LECTURE 19
Jeans equations in spherical coordinates: \((r, \theta, \phi)\).

\[
\begin{align*}
\frac{\partial (\rho \langle v_r \rangle)}{\partial t} + \frac{\partial (\rho \langle v_r^2 \rangle)}{\partial r} + \frac{\rho}{r} \left[ 2\langle v_r^2 \rangle - \langle v_r \rangle - \langle v_r^2 \rangle \right] + \rho \frac{\partial \Phi}{\partial r} &= 0 \\
\frac{\partial (\rho \langle v_\theta \rangle)}{\partial t} + \frac{\partial (\rho \langle v_r v_\theta \rangle)}{\partial r} + \frac{\rho}{r} \left[ 3\langle v_r v_\theta \rangle + (\langle v_\theta^2 \rangle - \langle v_\phi^2 \rangle) \cot \theta \right] &= 0 \\
\frac{\partial (\rho \langle v_\phi \rangle)}{\partial t} + \frac{\partial (\rho \langle v_r v_\phi \rangle)}{\partial r} + \frac{\rho}{r} \left[ 3\langle v_r v_\phi \rangle + 2\langle v_\theta v_\phi \rangle \cot \theta \right] &= 0
\end{align*}
\]

Upon inspection, these are 3 equations for a total of 9 unknowns…..no closure.

To proceed, it is common to make the following assumptions:

1. System is static \(\rightarrow\) time derivatives vanish

2. Kinematics are also spherical symmetric \(\rightarrow\) no streaming motions
   \(\rightarrow\) mixed 2nd order motions vanish

Only one Jeans eq. remains:

\[
\frac{\partial (\rho \sigma_r^2)}{\partial r} + \frac{2\rho}{r} \left[ \sigma_r^2 - \sigma_\theta^2 \right] + \rho \frac{\partial \Phi}{\partial r} = 0
\]
Jeans equations in spherical coordinates: 

\[ \frac{\partial (\rho \sigma^2_r)}{\partial r} + \frac{2\rho}{r} [\sigma^2_r - \sigma^2_\theta] + \rho \frac{\partial \Phi}{\partial r} = 0 \]

One equation with two unknowns….

Upon defining the anisotropy parameter

\[ \beta(r) \equiv 1 - \frac{\sigma^2_\theta(r) + \sigma^2_\phi(r)}{2\sigma^2_r(r)} = 1 - \frac{\sigma^2_\theta(r)}{\sigma^2_r(r)} \]

The spherical Jeans equation can be written as

\[ \frac{1}{\rho} \frac{\partial \langle v^2_r \rangle}{\partial r} + 2 \frac{\beta \langle v^2_r \rangle}{r} = -\frac{d\Phi}{dr} \]

which can be solved for any fixed \( \beta \)

Using that \( \frac{d\Phi}{dr} = GM(r)/r \)

this can be written as

\[ M(r) = -\frac{r \langle v^2_r \rangle}{G} \left[ \frac{d\ln \rho}{d\ln r} + \frac{d\ln \langle v^2_r \rangle}{d\ln r} + 2\beta \right] \]

For comparison:

Hydrostatic eq. for collisional fluid

\[ M(r) = -\frac{k_B T(r) r}{\mu m_p G} \left[ \frac{d\ln \rho}{d\ln r} + \frac{d\ln T}{d\ln r} \right] \]
Jeans modeling of spherical systems

For a spherical system the surface brightness $\Sigma(R)$ is related to the 3D luminosity density $\nu(r)$ according to

$$\Sigma(R) = 2 \int_{R}^{\infty} \frac{\nu r \, dr}{\sqrt{r^2 - R^2}}$$

Using the Abel transform, we can solve for the inverse relation, and thus obtain the luminosity density $\nu(r)$ directly from the surface brightness $\Sigma(R)$

$$\nu(r) = -\frac{1}{\pi} \int_{r}^{\infty} \frac{d\Sigma}{dR} \frac{dR}{\sqrt{R^2 - r^2}}$$

