Magnetohydrodynamics

MHD equations can be derived heuristically by taking moment equations from the Vlasov equation to which we add a collision term

$$\frac{\partial f_\mathrm{a}}{\partial t} + \vec{v}\cdot\frac{\partial f_\mathrm{a}}{\partial \vec{x}} + \frac{q_\mathrm{a}}{m}\left(\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right)\cdot\frac{\partial f_\mathrm{a}}{\partial \vec{v}} = \left(\frac{\partial f_\mathrm{a}}{\partial t}\right)_\mathrm{coll}$$

a = e or i

$$\int d\vec{v}\left(\frac{\partial f_\mathrm{a}}{\partial t}\right)_\mathrm{coll} = 0 \quad \Rightarrow \quad \frac{\partial n_\mathrm{a}}{\partial t} + \nabla\cdot(n_\mathrm{a}\vec{u}_\mathrm{a}) = 0$$

particles conserved in collisions

continuity equation

$$\vec{C}_\mathrm{a} = m_\mathrm{a}\int d\vec{v}\,\vec{v}\left(\frac{\partial f_\mathrm{a}}{\partial t}\right)_\mathrm{coll} \quad \Rightarrow \quad m_\mathrm{a} n_\mathrm{a}\left[\frac{\partial \vec{u}_\mathrm{a}}{dt} + (\vec{u}_\mathrm{a}\cdot\nabla)\vec{u}_\mathrm{a}\right] = -\nabla P_\mathrm{a} + q_\mathrm{a} n_\mathrm{a}\left(\vec{E} + \frac{\vec{u}_\mathrm{a}}{c}\times\vec{B}\right) + \vec{C}_\mathrm{a}$$

electron/ion momentum NOT conserved

momentum equations

Due to collisions between electrons and much heavier ions, momentum is transferred between these different species. Note, though, that total momentum of electrons plus ions IS conserved: $C_e = -C_i$

Magnetohydrodynamics

Since in MHD we treat the plasma as a single fluid, we now define the relevant quantities

total mass density	$\rho \equiv m_\mathrm{e}\, n_\mathrm{e} + m_\mathrm{i}\, n_\mathrm{i}$
total charge density	$\rho_\mathrm{c} \equiv q_\mathrm{e}\, n_\mathrm{e} + q_\mathrm{i}\, n_\mathrm{i} = e(n_\mathrm{i} - n_\mathrm{e})$
com fluid velocity	$\vec{u} \equiv \frac{1}{\rho}\,(m_\mathrm{i}\, n_\mathrm{i}\, \vec{u}_\mathrm{i} + m_\mathrm{e}\, n_\mathrm{e}\, \vec{u}_\mathrm{e})$
current density	$\vec{J} = q_\mathrm{e}\, n_\mathrm{e}\, \vec{u}_\mathrm{e} + q_\mathrm{i}\, n_\mathrm{i}\, \vec{u}_\mathrm{i}$
total pressure	$P = P_\mathrm{e} + P_\mathrm{i}$

m_e x continuity eq. of electrons + m_i x continuity eq. of ions yields

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0}$$

mass conservation

q_e x continuity eq. of electrons + q_i x continuity eq. of ions yields

$$\boxed{\frac{\partial \rho_\mathrm{c}}{\partial t} + \nabla \cdot \vec{J} = 0}$$

charge conservation

Magnetohydrodynamics

MHD

$$\rho\frac{d\vec{u}}{dt} = -\nabla P + \rho_c\vec{E} + \frac{1}{c}\vec{J}\times\vec{B}$$

negligible

$$\rho\frac{d\varepsilon}{dt} = -P\,\nabla\cdot\vec{u} - \frac{J^2}{\sigma}$$

ignoring radiation

Hydrodynamics

$$\frac{d\vec{u}}{dt} = -\frac{\nabla P}{\rho} - \nabla\Phi$$

Momentum eqs

$$\frac{d\varepsilon}{dt} = -\frac{P}{\rho}\nabla\cdot\vec{u} - \frac{\mathcal{L}}{\rho}$$

Energy eq.

Ohmic dissipation

Here $J^2 = \vec{J}\cdot\vec{J}$ and σ is the **electric conductivity**. The Ohmic dissipation term describes how collisions convert magnetic energy into thermal energy.

