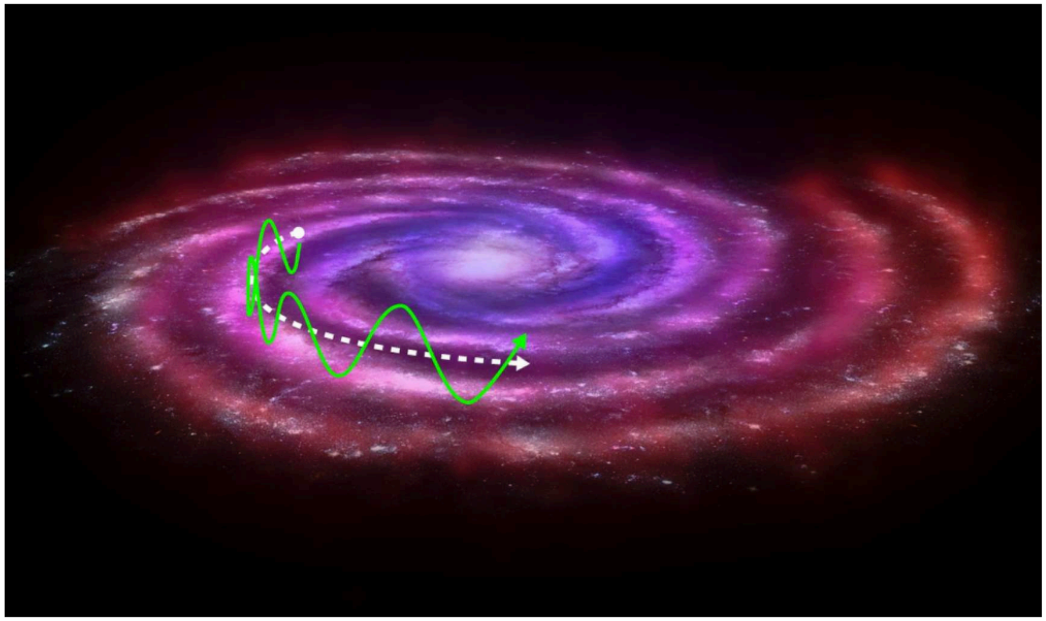
LECTURE 18

Part IV: Collisionless Dynamics



11	Tue 11/07	**Collisionless Dynamics:** Jeans equations & dynamical modeling
11	Thu 11/09	**Collisionless Dynamics:** virial theorem & gravothermal catastrophe
12	Tue 11/14	**Collisionless Dynamics:** collisions & encounters

Jeans Equations and Dynamical Modelling

Collisionless dynamics is governed by the Collisionless Boltmann Equation (CBE)

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + v_i \frac{\partial f}{\partial x_i} - \frac{\partial \Phi}{\partial x_i} \frac{\partial f}{\partial v_i} = 0$$

By taking velocity moments of the CBE, we end up with the momentum equations:

$$\frac{\partial u_i}{\partial t} + u_i \frac{\partial u_j}{\partial x_i} = \frac{1}{\rho} \frac{\partial \hat{\sigma}_{ij}}{\partial x_i} - \frac{\partial \Phi}{\partial x_i}$$

These are called the Jeans equations, and are basically exactly the same as the Euler equations or the Navier-Stokes equations, except that the stress tensor is different

Stress Tensor: $\hat{\sigma}_{ij} = -\rho \langle w_i w_j \rangle = \rho \langle v_i \rangle \langle v_j \rangle - \rho \langle v_i v_j \rangle$

Velocity Dispersion Tensor: $\sigma_{ij}^2 = \langle v_i v_j \rangle - \langle v_i \rangle \langle v_j \rangle = -\frac{\hat{\sigma}_{ij}}{\rho}$

Note that, for consistency with most literature on galactic dynamics, we write $<v_i>$ rather than u_i

The Issue of Closure:

For a collisional fluid, we have that $\hat{\sigma}_{ij} = -\rho\sigma^2_{ij} = -P\delta_{ij} + \tau_{ij}$

with the deviatoric stress tensor depending on kinetic and bulk viscocity.

