vec{q}}, t)\,\mathrm{d}t$$ where $\mathcal{L} = K - W$ is the Lagrangian



Different paths between the begining and end positions have a different value for the action

Principle of least action (aka Hamilton's principle) states that path taken by particle is the one for which the action is an extremum, i.e., $\delta S = 0$.

Using calculus of variations,
this implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0\,, \qquad (i = 1, 2, ..., n)$$

These n second-order differential equations are the Euler-Lagrange equations

If q_i is a Cartesian coordinate, then $\partial \mathcal{L}/\partial \dot{q}_i = m\dot{q}_i = p_i$ is a momentum

This suggests we define the conjugate momentum

$$p_i \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

If the Lagrangian of a system does not contain a given coordinate q_j,
then this coordinate is said to be cyclic or ignorable.

In that case, we have that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right) = \frac{\mathrm{d}p_j}{\mathrm{d}t} = 0$$

The generalized momentum conjugate to a cyclic coordinate
is conserved (i.e.,, is a constant of motion) !

Noether's Theorem

With each continous symmetry of the Lagrangian corresponds a conserved quantity

!

Consider a one-parameter family of maps $q_i(t) \to Q_i(\lambda, t)\,,\qquad \lambda \in \mathbb{R}$

such that $Q_i(0,t) = q_i(t)$, and with λ a continous variable that characterizes the coordinate transformation $q_i \to Q_i$

This transformation is said to be a continuous symmetry of the Lagrangian if

$$\frac{\partial}{\partial \lambda}\mathcal{L}\left(Q_i(\lambda,t),\dot{Q}(\lambda,t),t\right)_{\lambda=0} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = \frac{\partial \mathcal{L}}{\partial Q_i}\frac{\partial Q_i}{\partial \lambda} + \frac{\partial \mathcal{L}}{\partial \dot{Q}_i}\frac{\partial \dot{Q}_i}{\partial \lambda} \quad \Rightarrow \quad \left(\frac{\partial \mathcal{L}}{\partial \lambda}\right)_{\lambda=0} = \frac{\partial \mathcal{L}}{\partial q_i}\frac{\partial Q_i}{\partial \lambda} + \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\frac{\partial \dot{Q}_i}{\partial \lambda}$$

Using Euler-Lagrange eq. $\left(\frac{\partial \mathcal{L}}{\partial \lambda}\right)_{\lambda=0} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right)\frac{\partial Q_i}{\partial \lambda} + \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\frac{\partial \dot{Q}_i}{\partial \lambda} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\frac{\partial Q_i}{\partial \lambda}\right)$

Hence, if $(\partial \mathcal{L}/\partial \lambda)_{\lambda=0} = 0$, then $\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\frac{\partial Q_i}{\partial \lambda} = \sum_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\frac{\partial Q_i}{\partial \lambda}$ is a conserved quantity.

↑ Einstein summation convention

Q.E.D.

Example

Consider a closed system of N particles with Lagrangian $\mathcal{L} = \frac{1}{2}\sum_i m_i \dot{\vec{r}}_i^2 - \sum_{i \neq j} V(|\vec{r}_i - \vec{r}_j|)$

This Lagrangian is symmetric under the continuous transformation $\vec{r}_i(t) \rightarrow \vec{r}_i(t) + \lambda\vec{n}$ for any $\lambda \in \mathbb{R}$ and any vector $\vec{n}$

According to Noether's theorem, this implies that $\sum_i \frac{\partial \mathcal{L}}{\partial \dot{\vec{r}}_i} \cdot \vec{n} = \sum_i \vec{p}_i \cdot \vec{n}$ is conserved

This is simply the component of the total linear momentum in the direction of $\vec{n}$

Implications of Noether's Theorem

Invariance of $\mathcal{L}$ under	time	translation	⇔	conservation of energy
Invariance of $\mathcal{L}$ under	spatial	translation	⇔	conservation of linear momentum
Invariance of $\mathcal{L}$ under	rotational	translation	⇔	conservation of angular momentum

