LECTURE 19

Jeans equations in spherical coordinates: (r, θ, ϕ)

$$\boxed{\begin{aligned}
&\frac{\partial(\rho\langle v_r\rangle)}{\partial t} + \frac{\partial(\rho\langle v_r^2\rangle)}{\partial r} + \frac{\rho}{r}\left[2\langle v_r^2\rangle - \langle v_\theta^2\rangle - \langle v_\phi^2\rangle\right] + \rho\frac{\partial\Phi}{\partial r} = 0 \\
&\frac{\partial(\rho\langle v_\theta\rangle)}{\partial t} + \frac{\partial(\rho\langle v_r v_\theta\rangle)}{\partial r} + \frac{\rho}{r}\left[3\langle v_r v_\theta\rangle + \left(\langle v_\theta^2\rangle - \langle v_\phi^2\rangle\right)\cot\theta\right] = 0 \\
&\frac{\partial(\rho\langle v_\phi\rangle)}{\partial t} + \frac{\partial(\rho\langle v_r v_\phi\rangle)}{\partial r} + \frac{\rho}{r}\left[3\langle v_r v_\phi\rangle + 2\langle v_\theta v_\phi\rangle\cot\theta\right] = 0
\end{aligned}}$$

Upon inspection, these are 3 equations for a total of 9 unknowns…..no closure.

To proceed, it is common to make the following assumptions:

1. System is static → time derivatives vanish
2. Kinematics are also spherical symmetric → no streaming motions
   → mixed 2nd order motions vanish

Only one Jeans eq. remains:

$$\frac{\partial(\rho\sigma_r^2)}{\partial r} + \frac{2\rho}{r}\left[\sigma_r^2 - \sigma_\theta^2\right] + \rho\frac{\partial\Phi}{\partial r} = 0$$

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One equation with two unknowns….

Upon defining the anisotropy parameter $\beta(r) \equiv 1 - \frac{\sigma_\theta^2(r) + \sigma_\phi^2(r)}{2\sigma_r^2(r)} = 1 - \frac{\sigma_\theta^2(r)}{\sigma_r^2(r)}$

The spherical Jeans equation can be written as

$$\boxed{\frac{1}{\rho}\frac{\partial(\rho\langle v_r^2\rangle)}{\partial r} + 2\frac{\beta\langle v_r^2\rangle}{r} = -\frac{\mathrm{d}\Phi}{\mathrm{d}r}}$$

which can be solved for any fixed ***β***

Using that $\mathrm{d}\Phi/\mathrm{d}r = GM(r)/r$
this can be written as

$$\boxed{M(r) = -\frac{r\langle v_r^2\rangle}{G}\left[\frac{\mathrm{d}\ln\rho}{\mathrm{d}\ln r} + \frac{\mathrm{d}\ln\langle v_r^2\rangle}{\mathrm{d}\ln r} + 2\beta\right]}$$

For comparison:

Hydrostatic eq. for collisional fluid

$$\boxed{M(r) = -\frac{k_{\mathrm{B}}T(r)\,r}{\mu m_{\mathrm{p}}G}\left[\frac{\mathrm{d}\ln\rho}{\mathrm{d}\ln r} + \frac{\mathrm{d}\ln T}{\mathrm{d}\ln r}\right]}$$

Jeans modeling of spherical systems

For a spherical system the surface brightness Σ(R) is related to the 3D luminosity density ***ν***(r) acccording to