Using the equation of state $P=P(\rho,T)$ and constitutive equations for the transport coefficients $\mu=\mu(T)$ and $\eta=\eta(T)$ we achieve closure: #variables = #equations

For a collisionless fluid, no constitutive equations or equation of state exist…

Hence, the stress tensor, which is manifest symmetric, has 6 unknowns and the Jeans equations (together with continuity equation) does **not** form a closed set

Adding higher-order moment equations of the CBE (i.e., equivalent of energy equation) does not help; although this adds equations, it adds even more unknowns such as $\langle v_i v_j v_k \rangle$, etc

The set of CBE moment equations never closes…

The Issue of Closure:

The velocity dispersion tensor is a **local** quantity: $\sigma_{ij}^2 = \sigma_{ij}^2(\vec{x})$

At each location, it can be diagonalized to the local velocity ellipsoid, whose principal axes are defined by the orthogonal eigenvectors, with corresponding eigenvalues $\sigma_i^2=\sigma_{ii}^2$

These represent the anisotropic pressure-like forces that counteract the gravitational force

In general, we should expect $\sigma_1^2 \neq \sigma_3^2 \neq \sigma_3^2$, which implies that the system will be triaxial

In order to be able to solve the Jeans equations (i.e., to achieve closure), it is common to impose certain symmetries. A typical example is to assume that the system is isotropic, in which case $\sigma_1^2=\sigma_3^2=\sigma_3^2$: the stress tensor in that case has only one unknown and the Jeans equations are closed.

Jeans equations in cylindrical coordinates: (R, ϕ, z)

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \dot{R}\frac{\partial f}{\partial R} + \dot{\phi}\frac{\partial f}{\partial \phi} + \dot{z}\frac{\partial f}{\partial z} + \dot{v}_R\frac{\partial f}{\partial v_R} + \dot{v}_\phi\frac{\partial f}{\partial v_\phi} + \dot{v}_z\frac{\partial f}{\partial v_z}$$

$$\begin{aligned} \dot{v}_R &= -\frac{\partial \Phi}{\partial R} + \frac{v_\phi^2}{R} \\ \dot{v}_\phi &= -\frac{1}{R}\frac{\partial \Phi}{\partial R} + \frac{v_R v_\phi}{R} \\ \dot{v}_z &= -\frac{\partial \Phi}{\partial z} \end{aligned}$$

see lecture notes

$$\boxed{\frac{\partial f}{\partial t} + v_R\frac{\partial f}{\partial R} + \frac{v_\phi}{R}\frac{\partial f}{\partial \phi} + v_z\frac{\partial f}{\partial z} + \left[\frac{v_\phi^2}{R} - \frac{\partial \Phi}{\partial R}\right]\frac{\partial f}{\partial v_R} - \frac{1}{R}\left[v_R v_\phi + \frac{\partial \Phi}{\partial \phi}\right]\frac{\partial f}{\partial v_\phi} - \frac{\partial \Phi}{\partial z}\frac{\partial f}{\partial v_z} = 0}$$

Jeans equations in cylindrical coordinates: (R, ϕ, z)

$$\frac{\partial f}{\partial t} + v_R \frac{\partial f}{\partial R} + \frac{v_\phi}{R}\frac{\partial f}{\partial \phi} + v_z \frac{\partial f}{\partial z} + \left[\frac{v_\phi^2}{R} - \frac{\partial \Phi}{\partial R}\right]\frac{\partial f}{\partial v_R} - \frac{1}{R}\left[v_R v_\phi + \frac{\partial \Phi}{\partial \phi}\right]\frac{\partial f}{\partial v_\phi} - \frac{\partial \Phi}{\partial z}\frac{\partial f}{\partial v_z} = 0$$

The **Jeans equations** follow from multiplication with v_R, v_ϕ, and v_z and integrating over velocity space. Note that the **cylindrical symmetry** requires that all derivatives with respect to ϕ vanish. The remaining terms are:

$$\begin{aligned}
\int v_R \frac{\partial f}{\partial t} \mathrm{d}^3\vec{v} &= \frac{\partial}{\partial t}\int v_R f \mathrm{d}^3\vec{v} = \frac{\partial(\rho\langle v_R\rangle)}{\partial t} \\
\int v_R^2 \frac{\partial f}{\partial R} \mathrm{d}^3\vec{v} &= \frac{\partial}{\partial R}\int v_R^2 f \mathrm{d}^3\vec{v} = \frac{\partial(\rho\langle v_R^2\rangle)}{\partial R} \\
\int v_R v_z \frac{\partial f}{\partial z} \mathrm{d}^3\vec{v} &= \frac{\partial}{\partial z}\int v_R v_z f \mathrm{d}^3\vec{v} = \frac{\partial(\rho\langle v_R v_z\rangle)}{\partial z} \\
\int \frac{v_R v_\phi^2}{R}\frac{\partial f}{\partial v_R} \mathrm{d}^3\vec{v} &= \frac{1}{R}\left[\int \frac{\partial(v_R v_\phi^2 f)}{\partial v_R}\mathrm{d}^3\vec{v} - \int \frac{\partial(v_R v_\phi^2)}{\partial v_R} f \mathrm{d}^3\vec{v}\right] = -\rho\frac{\langle v_\phi^2\rangle}{R} \\
\int v_R \frac{\partial \Phi}{\partial R}\frac{\partial f}{\partial v_R} \mathrm{d}^3\vec{v} &= \frac{\partial \Phi}{\partial R}\left[\int \frac{\partial(v_R f)}{\partial v_R}\mathrm{d}^3\vec{v} - \int \frac{\partial v_R}{\partial v_R} f \mathrm{d}^3\vec{v}\right] = -\rho\frac{\partial \Phi}{\partial R} \\
\int \frac{v_R^2 v_\phi}{R}\frac{\partial f}{\partial v_\phi} \mathrm{d}^3\vec{v} &= \frac{1}{R}\left[\int \frac{\partial(v_R^2 v_\phi f)}{\partial v_\phi}\mathrm{d}^3\vec{v} - \int \frac{\partial(v_R^2 v_\phi)}{\partial v_\phi} f \mathrm{d}^3\vec{v}\right] = -\rho\frac{\langle v_R^2\rangle}{R} \\
\int v_R \frac{\partial \Phi}{\partial z}\frac{\partial f}{\partial v_z} \mathrm{d}^3\vec{v} &= \frac{\partial \Phi}{\partial z}\left[\int \frac{\partial(v_R f)}{\partial v_z}\mathrm{d}^3\vec{v} - \int \frac{\partial v_z}{\partial v_R} f \mathrm{d}^3\vec{v}\right] = 0
\end{aligned}$$

Jeans equations in cylindrical coordinates: (R, ϕ, z)

$$\frac{\partial(\rho\langle v_R\rangle)}{\partial t} + \frac{\partial(\rho\langle v_R^2\rangle)}{\partial R} + \frac{\partial(\rho\langle v_R v_z\rangle)}{\partial z} + \rho\left[\frac{\langle v_R^2\rangle - \langle v_\phi^2\rangle}{R} + \frac{\partial\Phi}{\partial R}\right] = 0$$

$$\frac{\partial(\rho\langle v_\phi\rangle)}{\partial t} + \frac{\partial(\rho\langle v_R v_\phi\rangle)}{\partial R} + \frac{\partial(\rho\langle v_\phi v_z\rangle)}{\partial z} + 2\rho\frac{\langle v_R v_\phi\rangle}{R} = 0$$

$$\frac{\partial(\rho\langle v_z\rangle)}{\partial t} + \frac{\partial(\rho\langle v_R v_z\rangle)}{\partial R} + \frac{\partial(\rho\langle v_z^2\rangle)}{\partial z} + \rho\left[\frac{\langle v_R v_z\rangle}{R} + \frac{\partial\Phi}{\partial z}\right] = 0$$

Upon inspection, these are 3 equations for a total of 9 unknowns…..no closure.

To proceed, it is common to make the following assumptions:

1 System is static $\Rightarrow$ the $\frac{\partial}{\partial t}$-terms are zero and $\langle v_R\rangle = \langle v_z\rangle = 0$.

2 Velocity dispersion tensor is diagonal $\Rightarrow \langle v_i v_j\rangle = 0$ (if $i \neq j$).

3 Meridional isotropy $\Rightarrow \langle v_R^2\rangle = \langle v_z^2\rangle = \sigma_R^2 = \sigma_z^2 \equiv \sigma^2$.

Jeans equations in cylindrical coordinates: (R, ϕ, z)

The Jeans equations now reduce to:

$$\frac{\partial(\rho\sigma^2)}{\partial R} + \rho\left[\frac{\sigma^2 - \langle v_\phi^2\rangle}{R} + \frac{\partial\Phi}{\partial R}\right] = 0$$

$$\frac{\partial(\rho\sigma^2)}{\partial z} + \rho\frac{\partial\Phi}{\partial z} = 0$$

2 equations with 2 unknowns….closure.