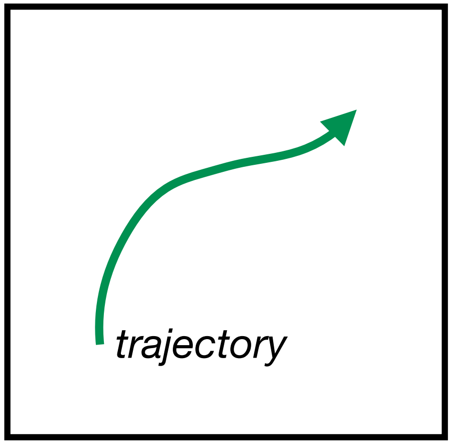
Hamiltonian Dynamics

Let $\vec{q} = (q_1, q_2, ..., q_n)$ be the vector of generalized coordinates that specifies the positions of all *N* particles in configuration space

The state of a system is specified by both $\vec{q}$ and $\vec{p} = (p_1, p_2, ..., p_n)$ where

$$\dot{p}_i = \frac{\partial \mathcal{L}}{\partial q_i}\,, \qquad (i = 1, 2, ..., n)$$

is the conjugate momentum



The state of a system $(\vec{q}, \vec{p})$ lives in $2n$-dimensional phase-space $\mathbf{\Gamma}$

The state-variables $(\vec{q}, \vec{p})$ are known as canonical coordinates

Note; no trajectories in $\mathbf{\Gamma}$-space can ever cross each other!!

Hamilton introduced a new function of the canonical phase-space coordinates which is generated from the Lagrangian via a Legendre transformation

$$\boxed{\mathcal{H}(q_i, p_i, t) = \dot{q}_i p_i - \mathcal{L}(q_i, \dot{q}_i, t)}$$

index form

$$\boxed{\mathcal{H}(\vec{q}, \vec{p}, t) = \dot{\vec{q}} \cdot \vec{p} - \mathcal{L}(\vec{q}, \dot{\vec{q}}, t)}$$

vector form

This serves as the definition of the Hamiltonian

(see App E of lecture notes for description of Legendre transformations)

Hamiltonian Dynamics

$$\mathcal{H}(q_i, p_i, t) = \dot{q}_i p_i - \mathcal{L}(q_i, \dot{q}_i, t)$$

The total derivative of the Hamiltonian is $\mathrm{d}\mathcal{H} = \frac{\partial \mathcal{H}}{\partial q_i}\,\mathrm{d}q_i + \frac{\partial \mathcal{H}}{\partial p_i}\,\mathrm{d}p_i + \frac{\partial \mathcal{H}}{\partial t}\,\mathrm{d}t$ **[A]**

An alternative expression can be obtained using the Legendre transformation:

$$\begin{aligned}
\mathrm{d}\mathcal{H} &= \dot{q}_i\,\mathrm{d}p_i + p_i\,\mathrm{d}\dot{q}_i - \mathrm{d}\mathcal{L} \\
&= \dot{q}_i\,\mathrm{d}p_i + p_i\,\mathrm{d}\dot{q}_i - \left(\frac{\partial \mathcal{L}}{\partial q_i}\,\mathrm{d}q_i + \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\,\mathrm{d}\dot{q}_i + \frac{\partial \mathcal{L}}{\partial t}\,\mathrm{d}t\right) \\
&= \dot{q}_i\,\mathrm{d}p_i - \frac{\partial \mathcal{L}}{\partial q_i}\,\mathrm{d}q_i - \frac{\partial \mathcal{L}}{\partial t}\,\mathrm{d}t \\
&= \dot{q}_i\,\mathrm{d}p_i - \dot{p}_i\,\mathrm{d}q_i - \frac{\partial \mathcal{L}}{\partial t}\,\mathrm{d}t
\end{aligned}$$

[B]

Equating expressions **[A]** and **[B]** we obtain Hamilton's equations

$$\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i}\,, \qquad \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i}\,, \qquad \frac{\partial \mathcal{L}}{\partial t} = -\frac{\partial \mathcal{H}}{\partial t}$$

Hamiltonian Dynamics

The first of these are the $2n$ first-order differential equations that replace the n second-order Euler-Lagrange equations as the equations of motion:

$$\boxed{\dot{\vec{q}} = \frac{\partial \mathcal{H}}{\partial \vec{p}}\,, \qquad \dot{\vec{p}} = -\frac{\partial \mathcal{H}}{\partial \vec{q}}}$$

!