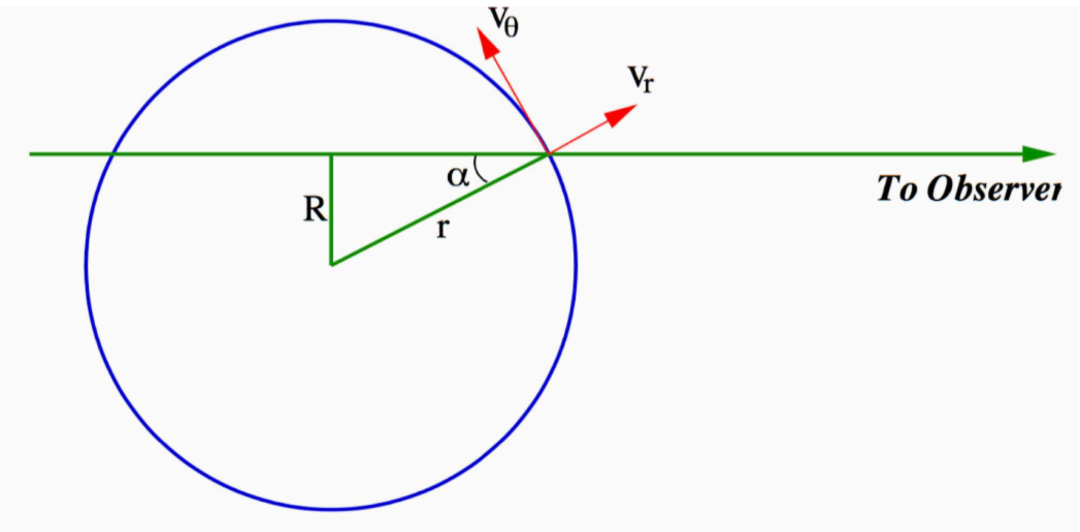
$$\Sigma(R) = 2 \int_R^\infty \frac{\nu \, r \, \mathrm{d}r}{\sqrt{r^2 - R^2}}$$

Using the Abel transform, we can solve for the inverse relation, and thus obtain the luminosity density ***ν***(r) directly from the surface brightness Σ(R)

$$\nu(r) = -\frac{1}{\pi} \int_r^\infty \frac{\mathrm{d}\Sigma}{\mathrm{d}R} \frac{\mathrm{d}R}{\sqrt{R^2 - r^2}}$$

The stellar mass density then follows from $\rho(r) = \Upsilon(r) \times \nu(r)$, with $\Upsilon(r)$ the stellar mass-to-light ratio

Jeans modeling of spherical systems



Similarly, the line-of-sight velocity dispersion, $\sigma_p{}^2(R)$, which can be inferred from spectroscopy, is related to both internal dynamics and luminosity density according to

$$\begin{aligned}
\Sigma(R)\sigma_p^2(R) &= 2\int_R^\infty \langle (v_r\cos\alpha - v_\theta\sin\alpha)^2\rangle \frac{\nu\, r\,\mathrm{d}r}{\sqrt{r^2-R^2}} \\
&= 2\int_R^\infty \left(\langle v_r^2\rangle\cos^2\alpha + \langle v_\theta^2\rangle\sin^2\alpha\right)\frac{\nu\, r\,\mathrm{d}r}{\sqrt{r^2-R^2}} \\
&= 2\int_R^\infty \left(1-\beta\frac{R^2}{r^2}\right)\frac{\nu\,\langle v_r^2\rangle\, r\,\mathrm{d}r}{\sqrt{r^2-R^2}}
\end{aligned}$$

Jeans modeling of spherical systems

Example

Assume isotropy, $\beta(r)=0$. In that case we can use the Abel transform to obtain

$$\nu(r)\langle v_r^2\rangle(r) = -\frac{1}{\pi}\int_r^\infty \frac{\mathrm{d}(\Sigma\sigma_p^2)}{\mathrm{d}R}\frac{\mathrm{d}R}{\sqrt{R^2-r^2}}$$

and the enclosed mass follows from the Jeans equations

$$M(r) = -\frac{r\langle v_r^2\rangle}{G}\left[\frac{\mathrm{d}\ln\nu}{\mathrm{d}\ln r}+\frac{\mathrm{d}\ln\langle v_r^2\rangle}{\mathrm{d}\ln r}\right]$$

from which one finally obtains the radially dependent mass-to-light ratio

$$\Upsilon(r) = \frac{M(r)}{4\pi\int_0^r \nu(r)\, r^2\,\mathrm{d}r}$$

which can be used to constrain a potential central Black Hole and/or the contribution of a dark matter halo.