Jeans Modelling:

Observations:

surface brightness $\Sigma(\mathbf{x},\mathbf{y})$
rotation velocity $v_{rot}(\mathbf{x},\mathbf{y})$
velocity dispersion $\sigma_{los}(\mathbf{x},\mathbf{y})$

Assumptions

inclination angle
mass-to-light ratio
dark matter distribution
central black hole mass

$\Sigma(\mathbf{x},\mathbf{y})$ → $\rho(R,z)$ → $\Phi(R,z)$ → $\sigma(R,z)$ → $\langle v_\phi^2\rangle$

deprojection
M/L ratio

Poisson eq.
dark matter
black hole

2nd Jeans eq.

1st Jeans eq.

Jeans equations in cylindrical coordinates: (R, ϕ, z)

$\Sigma(\mathbf{x},\mathbf{y})$ → $\rho(R,z)$ → $\Phi(R,z)$ → $\sigma(R,z)$ → $\langle v_\phi^2 \rangle$

deprojection M/L ratio — Poisson eq. dark matter black hole — 2nd Jeans eq. — 1st Jeans eq.

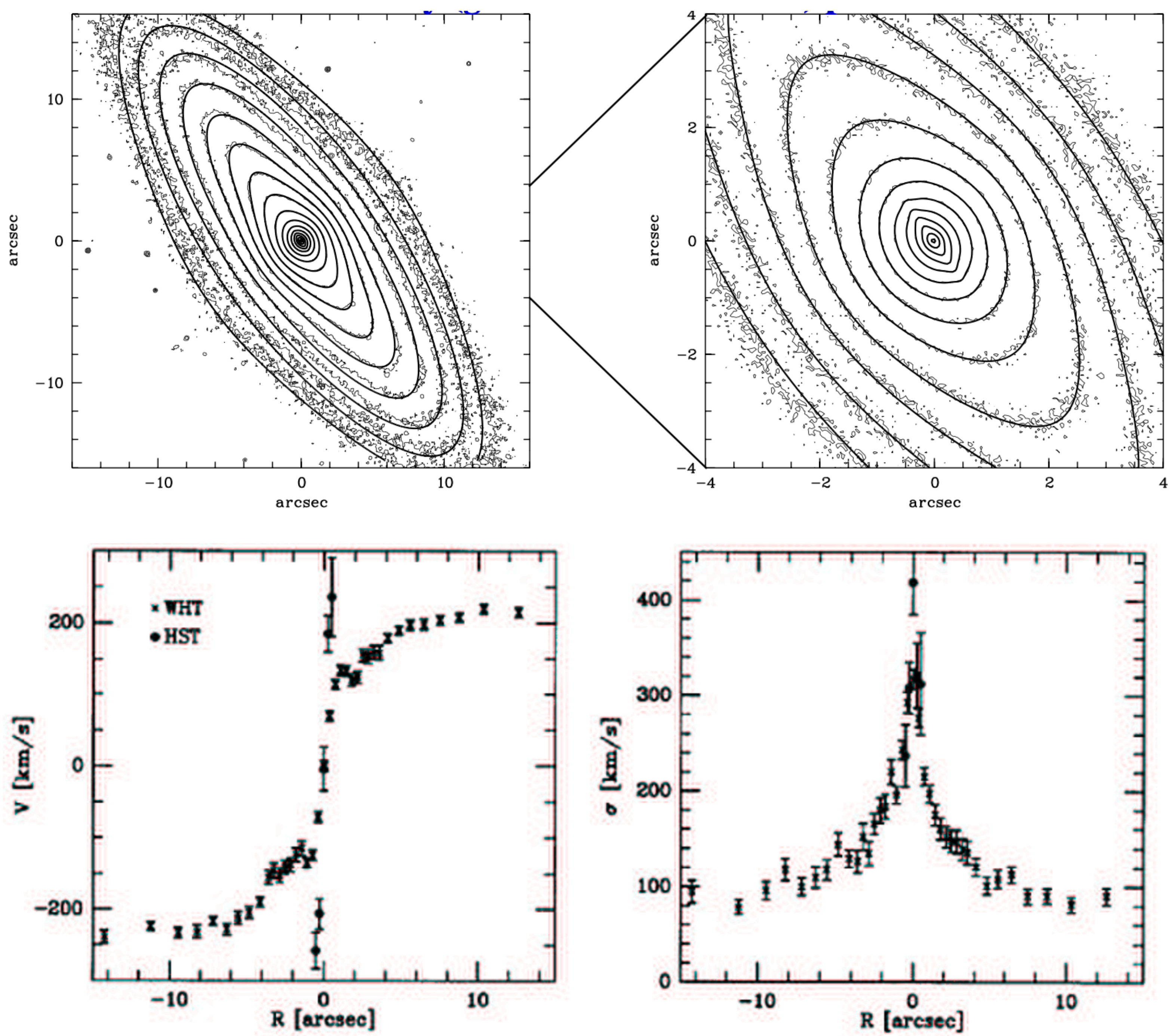
Note that $\langle v_\phi^2 \rangle = \langle v_\phi \rangle^2 + \sigma_\phi^2$; we still don't know how the rms motion in the azimuthal direction splits in bulk motion (rotation) and random motion (dispersion)