If the equations of transformation that define the generalized coordinates do not depend explicitely on time, and the potential is independent of velocity (i.e., no friction) then the Hamiltonian is equal to the total energy (see lecture notes for details)

If the potential is also time-independent, then time does not appear explicitely in the Lagrangian and we have that

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = \frac{\partial \mathcal{H}}{\partial t} = -\frac{\partial \mathcal{L}}{\partial t} = 0$$

and we thus see that the total energy of the system is conserved.

Hamilton-Jacobi Theory

Given two functions $A(q_i,p_i)$ and $B(q_i,p_i)$ of the canonical phase-space coordinates q_i and p_i the Poisson bracket of A and B is defined as

$$\{A,B\} = \frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i}$$

index form

$$\{A,B\} = \frac{\partial A}{\partial \vec{q}}\cdot\frac{\partial B}{\partial \vec{p}} - \frac{\partial A}{\partial \vec{p}}\cdot\frac{\partial B}{\partial \vec{q}}$$

vector form

The Poisson brackets obey the following relations:

$$\begin{aligned}
\{A,B\} &= -\{B,A\} \\
\{\alpha A+\beta B,C\} &= \alpha\{A,C\}+\beta\{B,C\} \quad \forall \alpha,\beta \in \mathbb{R} \\
\{AB,C\} &= A\{B,C\}+B\{A,C\} \\
\{\{A,B\},C\}+\{\{B,C\},A\}+\{\{C,A\},B\} &= 0
\end{aligned}$$

(Jacobi identity)

Let A and B be the canonical variables ***q*** and ***p*** themselves, then

$$\{q_i,q_j\}=0, \qquad \{p_i,p_j\}=0, \qquad \{q_i,p_j\}=\delta_{ij}$$

These are known as the canonical commutation relations. Any set (***Q***,***P***) of canonical variable has to obey these relations otherwise they are not a canonical set

For any function $f(\boldsymbol{q},\boldsymbol{p},t)$

$$\begin{aligned}\frac{\mathrm{d}f}{\mathrm{d}t} &= \frac{\partial f}{\partial q_i}\dot{q}_i + \frac{\partial f}{\partial p_i}\dot{p}_i + \frac{\partial f}{\partial t} \\ &= \frac{\partial f}{\partial q_i}\frac{\partial \mathcal{H}}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial \mathcal{H}}{\partial q_i} + \frac{\partial f}{\partial t} \\ &= \{f,\mathcal{H}\} + \frac{\partial f}{\partial t}\end{aligned}$$

Poisson's equation of motion

If for f we subsitute one of the canonical variables itself, then we see that

$$\boxed{\dot{q}_i = \{q_i,\mathcal{H}\}\,, \qquad \dot{p}_i = \{p_i,\mathcal{H}\}}$$

If we now introduce the $2n$-dimensional vector $\boldsymbol{w} = (\boldsymbol{q},\boldsymbol{p})$, Hamilton's equations of motion can be written in the short-hand form

$$\boxed{\dot{\vec{w}} = \{\vec{w},\mathcal{H}\}}$$!