The stellar mass density then follows from $\rho(r) = \Upsilon(r) \times \nu(r)$, with $\Upsilon(r)$ the stellar mass-to-light ratio.
Jeans modeling of spherical systems

Similarly, the line-of-sight velocity dispersion, \( \sigma_p^2(R) \), which can be inferred from spectroscopy, is related to both internal dynamics and luminosity density according to

\[
\Sigma(R) \sigma_p^2(R) = 2 \int_0^\infty \left( \langle v_r \cos \alpha - v_\theta \sin \alpha \rangle^2 \right) \frac{\nu r \, dr}{\sqrt{r^2 - R^2}}
\]

\[
= 2 \int_0^\infty \left( \langle v_r^2 \rangle \cos^2 \alpha + \langle v_\theta^2 \rangle \sin^2 \alpha \right) \frac{\nu r \, dr}{\sqrt{r^2 - R^2}}
\]

\[
= 2 \int_0^\infty \left( 1 - \beta \frac{R^2}{r^2} \right) \frac{\nu \langle v_r^2 \rangle \, r \, dr}{\sqrt{r^2 - R^2}}
\]
Assume isotropy, $\beta(r)=0$. In that case we can use the Abel transform to obtain

$$
\nu(r) \langle v_r^2 \rangle (r) = -\frac{1}{\pi} \int_r^\infty \frac{d(\Sigma \sigma_p^2)}{dR} \frac{dR}{\sqrt{R^2 - r^2}}
$$

and the enclosed mass follows from the Jeans equations

$$
M(r) = -\frac{r \langle v_r^2 \rangle}{G} \left[ \frac{d \ln \nu}{d \ln r} + \frac{d \ln \langle v_r^2 \rangle}{d \ln r} \right]
$$

from which one finally obtains the radially dependent mass-to-light ratio

$$
\Upsilon(r) = \frac{M(r)}{4\pi \int_0^r \nu(r) r^2 dr}
$$

which can be used to constrain a potential central Black Hole and/or the contribution of a dark matter halo.

...But any such constraints are ONLY valid under the assumption of isotropy...
Mass-Anisotropy Degeneracy

\[ M(r) = - \frac{r \langle v_r^2 \rangle}{G} \left[ \frac{d \ln \rho}{d \ln r} + \frac{d \ln \langle v_r^2 \rangle}{d \ln r} + 2 \beta \right] \]

Typically, constraints on the mass profile are degenerate with constraints/assumptions about the anisotropy profile.

Breaking this degeneracy typically requires going to higher order Jeans equations, that can predict the kurtosis (or the Gauss-Hermite moment \( h_4 \)) of the line-of-sight velocity distribution (LOSVD)

\[ \Phi(v) \]

\[ v \]

\[ h_3 = -0.1 \]
\[ h_3 = 0 \]
\[ h_3 = 0.1 \]

\[ h_4 = -0.1 \]
\[ h_4 = 0 \]
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van der Marel & Franx (1993)
Radial anisotropy typically results in LOSVDs that are **more** peaked than a Gaussian ($h_4 > 0$).

Azimuthal anisotropy typically results in LOSVDs that are **less** peaked than a Gaussian ($h_4 < 0$).

Making model predictions for $h_4$ (and other Gauss-Hermite moments) requires using higher-order Jeans equations.

At this point it becomes easier to use Schwarschild’s orbit superposition method.
RECALL: An integral of motion is a function $I(x, \vec{v})$ of the phase-space coordinates that is constant along all orbits, i.e.,

$$\frac{dI}{dt} = \frac{\partial I}{\partial x_i} \frac{dx_i}{dt} + \frac{\partial I}{\partial v_i} \frac{dv_i}{dt} = \vec{v} \cdot \vec{\nabla} I - \vec{\nabla} \Phi \cdot \frac{\partial I}{\partial \vec{v}} = 0$$

Compare this to the CBE for a steady-state (static) system:

$$\vec{v} \cdot \vec{\nabla} f - \vec{\nabla} \Phi \cdot \frac{\partial f}{\partial \vec{v}} = 0$$

Thus the condition for $I$ to be an integral of motion is identical with the condition for $I$ to be a steady-state solution of the CBE. Hence:
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**Jeans Theorem** Any steady-state solution of the CBE depends on the phase-space coordinates only through integrals of motion. Any function of these integrals is a steady-state solution of the CBE.
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**Jeans Theorem** Any steady-state solution of the CBE depends on the phase-space coordinates only through integrals of motion. Any function of these integrals is a steady-state solution of the CBE.