Since both the momentum and energy equations contain the current density, we need to complement our set of equations with an evolution equation for the current

Multiplying momentum eqs with charge, one obtains the **Generalized Ohm's law**

$$\frac{m_e m_i}{\rho e^2}\frac{\partial\vec{J}}{\partial t} = \frac{m_i}{2\rho e}\nabla P + \vec{E} + \frac{1}{c}\vec{u}\times\vec{B} - \frac{m_i}{\rho e c}\vec{J}\times\vec{B} + \frac{m_i}{\rho e}\vec{C}_i$$

Magnetohydrodynamics

Heuristic derivation of the collision term:

$$\vec{C}_\mathrm{e} = -m_\mathrm{e}\, n_\mathrm{e}\, \nu_\mathrm{L}\, (\vec{u}_\mathrm{e} - \vec{u}_\mathrm{i})$$

$$\vec{J} = q_\mathrm{e}\, n_\mathrm{e}\, \vec{u}_\mathrm{e} + q_\mathrm{i}\, n_\mathrm{i}\, \vec{u}_\mathrm{i} \simeq n_\mathrm{e}\, e\, (\vec{u}_\mathrm{i} - \vec{u}_\mathrm{e})$$

➡

$$\vec{C}_\mathrm{e} = +n_\mathrm{e}\, e\, \eta\, \vec{J}$$

$$\eta = \frac{m_\mathrm{e}\, \nu_\mathrm{L}}{n_\mathrm{e}\, e^2}$$

electric resistivity

The electric resisticity is the inverse of the electric conductivity $\sigma = \eta^{-1}$

Substituting expression for collision frequency ➡ $\eta = 1.69 \ln \Lambda \, \frac{m_\mathrm{e}^{1/2}\, e^2}{(k_\mathrm{B}\, T)^{3/2}}$

$$\boxed{\frac{m_\mathrm{e}\, m_\mathrm{i}}{\rho e^2} \frac{\partial \vec{J}}{\partial t} = \frac{m_\mathrm{i}}{2\rho e} \nabla P + \vec{E} + \frac{1}{c} \vec{u} \times \vec{B} - \frac{m_\mathrm{i}}{\rho e c} \vec{J} \times \vec{B} + \frac{m_\mathrm{i}}{\rho e} \vec{C}_\mathrm{i}}$$

Assuming plasma is cold (P~0), that current is small compared to velocities, and that $\partial J/\partial t$ is small compared to collision term, the generalized Ohm's law simplifies to

$$\boxed{\vec{J} = \sigma \left(\vec{E} + \frac{\vec{u}}{c} \times \vec{B} \right)}$$

Ohm's law

Magnetohydrodynamics

The MHD equations derived thus far (mass continuity, charge continuity, momentum conservation and Ohm's law) need to be complemented with Maxwell equations:

$$\begin{aligned}
\nabla \cdot \vec{E} &= 4\pi(n_\mathrm{i} - n_\mathrm{e})\, e \\
\nabla \cdot \vec{B} &= 0 \\
\nabla \times \vec{E} &= -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \\
\nabla \times \vec{B} &= \frac{4\pi}{c}(n_\mathrm{i}\vec{u}_\mathrm{i} - n_\mathrm{e}\vec{u}_\mathrm{e})\, e + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}
\end{aligned}$$

Gauss' law of magnetism

Faraday's law of induction

Ampère's circuital law

displacement current

Typically, the displacement current is negligle: hence, combining Ampère's circuital law with Ohm's law yields

$$\vec{E} = \frac{c}{4\pi\sigma}(\nabla \times \vec{B}) - \frac{\vec{u}}{c} \times \vec{B}$$

Hence, the electric field follows from the magnetic field (E is not an independent variable)

Magnetohydrodynamics

Plugging this expression for the electric field in Faraday's law of induction yields that

$$\frac{\partial \vec{B}}{\partial t} = -\frac{c^2}{4\pi\sigma}\nabla\times(\nabla\times\vec{B}) + \nabla\times(\vec{u}\times\vec{B})$$

Using the vector identity $\nabla\times(\nabla\times\vec{B}) = \nabla(\nabla\cdot\vec{B}) - \nabla^2\vec{B}$ this can be written as:

$$\boxed{\frac{\partial \vec{B}}{\partial t} = \nabla\times(\vec{u}\times\vec{B}) + \lambda\nabla^2\vec{B}}$$

induction equation

$$\lambda \equiv \frac{c^2}{4\pi\sigma}$$

magnetic diffusivity

Note the similarity with the vorticity equation:

$$\frac{\partial \vec{w}}{\partial t} = \nabla\times(\vec{u}\times\vec{\omega}) - \nabla\times\left(\frac{\nabla P}{\rho}\right) + \nu\nabla^2\vec{\omega}$$