Note that the generalized coordinates and conjugate momenta can be treated on equal footing…

If the function has no explicit time dependence, such that $f = f(\boldsymbol{q},\boldsymbol{p})$, and the function Poisson commutes with the Hamiltonian, i.e.,

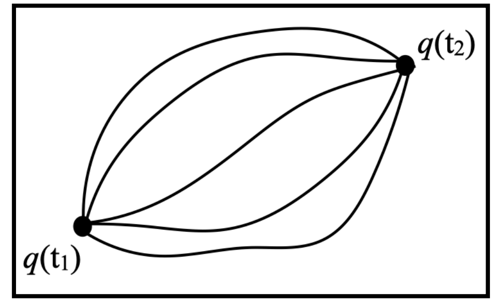
$$\{f,\mathcal{H}\} = 0$$

then f is a constant of motion (i.e., $\mathrm{d}f/\mathrm{d}t = 0$). It is called an integral of motion.

Two integrals of motion that Poisson commute with each other are said to be in involution

Canonical Transformations

Consider the following transformation of the Lagrangian $\mathcal{L} \to \mathcal{L}' = \mathcal{L} + \frac{\mathrm{d}F}{\mathrm{d}t}$
where $F = F(\boldsymbol{q},t)$



Under this transformation the action integral becomes

$$S' = \int_{t_1}^{t_2} \mathcal{L}' \, \mathrm{d}t = \int_{t_1}^{t_2} \mathcal{L} \mathrm{d}t + \int_{t_1}^{t_2} \frac{\mathrm{d}F}{\mathrm{d}t} \, \mathrm{d}t = S + F(t_2) - F(t_1)$$

Variations of the action leave the start and end points fixed, hence we have that $\delta S' = \delta S$ which implies that the equations of motion remain invariant.

Hence, we see that there is some non-uniqueness to the Lagrangian, which we can use to our advantage…

Canonical transformations are transformations of the form $(\boldsymbol{q},\boldsymbol{p}) \to (\boldsymbol{Q},\boldsymbol{P})$ between two canonical coordinate systems that leaves the equations of motion invariant.

Let $\mathcal{L}(\vec{q}, \dot{\vec{q}}, t)$ and $\mathcal{L}'(\vec{Q}, \dot{\vec{Q}}, t)$ be the corresponding Lagrangians. Then in order for the equations of motion to be invariant we require that

$$\mathcal{L}(\vec{q}, \vec{p}, t) \quad = \quad \mathcal{L}'(\vec{Q}, \vec{P}, t) + \frac{\mathrm{d}F}{\mathrm{d}t}$$

Canonical Transformations

Using the definition of the Hamiltonian, based on the Legendre transform, we have that

$$\begin{aligned} \mathcal{L}(\vec{q},\vec{p},t) &= \vec{p}\cdot\dot{\vec{q}} - \mathcal{H}(\vec{q},\vec{p},t) \\ \mathcal{L}'(\vec{Q},\vec{P},t) &= \vec{P}\cdot\dot{\vec{Q}} - \mathcal{H}'(\vec{Q},\vec{P},t) \end{aligned}$$

Hence,

$$\begin{aligned} \mathcal{L}(\vec{q},\vec{p},t) &= \mathcal{L}'(\vec{Q},\vec{P},t) + \frac{\mathrm{d}F}{\mathrm{d}t} \\ \Leftrightarrow \frac{\mathrm{d}F}{\mathrm{d}t} &= \vec{p}\cdot\dot{\vec{q}} - \mathcal{H}(\vec{q},\vec{p},t) - \left[\vec{P}\cdot\dot{\vec{Q}} - \mathcal{H}'(\vec{Q},\vec{P},t)\right] \\ \Leftrightarrow \mathrm{d}F &= p_i\mathrm{d}q_i - P_i\mathrm{d}Q_i + (\mathcal{H}' - \mathcal{H})\mathrm{d}t \qquad \text{[A]} \end{aligned}$$

If we take $F = F(\vec{q},\vec{Q},t)$ then we also can write that

$$\mathrm{d}F = \frac{\partial F}{\partial q_i}\mathrm{d}q_i + \frac{\partial F}{\partial Q_i}\mathrm{d}Q_i + \frac{\partial F}{\partial t}\mathrm{d}t \qquad \text{[B]}$$