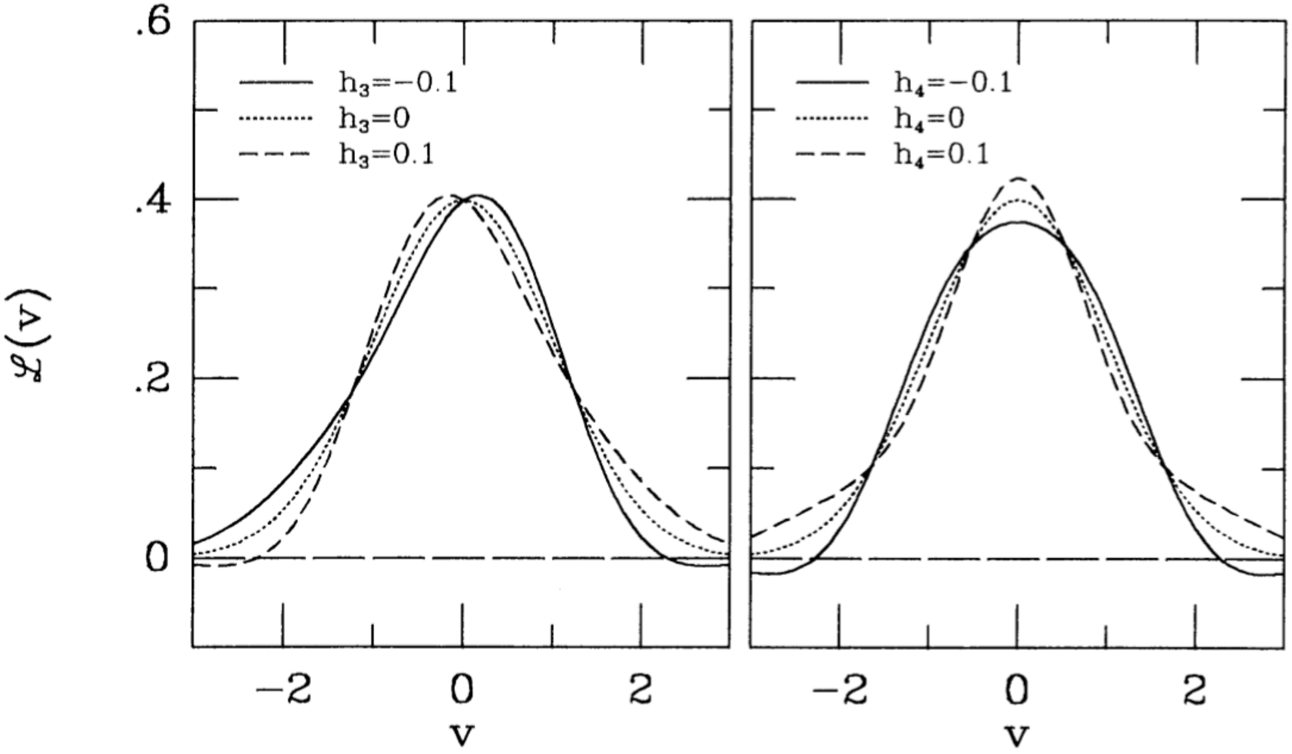
…But any such constraints are ONLY valid under the assumption of isotropy…

Mass-Anisotropy Degeneracy

$$\boxed{M(r) = -\frac{r\langle v_r^2\rangle}{G}\left[\frac{\mathrm{d}\ln\rho}{\mathrm{d}\ln r} + \frac{\mathrm{d}\ln\langle v_r^2\rangle}{\mathrm{d}\ln r} + 2\beta\right]}$$

Typically, constraints on the mass profile are degenerate with constraints/assumptions about the anisotropy profile.