PROOF: Let $f$ be any function of the $n$ integrals of motion $I_1, I_2, \ldots I_n$ then

$$\frac{df}{dt} = \sum_{k=1}^{n} \frac{\partial f}{\partial I_k} \frac{dI_k}{dt} = 0$$

which proofs that $f$ satisfies the CBE.
More useful than the Jeans Theorem is the Strong Jeans Theorem, which is due to Lynden-Bell (1962).
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**Strong Jeans Theorem** The DF of a steady-state system in which almost all orbits are regular can be written as a function of the independent isolating integrals of motion, or of the action-integrals.

Note that a regular orbit in a system with $n$ degrees of freedom is uniquely, and completely, specified by the values of the $n$ isolating integrals of motion in involution. Thus the DF can be thought of as a function that expresses the probability for finding a star on each of the phase-space tori.
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We first consider an application of the **Jeans Theorem** to Spherical Systems. As we have seen, any orbit in a spherical potential admits four isolating integrals of motion: \( E, L_x, L_y, L_z \).

Therefore, according to the **Strong Jeans Theorem**, the DF of any\(^\dagger\) steady-state spherical system can be expressed as \( f = f(E, \vec{L}) \).

\(\dagger\) except for point masses and uniform spheres, which have five isolating integrals of motion
If the system is spherically symmetric in all its properties, then $f = f(E, L^2)$ rather than $f = f(E, \vec{L})$: i.e., the DF can only depend on the magnitude of the angular momentum vector, not on its direction.
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Contrary to what one might naively expect, this is not true in general. In fact, as beautifully illustrated by Lynden-Bell (1960), a spherical system can rotate without being oblate.
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Consider a spherical system with $f(E, \vec{L}) = f(E, -\vec{L})$. In such a system, for each star $S$ on an orbit $\mathcal{O}$, there is exactly one star on the same orbit $\mathcal{O}$ but counterrotating with respect to $S$. Consequently, this system is perfectly spherically symmetric in all its properties.
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Now consider all stars in the \( z = 0 \)-plane, and revert the sense of all those stars with \( L_z < 0 \). Clearly this does not influence \( \rho(r) \), but it does give the system a net sense of rotation around the \( z \)-axis.

Thus, although a system with \( f = f(E, L^2) \) is not the most general case, systems with \( f = f(E, \vec{L}) \) are rarely considered in galactic dynamics.
An even simpler case to consider is the one in which $f = f(E)$.

Since $E = \Phi(\mathbf{r}) + \frac{1}{2} [v_r^2 + v_\theta^2 + v_\phi^2]$ we have that

$$\langle v_r^2 \rangle = \frac{1}{\rho} \int dv_r dv_\theta dv_\phi \, v_r^2 \, f \left( \Phi + \frac{1}{2} [v_r^2 + v_\theta^2 + v_\phi^2] \right)$$

$$\langle v_\theta^2 \rangle = \frac{1}{\rho} \int dv_r dv_\theta dv_\phi \, v_\theta^2 \, f \left( \Phi + \frac{1}{2} [v_r^2 + v_\theta^2 + v_\phi^2] \right)$$