Equating [A] and [B] we obtain the transformation rules:

$$\boxed{p_i = \frac{\partial F}{\partial q_i}\,, \qquad P_i = -\frac{\partial F}{\partial Q_i}\,, \qquad \mathcal{H}' = \mathcal{H} + \frac{\partial F}{\partial t}}$$

The function $F = F(\vec{q},\vec{Q},t)$ is called the generating function of the canonical transformation $(\vec{q},\vec{p}) \rightarrow (\vec{Q},\vec{P})$

(see lecture notes for details)

Canonical Transformations

Canonical transformations leave the equations of motion, and therefore also the Poisson brackets, invariant.

$$\{A, B\}_{\vec{q},\vec{p}} = \{A, B\}_{\vec{Q},\vec{P}}$$

QUESTION: is there a particular canonical transformation for which the equations of motion become particularly simple?

YES: if we can find a canonical transformation $(\vec{q},\vec{p}) \rightarrow (\vec{Q},\vec{P})$ for which the new Hamiltonian $\mathcal{H}'(\vec{Q},\vec{P}) = \mathcal{H}'(\vec{P})$, i.e., all new generalized coordinates are cyclic, then the equations of motion become

$$\dot{P}_i = -\frac{\partial \mathcal{H}'}{\partial Q_i} = 0\,, \qquad \dot{Q}_i = \frac{\partial \mathcal{H}'}{\partial P_i} = \text{constant}$$

Hence, the new generalized momenta are all integrals of motion, and the solution for the dynamics is trivially given by

$$\boxed{Q_i(t) = \omega_i\, t + Q_i(0)\,, \qquad P_i(t) = P_i(0) \qquad \text{with } \omega_i \equiv \partial\mathcal{H}'/\partial P_i}$$

Hamilton-Jacobi equation[+]

How can we find the generator of the canonical transformation for which all Q are cyclic?

Consider a generator of the second kind, $F{=}F_2(\boldsymbol{q},\boldsymbol{P})$ without explicit time-dependence for which the transformation rules are:

$$p_i = \frac{\partial F_2}{\partial q_i}\,, \qquad Q_i = \frac{\partial F_2}{\partial P_i}\,, \qquad \mathcal{H}' = \mathcal{H} + \frac{\partial F_2}{\partial t}$$

We have that $\mathcal{H}(\vec{q},\vec{p}) \;=\; \mathcal{H}'(\vec{P}) \;=\; E$

Substituting the transformation rule for p_i in the original Hamiltonian we obtain

$$\boxed{\mathcal{H}\left(q_i, \frac{\partial F_2}{\partial q_i}\right) = E} \quad \textbf{!}$$

This is the Hamilton-Jacobi equation, which is a PDE of n variables q_i

The solution for the generating function is called Hamilton's characteristic function, and is typically indicated by the symbol $W(\boldsymbol{q},\boldsymbol{P})$, rather than $F_2(\boldsymbol{q},\boldsymbol{P})$

[+] here we focus exclusively on conservative systems for which $\partial\mathcal{H}/\partial t = 0$, and we assume that $\mathcal{H} = E$

Hamilton-Jacobi equation

So, we have a complete solution for the dynamics of a Hamiltonian system, if we can solve the Hamilton-Jacobi equation…

Unfortunately, solving an n-dimensional PDE is extremely hard…

However, if Hamilton's characteristic function is separable, which means can be written as

$$W(\vec{q}, \vec{P}) = \sum_{i=1}^{n} W_i(q_i, \vec{P})$$

then the Hamilton-Jacobi equation reduces to a set of n first-order ODEs, which are easily solved by quadrature (i.e., can be written as n integral equations).

If the Hamilton-Jacobi equation is separable, we say that the Hamiltonian is integrable

An integrable Hamiltonian with n degrees of freedom has n integrals of motion in involution

these are the n generalized coordinates P_i

Liouville's Theorem of Integrable Systems

If a system with n degrees of freedom has n mutually Poisson commuting integrals of motion $I_1, I_2, \ldots, I_n$ then the system is integrable.