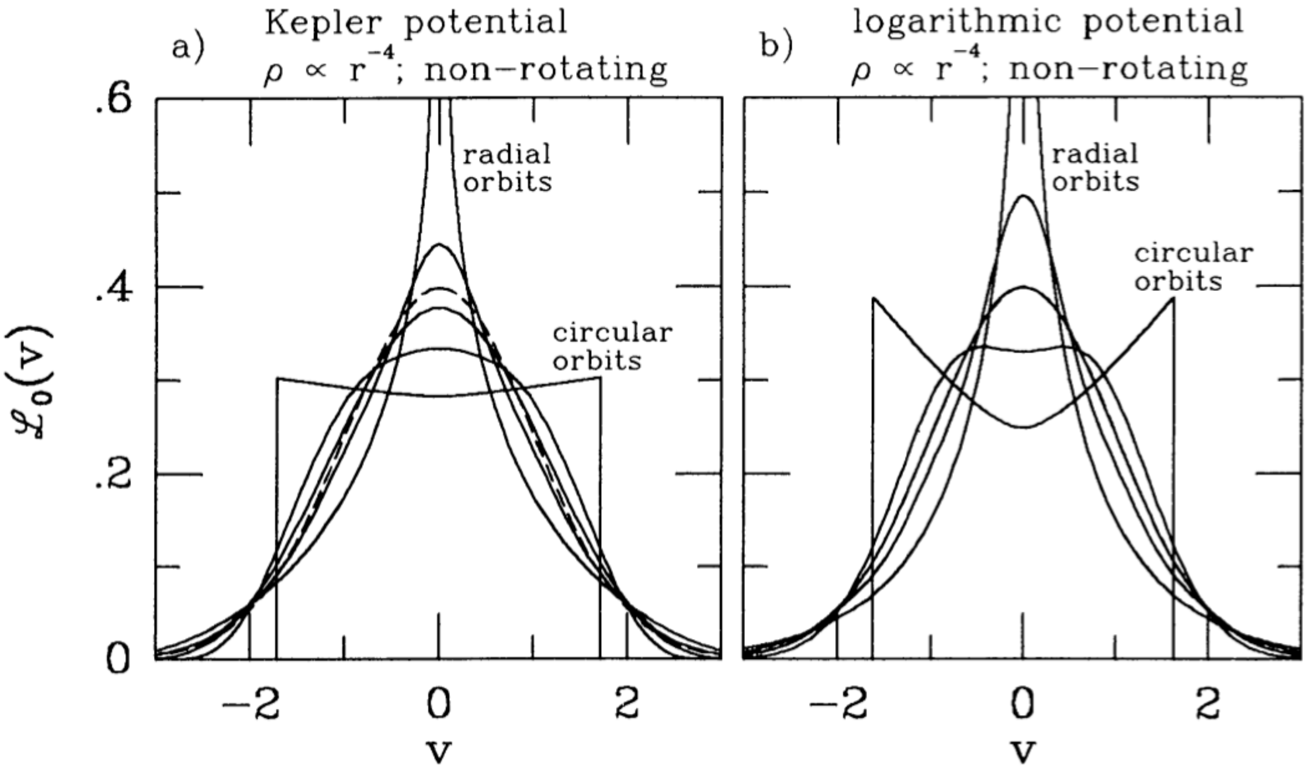
Breaking this degeneracy typically requires going to higher order Jeans equations, that can predict the kurtosis (or the Gauss-Hermite moment h_4) of the line-of-sight velocity distribution (LOSVD)



van der Marel & Franx (1993)

Radial anisotropy typically results in LOSVDs that are more peaked than a Gaussian ($h_4 > 0$)

Azimuthal anisotropy typically results in LOSVDs that are less peaked than a Gaussian ($h_4 < 0$)



van der Marel & Franx (1993)

Making model predictions for h_4 (and other Gauss-Hermite moments) requires using higher-order Jeans equations….

At this point it becomes easier to use Schwarzschild’s orbit superposition method

RECALL: An **integral of motion** is a function $I(\vec{x}, \vec{v})$ of the phase-space coordinates that is constant along **all** orbits, i.e.,

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{\partial I}{\partial x_i}\frac{\mathrm{d}x_i}{\mathrm{d}t} + \frac{\partial I}{\partial v_i}\frac{\mathrm{d}v_i}{\mathrm{d}t} = \vec{v} \cdot \vec{\nabla} I - \vec{\nabla}\Phi \cdot \frac{\partial I}{\partial \vec{v}} = 0$$

Compare this to the **CBE** for a steady-state (static) system:

$$\vec{v} \cdot \vec{\nabla} f - \vec{\nabla}\Phi \cdot \frac{\partial f}{\partial \vec{v}} = 0$$

Thus the condition for I to be an **integral of motion** is identical with the condition for I to be a **steady-state** solution of the **CBE**. Hence:

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PROOF: Let f be any function of the n integrals of motion $I_1, I_2, ...I_n$ then

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \sum_{k=1}^{n} \frac{\partial f}{\partial I_k}\frac{\mathrm{d}I_k}{\mathrm{d}t} = 0$$

which proofs that f satisfies the CBE.

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Strong Jeans Theorem The DF of a steady-state system in which almost all orbits are regular can be written as a function of the independent isolating integrals of motion, or of the action-integrals.

Note that a regular orbit in a system with n degrees of freedom is uniquely, and completely, specified by the values of the n isolating integrals of motion in involution. Thus the DF can be thought of as a function that expresses the probability for finding a star on each of the phase-space tori.