Magnetic diffusivity is to magnetic field as viscosity is to vorticity

Magnetohydrodynamics

Recall the momentum equations:

$$\rho \frac{\mathrm{d}\vec{u}}{\mathrm{d}t} = -\nabla P + \rho_c \vec{E} + \frac{1}{c} \vec{J} \times \vec{B}$$

negligible

Using Ampère's circuital law w/o the displacement current: $\nabla \times \vec{B} = \frac{4\pi}{c}\vec{J} + \cancel{\frac{1}{c}\frac{\partial \vec{E}}{\partial t}}$

we have that: $$\frac{1}{c}(\vec{J} \times \vec{B}) = \frac{1}{4\pi}(\nabla \times \vec{B}) \times \vec{B} = \frac{1}{4\pi}\left[(\vec{B} \cdot \nabla)\vec{B} - \nabla\left(\frac{B^2}{2}\right)\right]$$

Next, using that $$\left(\vec{B} \cdot \nabla\right)\vec{B} = B_j \frac{\partial B_i}{\partial x_j} = \frac{\partial B_i B_j}{\partial x_j} - B_i \frac{\partial B_j}{\partial x_j} = \frac{\partial B_i B_j}{\partial x_j}$$

$\nabla \cdot \vec{B} = 0$

we can write the momentum equations (in index form) as

$$\rho \frac{\mathrm{d}u_i}{\mathrm{d}t} = -\frac{\partial P}{\partial x_i} - \frac{\partial}{\partial x_i}\left(\frac{B^2}{8\pi}\right) + \frac{\partial}{\partial x_j}\left(\frac{B_i B_j}{4\pi}\right) = +\frac{\partial}{\partial x_j}\left[\sigma_{ij} - M_{ij}\right]$$

Here

$$\sigma_{ij} = -P\delta_{ij}$$ stress tensor (w/o viscosity)

$$M_{ij} \equiv \frac{B^2}{8\pi}\delta_{ij} - \frac{B_i B_j}{4\pi}$$ magnetic stress tensor

Magnetohydrodynamics

we can write the momentum equations (in index form) as

$$\rho \frac{\mathrm{d}u_i}{\mathrm{d}t} = -\frac{\partial P}{\partial x_i} - \frac{\partial}{\partial x_i}\left(\frac{B^2}{8\pi}\right) + \frac{\partial}{\partial x_j}\left(\frac{B_i B_j}{4\pi}\right) = +\frac{\partial}{\partial x_j}\left[\sigma_{ij} - M_{ij}\right]$$

Here

$$\sigma_{ij} = -P\delta_{ij}$$ stress tensor (w/o viscosity)

$$M_{ij} \equiv \frac{B^2}{8\pi}\delta_{ij} - \frac{B_i B_j}{4\pi}$$ magnetic stress tensor

The diagonal elements of the magnetic stress tensor represent magnetic pressure

The off-diagonal elements of the magnetic stress tensor represent magnetic tension

Magnetohydrodynamics

Continuity Eq.	$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\rho \nabla \cdot \vec{u}$
Momentum Eqs.	$\rho \frac{\mathrm{d}\vec{u}}{\mathrm{d}t} = -\nabla P + \frac{1}{c} \vec{J} \times \vec{B}$
Energy Eq.	$\rho \frac{\mathrm{d}\varepsilon}{\mathrm{d}t} = -P \nabla \cdot \vec{u} - \frac{J^2}{\sigma}$
Ohm's Law	$\vec{J} = \sigma \left(\vec{E} + \frac{\vec{u}}{c} \times \vec{B} \right)$
Induction Eq.	$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{u} \times \vec{B}) + \lambda \nabla^2 \vec{B}$
Constitutive Eqs.	$\lambda = \frac{c^2}{4\pi\sigma}, \qquad \sigma^{-1} = \eta \propto \frac{m_{\mathrm{e}}^{1/2} e^2}{(k_{\mathrm{B}} T)^{3/2}}$

The equations of resistive MHD

A fluid that obeys these equations is called a magnetofluid

Note that Ohm's law is not required for closure. After all, we can obtain the current directly from Ampère's circuital law w/o the displacement current; $\nabla \times \vec{B} = (4\pi/c)\vec{J}$.