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Since these equations differ only in the labelling of one of the variables of integration, it is immediately evident that \( \langle v_r^2 \rangle = \langle v_\theta^2 \rangle = \langle v_\phi^2 \rangle \).
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Note that from

\[
\langle v_i \rangle = \frac{1}{\rho} \int dv_r dv_\theta dv_\phi v_i f \left( \Phi + \frac{1}{2}[v_r^2 + v_\theta^2 + v_\phi^2] \right)
\]

it is also immediately evident that \( \langle v_r \rangle = \langle v_\theta \rangle = \langle v_\phi \rangle = 0 \). Thus, similar as for a system with \( f = f(E, L^2) \) a system with \( f = f(E) \) has no net sense of rotation.
In what follows we define the relative potential \( \Psi \equiv -\Phi + \Phi_0 \) and relative energy \( \mathcal{E} = -E + \Phi_0 = \Psi - \frac{1}{2}\nu^2 \). In general one chooses \( \Phi_0 \) such that \( f > 0 \) for \( \mathcal{E} > 0 \) and \( f = 0 \) for \( \mathcal{E} \leq 0 \).

Now consider a self-consistent, spherically symmetric system with \( f = f(\mathcal{E}) \). Here self-consistent means that the potential is due to the system itself, i.e.,

\[
\nabla^2 \Psi = -4\pi G \rho = -4\pi G \int f(\mathcal{E}) d^3 \vec{r}
\]

(note the minus sign in the Poisson equation), which can be written as

\[
\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\Psi}{dr} \right) = -16\pi^2 G \int_0^{\Psi} f(\mathcal{E}) \sqrt{2(\Psi - \mathcal{E})} \, d\mathcal{E}
\]
Using that $\Psi$ is a monotonic function of $r$, so that $\rho$ can be regarded as a function of $\Psi$, we have

$$\rho(\Psi) = \int f d^3\mathbf{v} = 4\pi \int_0^\Psi f(\mathcal{E}) \sqrt{2(\Psi - \mathcal{E})} d\mathcal{E}$$

differentiating both sides with respect to $\Psi$ yields

$$\frac{1}{\sqrt{8\pi}} \frac{d\rho}{d\Psi} = \int_0^\Psi \frac{f(\mathcal{E}) d\mathcal{E}}{\sqrt{\Psi - \mathcal{E}}}$$

which is an Abel integral equation, whose solution is

$$f(\mathcal{E}) = \frac{1}{\sqrt{8\pi^2}} \frac{d}{d\mathcal{E}} \int_0^\mathcal{E} \frac{d\rho}{d\Psi} \frac{d\Psi}{\sqrt{\mathcal{E} - \Psi}}$$

This is called Eddington’s formula, which may also be written in the form

$$f(\mathcal{E}) = \frac{1}{\sqrt{8\pi^2}} \left[ \int_0^\mathcal{E} \frac{d^2\rho}{d\Psi^2} \frac{d\Psi}{\sqrt{\mathcal{E} - \Psi}} + \frac{1}{\sqrt{\mathcal{E}}} \left( \frac{d\rho}{d\Psi} \right)_{\Psi=0} \right]$$
Using Eddington’s formula

\[ f(\mathcal{E}) = \frac{1}{\sqrt{8\pi^2}} \frac{d}{d\mathcal{E}} \int_0^{\mathcal{E}} \frac{d\rho}{d\Psi} \frac{d\Psi}{\sqrt{\mathcal{E}-\Psi}} \]

we see that the requirement \( f(\mathcal{E}) \geq 0 \) is identical to the requirement that the function

\[ \int_0^{\mathcal{E}} \frac{d\rho}{d\Psi} \frac{d\Psi}{\sqrt{\mathcal{E}-\Psi}} \]

is an increasing function of \( \mathcal{E} \).

If a density distribution \( \rho(r) \) does not satisfy this requirement, then the model obtained by setting the anisotropy parameter \( \beta = 0 \) [i.e., by assuming that \( f = f(\mathcal{E}) \)] and solving the Jeans Equations is unphysical.