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We first consider an application of the Jeans Theorem to Spherical Systems
As we have seen, any orbit in a spherical potential admits four isolating integrals of motion: E, L_x, L_y, L_z.

Therefore, according to the Strong Jeans Theorem, the DF of any[†] steady-state spherical system can be expressed as $f = f(E, \vec{L})$.

[†] except for point masses and uniform spheres, which have five isolating integrals of motion

If the system is spherically symmetric in all its properties, then $f = f(E, L^2)$ rather than $f = f(E, \vec{L})$: ie., the DF can only depend on the magnitude of the angular momentum vector, not on its direction.

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Consider a spherical system with $f(E, \vec{L}) = f(E, -\vec{L})$. In such a system, for each star S on a orbit $\mathcal{O}$, there is exactly one star on the same orbit $\mathcal{O}$ but counterrotating with respect to S. Consequently, this system is perfectly spherically symmetric in all its properties.

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Now consider all stars in the $z = 0$-plane, and revert the sense of all those stars with $L_z < 0$. Clearly this does not influence $\rho(r)$, but it does give the system a net sense of rotation around the z-axis.

Thus, although a system with $f = f(E, L^2)$ is not the most general case, systems with $f = f(E, \vec{L})$ are rarely considered in galactic dynamics.

An even simpler case to consider is the one in which $f = f(E)$.

Since $E = \Phi(\vec{r}) + \frac{1}{2}[v_r^2 + v_\theta^2 + v_\phi^2]$ we have that

$$\langle v_r^2 \rangle = \frac{1}{\rho} \int \mathrm{d}v_r \mathrm{d}v_\theta \mathrm{d}v_\phi \, v_r^2 \, f\left(\Phi + \frac{1}{2}[v_r^2 + v_\theta^2 + v_\phi^2]\right)$$

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Note that from

$$\langle v_i \rangle = \frac{1}{\rho} \int \mathrm{d}v_r \mathrm{d}v_\theta \mathrm{d}v_\phi \, v_i \, f\left(\Phi + \frac{1}{2}[v_r^2 + v_\theta^2 + v_\phi^2]\right)$$

it is also immediately evident that $\langle v_r \rangle = \langle v_\theta \rangle = \langle v_\phi \rangle = 0$. Thus, similar as for a system with $f = f(E, L^2)$ a system with $f = f(E)$ has no net sense of rotation.

In what follows we define the relative potential $\Psi \equiv -\Phi + \Phi_0$ and relative energy $\mathcal{E} = -E + \Phi_0 = \Psi - \frac{1}{2}v^2$. In general one chooses Φ_0 such that $f > 0$ for $\mathcal{E} > 0$ and $f = 0$ for $\mathcal{E} \leq 0$

Now consider a self-consistent, spherically symmetric system with $f = f(\mathcal{E})$. Here self-consistent means that the potential is due to the system itself, i.e.,

$$\boxed{\nabla^2\Psi = -4\pi G\rho = -4\pi G \int f(\mathcal{E})\mathrm{d}^3\vec{v}}$$

(note the minus sign in the Poisson equation), which can be written as

$$\boxed{\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\Psi}{\mathrm{d}r}\right) = -16\pi^2 G \int_0^{\Psi} f(\mathcal{E})\,\sqrt{2(\Psi - \mathcal{E})}\,\mathrm{d}\mathcal{E}}$$

Using that Ψ is a monotonic function of r, so that ρ can be regarded as a function of Ψ, we have

$$\rho(\Psi) = \int f \mathrm{d}^3\vec{v} = 4\pi \int\limits_0^{\Psi} f(\mathcal{E})\sqrt{2(\Psi - \mathcal{E})}\mathrm{d}\mathcal{E}$$

differentiating both sides with respect to Ψ yields

$$\frac{1}{\sqrt{8}\pi}\frac{\mathrm{d}\rho}{\mathrm{d}\Psi} = \int\limits_0^{\Psi} \frac{f(\mathcal{E})\,\mathrm{d}\mathcal{E}}{\sqrt{\Psi - \mathcal{E}}}$$

which is an Abel integral equation, whose solution is

$$\boxed{f(\mathcal{E}) = \frac{1}{\sqrt{8}\pi^2}\frac{\mathrm{d}}{\mathrm{d}\mathcal{E}}\int\limits_0^{\mathcal{E}} \frac{\mathrm{d}\rho}{\mathrm{d}\Psi}\frac{\mathrm{d}\Psi}{\sqrt{\mathcal{E} - \Psi}}}$$