Magnetohydrodynamics

Motivated by the similarities between vorticity equation and the induction equation we define the magnetic Reynold number as

$$\mathcal{R}_{\mathrm{m}} = \frac{U\,L}{\lambda}$$

where we simply replace the kinetic viscocity with the magnetic resistivity

induction equation $$\boxed{\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{u} \times \vec{B}) + \lambda \nabla^2 \vec{B}}$$

[1] When $\mathcal{R}_{\mathrm{m}} \ll 1$ the second term dominates: $\frac{\partial B}{\partial t} \simeq \lambda\, \nabla^2 \vec{B}$

This is situation in laboratory plasmas (U and L small). Hence, laboratory plasmas decay due to magnetic diffusion (magn. fields are related to currents, which die away due to Ohmic dissipation unless one applies a source of voltage)

[2] When $\mathcal{R}_{\mathrm{m}} \gg 1$ the first term dominates: $\frac{\partial B}{\partial t} \simeq \nabla \times (\vec{u} \times \vec{B})$

This is situation in astrophysics (U and L large). This is the realm of ideal MHD, in which electrical resistivity and magnetic diffusivity can be ignored (i.e., electrictal conductivity is infinite) —> No Ohmic dissipation

Magnetohydrodynamics

Continuity Eq.	$\frac{d\rho}{dt} = -\rho \nabla \cdot \vec{u}$
Momentum Eqs.	$\rho \frac{d\vec{u}}{dt} = -\nabla P + \frac{1}{c}\vec{J} \times \vec{B}$
Energy Eq.	$\rho \frac{d\varepsilon}{dt} = -P \nabla \cdot \vec{u}$
Induction Eq.	$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{u} \times \vec{B})$
Ampère's law	$\nabla \times \vec{B} = \frac{4\pi}{c}\vec{J}$

The equations of ideal MHD

Almost all MHD simulations in astronomy assume these ideal MHD conditions

Going from hydrodynamics to ideal MHD, one merely adds magnetic fields, which give rise to magnetic pressure and magnetic tension. Collisions between electrons and ions can create currents, which spawn magnetic fields, which can become amplified…

Magnetohydrodynamics

An important implication of ideal MHD is that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_S \vec{B}\cdot\mathrm{d}^2s = 0$$

Alfvén's theorem of flux freezing

This indicates that the magnetic flux is conserved as it moves with the fluid.

This is equivalent to Helmholz's theorem that the circulation of an inviscid fluid is conserved.

Once more this demonstrates the close similarity between magnetic field lines and vortex lines…

Magnetohydrodynamics

A linear perturbation analysis of the ideal MHD equations yields the following dispersion relation:

$$\omega^2 \vec{u}_1 = (c_s^2 + u_A^2)(\vec{k} \cdot \vec{u}_1)\vec{k} + \vec{u}_A \cdot \vec{k} \left[(\vec{u}_A \cdot \vec{k})\vec{u}_1 - (\vec{u}_A \cdot \vec{u}_1)\vec{k} - (\vec{k} \cdot \vec{u}_1)\vec{u}_A \right]$$

$$\boxed{\vec{u}_A = \frac{\vec{B}_0}{\sqrt{4\pi\rho_0}}}$$

Alfvén velocity

This dispersion relation for hydromagnetic waves has several solutions

One of these is for transverse waves, in which the displacement, and thus the velocity perturbation, $\boldsymbol{u}_1$, is perpendicular to both $\boldsymbol{k}$ and $\boldsymbol{B}$

Under those conditions the dispersion relation reduces to: $\omega^2 = (\vec{u}_A \cdot \vec{k})^2$

These are called the Alfvén waves and have a group velocity $v_g = \partial\omega / \partial\vec{k} = \vec{u}_A$

Similar to waves in a rope/string, Alfven waves are transverse waves along magnetic field lines, for which the restoring force is the magnetic tension

Magnetohydrodynamics

In case of ideal MHD, magnetic resistivity is zero and there is no damping of the Alfvén waves. In reality (resistive MHD), Alfvén waves damp out due to magnetic diffusion, thereby transferring wave energy to thermal energy…

Other solutions to the dispersion relation of hydromagnetic waves are the so-called slow and fast mode waves.

Without going into any details; in slow and fast mode waves the restoring force are BOTH magnetic pressure and magnetic tension.

Any hydromagnetic wave can be represented as a superposition of Alfvén, fast and slow waves,

The End