There are limits to self-consistent, isotropic, spherical density distributions...
SPHERICAL MODELS: SUMMARY

In its most general form, the DF of a static, spherically symmetric model has the form $f = f(E, \vec{L})$. From the symmetry of individual orbits one can see that one always has to have

$$\langle v_r \rangle = \langle v_\theta \rangle = 0 \quad \langle v_r v_\phi \rangle = \langle v_r v_\theta \rangle = \langle v_\theta v_\phi \rangle = 0$$

This leaves four unknowns: $\langle v_\phi \rangle$, $\langle v_r^2 \rangle$, $\langle v_\theta^2 \rangle$, and $\langle v_\phi^2 \rangle$.
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If one makes the assumption that the system is spherically symmetric in all its properties then \( f(E, \vec{L}) \rightarrow f(E, L^2) \) and

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$$\langle v_\phi \rangle = 0, \quad \langle v_\theta^2 \rangle = \langle v_\phi^2 \rangle$$

In this case the only non-trivial Jeans equation is

$$\frac{1}{\rho} \frac{\partial (\rho \langle v_r^2 \rangle)}{\partial r} + 2 \frac{\beta \langle v_r^2 \rangle}{r} = - \frac{d\Phi}{dr}$$

with the anisotropy parameter defined by

$$\beta(r) = 1 - \frac{\langle v_r^2 \rangle(r)}{\langle v_r^2 \rangle(r)}$$
SPHERICAL MODELS: SUMMARY

Many different models, with different orbital anisotropies, can correspond to the same density distribution. Examples of models are:

- \( f(E, L^2) = f(E) \) \hspace{1cm} \text{isotropic model, i.e., } \beta = 0
- \( f(E, L^2) = g(E) \delta(L) \) \hspace{1cm} \text{radial orbits only, i.e. } \beta = 1
- \( f(E, L^2) = g(E) \delta[L - L_c(E)] \) \hspace{1cm} \text{circular orbits only, i.e., } \beta = -\infty
- \( f(E, L^2) = g(E) L^{-2\beta} \) \hspace{1cm} \text{constant anisotropy, i.e. } \beta(r) = \beta
- \( f(E, L^2) = g(E) h(L) \) \hspace{1cm} \text{anisotropy depends on circularity function } h
- \( f(E, L^2) = f(E + L^2 / 2r_a^2) \) \hspace{1cm} \text{center isotropic, outside radial}

Osipkov-Merritt models
Next we consider \textit{axisymmetric} systems. If we only consider systems for which most orbits are regular, then the strong Jeans Theorem states that, in the most general case, \( f = f(E, \, L_z, \, I_3) \).
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From the symmetries of the individual orbits, it is evident that in this case

\[
\langle v_R \rangle = \langle v_z \rangle = 0 \quad \langle v_R v_\phi \rangle = \langle v_z v_\phi \rangle = 0
\]

Note that, in this case, \( \langle v_R v_z \rangle \neq 0 \), which is immediately evident when considering a thin tube orbit. In other words, in general the velocity ellipsoid is not aligned with \((R, \phi, z)\).
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Thus, in a three-integral model with \( f = f(E, L_z, I_3) \) the stress tensor contains four unknowns: \( \langle v_R^2 \rangle, \langle v_\phi^2 \rangle, \langle v_z^2 \rangle \), and \( \langle v_R v_z \rangle \).
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\langle v_R \rangle = \langle v_z \rangle = 0 \quad \langle v_R v_\phi \rangle = \langle v_z v_\phi \rangle = 0
\]

Note that, in this case, \( \langle v_R v_z \rangle \neq 0 \), which is immediately evident when considering a thin tube orbit. In other words, in general the velocity ellipsoid is not aligned with \((R, \phi, z)\).

Thus, in a three-integral model with \( f = f(E, L_z, I_3) \) the stress tensor contains four unknowns: \( \langle v^2_R \rangle, \langle v^2_\phi \rangle, \langle v^2_z \rangle \), and \( \langle v_R v_z \rangle \).