This is called Eddington's formula, which may also be written in the form

$$\boxed{f(\mathcal{E}) = \frac{1}{\sqrt{8}\pi^2}\left[\int\limits_0^{\mathcal{E}} \frac{\mathrm{d}^2\rho}{\mathrm{d}\Psi^2}\frac{\mathrm{d}\Psi}{\sqrt{\mathcal{E} - \Psi}} + \frac{1}{\sqrt{\mathcal{E}}}\left(\frac{\mathrm{d}\rho}{\mathrm{d}\Psi}\right)_{\Psi=0}\right]}$$

Using Eddington's formula

$$\boxed{f(\mathcal{E}) = \frac{1}{\sqrt{8}\pi^2}\frac{\mathrm{d}}{\mathrm{d}\mathcal{E}}\int\limits_0^{\mathcal{E}} \frac{\mathrm{d}\rho}{\mathrm{d}\Psi}\frac{\mathrm{d}\Psi}{\sqrt{\mathcal{E}-\Psi}}}$$

we see that the requirement $f(\mathcal{E}) \geq 0$ is identical to the the requirement that the function

$$\int\limits_0^{\mathcal{E}} \frac{\mathrm{d}\rho}{\mathrm{d}\Psi}\frac{\mathrm{d}\Psi}{\sqrt{\mathcal{E}-\Psi}}$$

is an increasing function of $\mathcal{E}$.

If a density distribution $\rho(r)$ does not satisfy this requirement, then the model obtained by setting the anisotropy parameter $\beta = 0$ [i.e., by assuming that $f = f(\mathcal{E})$] and solving the Jeans Equations is unphysical.

There are limits to self-consistent, isotropic, spherical density distributions…

SPHERICAL MODELS: SUMMARY

In its most general form, the DF of a static, spherically symmetric model has the form $f = f(E, \vec{L})$. From the symmetry of individual orbits one can see that one always has to have

$$\boxed{\langle v_r \rangle = \langle v_\theta \rangle = 0 \qquad\qquad \langle v_r v_\phi \rangle = \langle v_r v_\theta \rangle = \langle v_\theta v_\phi \rangle = 0}$$

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If one makes the assumption that the system is spherically symmetric in all its properties then $f(E, \vec{L}) \longrightarrow f(E, L^2)$ and

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In this case the only non-trivial Jeans equation is

$$\boxed{\frac{1}{\rho} \frac{\partial(\rho \langle v_r^2 \rangle)}{\partial r} + 2 \frac{\beta \langle v_r^2 \rangle}{r} = -\frac{\mathrm{d}\Phi}{\mathrm{d}r}}$$

with the anisotropy parameter defined by

$$\boxed{\beta(r) = 1 - \frac{\langle v_r^2 \rangle(r)}{\langle v_r^2 \rangle(r)}}$$

SPHERICAL MODELS: SUMMARY

Many different models, with different orbital anisotropies, can correspond to the same density distribution. Examples of models are:

- $f(E,L^2) = f(E)$ **isotropic model, i.e., $\beta = 0$**
- $f(E,L^2) = g(E)\delta(L)$ **radial orbits only, i.e. $\beta = 1$**
- $f(E,L^2) = g(E)\delta[L - L_c(E)]$ **circular orbits only, i.e., $\beta = -\infty$**
- $f(E,L^2) = g(E)L^{-2\beta}$ **constant anisotropy, i.e. $\beta(r) = \beta$**
- $f(E,L^2) = g(E)h(L)$ **anisotropy depends on circularity function h**
- $f(E,L^2) = f(E + L^2/2r_a^2)$ **center isotropic, outside radial**

Osipkov-Merritt models

Next we consider **axisymmetric** systems. If we only consider systems for which most orbits are regular, then the **strong Jeans Theorem** states that, in the most general case, $f = f(E, L_z, I_3)$.

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From the symmetries of the individual orbits, it is evident that in this case

$$\boxed{\langle v_R \rangle = \langle v_z \rangle = 0 \qquad\qquad \langle v_R v_\phi \rangle = \langle v_z v_\phi \rangle = 0}$$

Note that, in this case, $\langle v_R v_z \rangle \neq 0$, which is immediately evident when considering a thin tube orbit. In other words, in general the velocity ellipsoid is not aligned with (R, ϕ, z).

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$$\boxed{\langle v_R \rangle = \langle v_z \rangle = 0 \qquad\qquad \langle v_R v_\phi \rangle = \langle v_z v_\phi \rangle = 0}$$

Note that, in this case, $\langle v_R v_z \rangle \neq 0$, which is immediately evident when considering a thin tube orbit. In other words, in general the velocity ellipsoid is not aligned with (R, ϕ, z).

Thus, in a three-integral model with $f = f(E, L_z, I_3)$ the stress tensor contains four unknowns: $\langle v_R^2 \rangle$, $\langle v_\phi^2 \rangle$, $\langle v_z^2 \rangle$, and $\langle v_R v_z \rangle$.

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In this case there are two non-trivial Jeans Equations:

$$\boxed{\begin{array}{l} \dfrac{\partial(\rho\langle v_R^2\rangle)}{\partial R} + \dfrac{\partial(\rho\langle v_R v_z\rangle)}{\partial z} + \rho\left[\dfrac{\langle v_R^2\rangle - \langle v_\phi^2\rangle}{R} + \dfrac{\partial\Phi}{\partial R}\right] = 0 \\ \dfrac{\partial(\rho\langle v_R v_z\rangle)}{\partial R} + \dfrac{\partial(\rho\langle v_z^2\rangle)}{\partial z} + \rho\left[\dfrac{\langle v_R v_z\rangle}{R} + \dfrac{\partial\Phi}{\partial z}\right] = 0 \end{array}}$$

which clearly doesn't suffice to solve for the four unknowns.

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It is not that difficult to show that, under these conditions,

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Now we have two unknowns left, $\langle v_R^2 \rangle$ and $\langle v_\phi^2 \rangle$, and the Jeans equations reduce to

$$\boxed{\begin{array}{l} \frac{\partial(\rho \langle v_R^2 \rangle)}{\partial R} + \rho \left[\frac{\langle v_R^2 \rangle - \langle v_\phi^2 \rangle}{R} + \frac{\partial \Phi}{\partial R} \right] = 0 \\ \frac{\partial(\rho \langle v_z^2 \rangle)}{\partial z} + \rho \frac{\partial \Phi}{\partial z} = 0 \end{array}}$$

which can be solved. Note, however, that the Jeans equations provide no information regarding how $\langle v_\phi^2 \rangle$ splits in streaming and random motions.

In practice one often follows Satoh (1980), and writes that $\langle v_\phi \rangle^2 = k \left[\langle v_\phi^2 \rangle - \langle v_R^2 \rangle \right]$. Here k is a free parameter, and the model is isotropic for $k = 1$.

$$\begin{aligned}\rho &= \frac{2\pi}{R}\int_0^{\Psi} \mathrm{d}\mathcal{E}\int_{L_z^2<2(\Psi-\mathcal{E})R^2} f(\mathcal{E},L_z)\mathrm{d}L_z \\ &= \frac{2\pi}{R}\int_0^{\Psi} \mathrm{d}\mathcal{E}\int_0^{R\sqrt{2(\Psi-\mathcal{E})}} \left[f(\mathcal{E},L_z)+f(\mathcal{E},-L_z)\right]\mathrm{d}L_z \\ &= \frac{4\pi}{R}\int_0^{\Psi} \mathrm{d}\mathcal{E}\int_0^{R\sqrt{2(\Psi-\mathcal{E})}} f_+(\mathcal{E},L_z)\mathrm{d}L_z\end{aligned}$$