In this case there are two non-trivial Jeans Equations:

\[
\frac{\partial (\rho \langle v^2_R \rangle)}{\partial R} + \frac{\partial (\rho \langle v_R v_z \rangle)}{\partial z} + \rho \left[ \frac{\langle v^2_R \rangle - \langle v^2_\phi \rangle}{R} + \frac{\partial \Phi}{\partial R} \right] = 0
\]

\[
\frac{\partial (\rho \langle v_R v_z \rangle)}{\partial R} + \frac{\partial (\rho \langle v^2_z \rangle)}{\partial z} + \rho \left[ \frac{\langle v_R v_z \rangle}{R} + \frac{\partial \Phi}{\partial z} \right] = 0
\]

which clearly doesn’t suffice to solve for the four unknowns.
To make progress, one therefore often makes the additional assumption that the DF has the two-integral form $f = f(E, L_z)$. 
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It is not that difficult to show that, under these conditions,

$$f = f(E, L_z) \implies \langle v_R^2 \rangle = \langle v_z^2 \rangle \text{ and } \langle v_R v_z \rangle = 0$$
To make progress, one therefore often makes the additional assumption that the DF has the **two-integral form** \( f = f(E, L_z) \).

It is not that difficult to show that, under these conditions,

\[
  f = f(E, L_z) \quad \Rightarrow \quad \langle v_R^2 \rangle = \langle v_z^2 \rangle \text{ and } \langle v_R v_z \rangle = 0
\]

Now we have two unknowns left, \( \langle v_R^2 \rangle \) and \( \langle v_\phi^2 \rangle \), and the Jeans equations reduce to

\[
  \frac{\partial (\rho \langle v_R^2 \rangle)}{\partial R} + \rho \left( \frac{\langle v_R^2 \rangle - \langle v_\phi^2 \rangle}{R} + \frac{\partial \Phi}{\partial R} \right) = 0
\]

\[
  \frac{\partial (\rho \langle v_z^2 \rangle)}{\partial z} + \rho \frac{\partial \Phi}{\partial z} = 0
\]

which can be solved. Note, however, that the Jeans equations provide no information regarding how \( \langle v_\phi^2 \rangle \) splits in streaming and random motions.

In practice one often follows Satoh (1980), and writes that

\[\langle v_\phi \rangle^2 = k \left[ \langle v_\phi^2 \rangle - \langle v_R^2 \rangle \right].\]

Here \( k \) is a free parameter, and the model is **isotropic** for \( k = 1 \).
Given a density distribution \( \rho(R, z) \), one can calculate the corresponding \( f_{+}(\varepsilon, L_z) \) using a complex contour integral equation (due to Hunter & Qian 1993) that is the equivalent of the Eddington’s formula for spherical systems.

\[
\rho = \frac{2\pi}{R} \int_0^\Psi d\varepsilon \int_{L_z < 2(\Psi - \varepsilon)R^2} f(\varepsilon, L_z) dL_z
= \frac{2\pi}{R} \int_0^\Psi d\varepsilon \int_0^{R\sqrt{2(\Psi - \varepsilon)}} [f(\varepsilon, L_z) + f(\varepsilon, -L_z)] dL_z
= \frac{4\pi}{R} \int_0^\Psi d\varepsilon \int_0^{R\sqrt{2(\Psi - \varepsilon)}} f_{+}(\varepsilon, L_z) dL_z
\]

where we have defined \( f_{+} \) as the part of the DF that is even in \( L_z \), i.e.,

\[
f(\varepsilon, L_z) = f_{+}(\varepsilon, L_z) + f_{-}(\varepsilon, L_z)
\]

\[
f_{\pm}(\varepsilon, L_z) \equiv \frac{1}{2} [f(\varepsilon, L_z) \pm f(\varepsilon, -L_z)]
\]

We thus see that the density depends only on the even part of the DF (i.e., the density contributed by a star does not depend on its sense of rotation). This also implies that there are infinitely many DFs \( f(E, L_z) \) that correspond to exactly the same \( \rho(R, z) \), namely all those that only differ in \( f_{-}(\varepsilon, L_z) \).

Given a density distribution \( \rho(R, z) \), one can calculate the corresponding \( f_{+}(\varepsilon, L_z) \) using a complex contour integral equation (due to Hunter & Qian 1993) that is the equivalent of the Eddington’s formula for spherical systems.

The odd part of the DF, \( f_{-}(\varepsilon, L_z) \), specifies the rotational streaming motion.