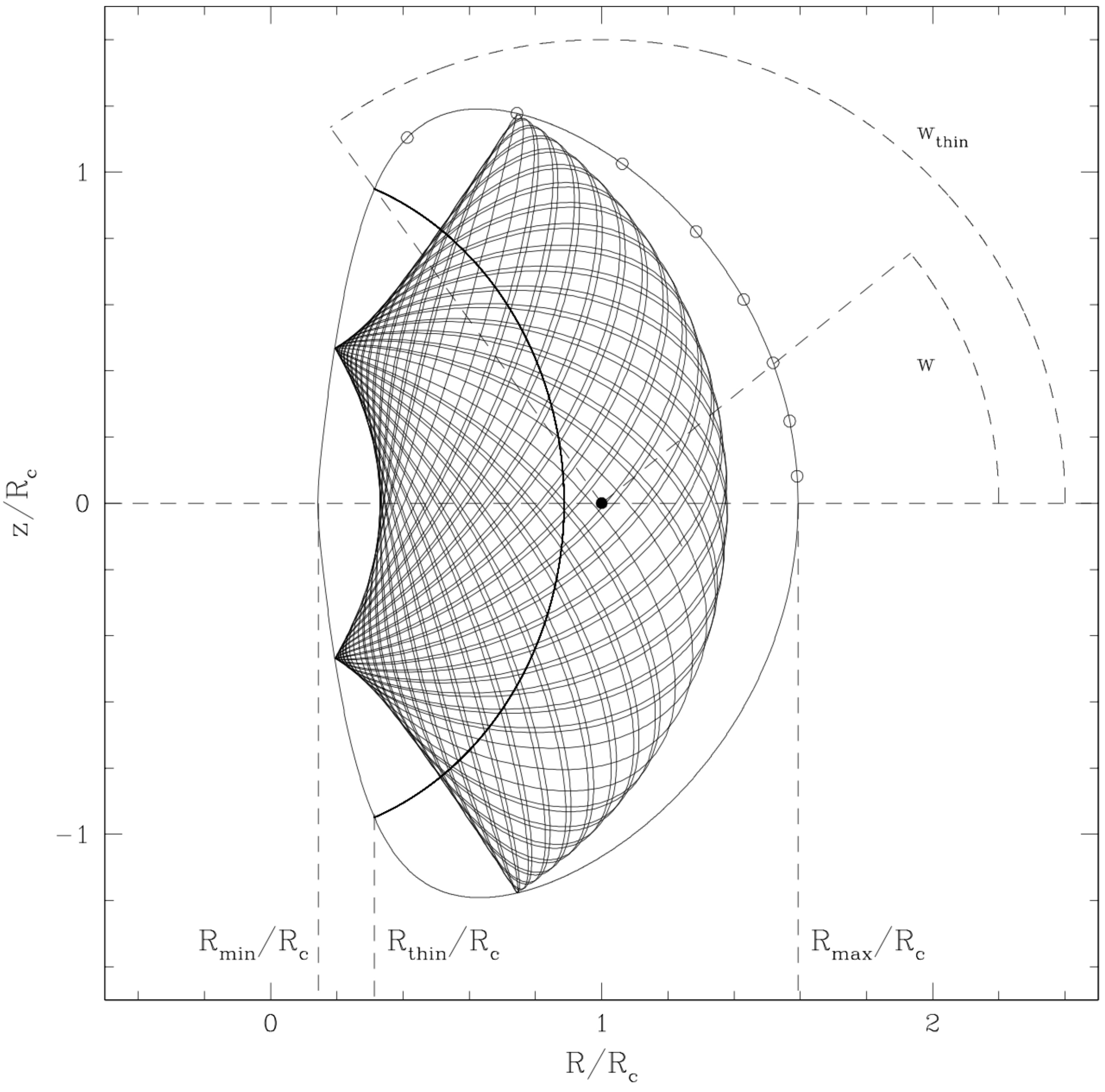
where we have defined f_+ as the part of the DF that is even in L_z, i.e.,

$$\begin{aligned}f(\mathcal{E},L_z) &= f_+(\mathcal{E},L_z)+f_-(\mathcal{E},L_z) \\ f_\pm(\mathcal{E},L_z) &\equiv \tfrac{1}{2}\left[f(\mathcal{E},L_z)\pm f(\mathcal{E},-L_z)\right]\end{aligned}$$

We thus see that the density depends only on the even part of the DF (i.e., the density contributed by a star does not depend on its sense of rotation). This also implies that there are infinitely many DFs $f(E,L_z)$ that correspond to exactly the same $\rho(R,z)$, namely all those that only differ in $f_-(\mathcal{E},L_z)$.

Given a density distribution $\rho(R,z)$, one can calculate the corresponding $f_+(\varepsilon,L_z)$ using a complex contour integral equation (due to Hunter & Qian 1993**) that is the equivalent of the Eddington's formula for spherical systems.**

The odd part of the DF, $f_-(\varepsilon,L_z)$, specifies the rotational streaming motion.



w_{thin}
w
R_{min}/R_c
R_{thin}/R_c
R_{max}/R_c
z/R_c
R/R_c
1
0
−1
0
1
2