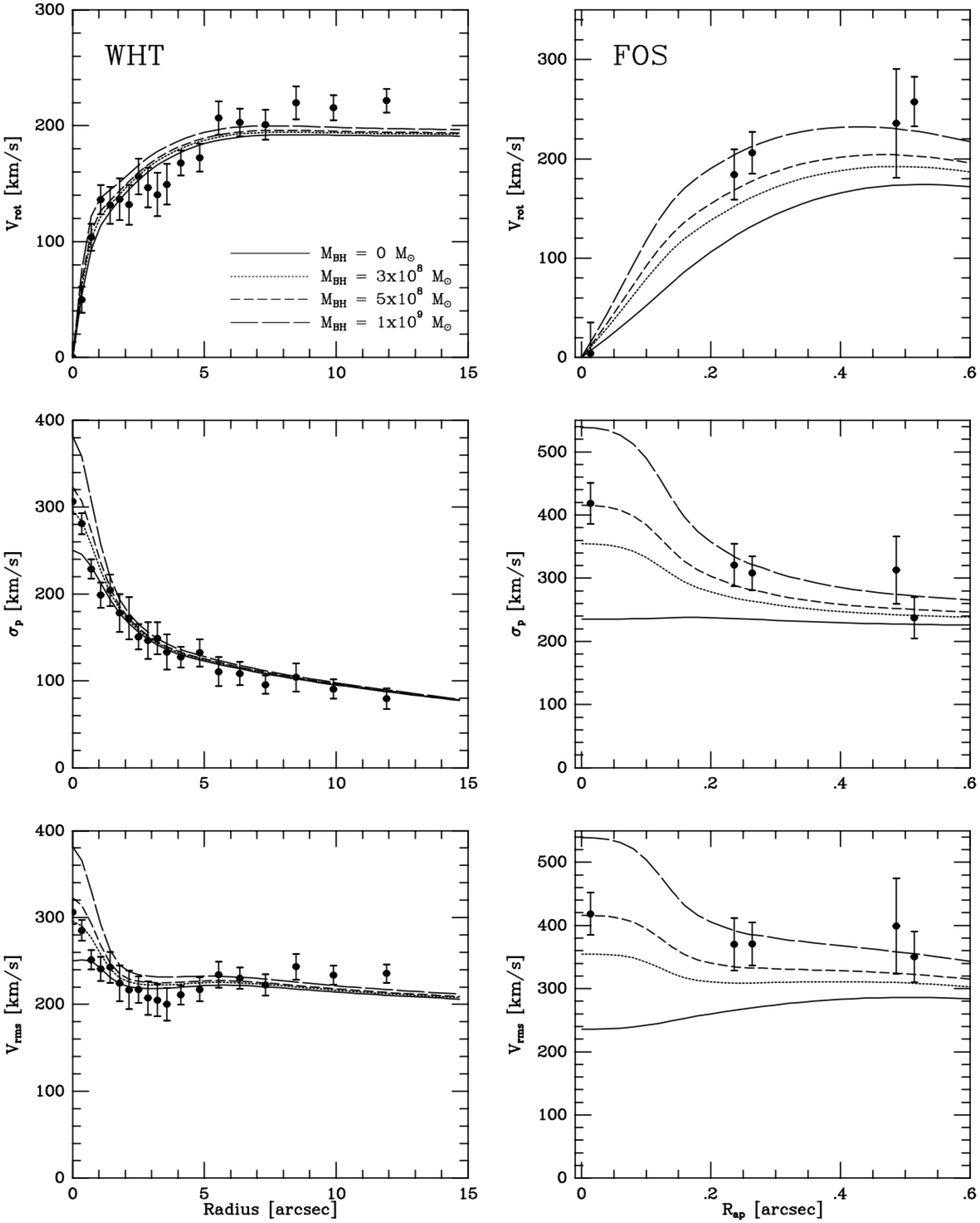
In practice, one often follows Satoh (1980) and writes $\langle v_\phi \rangle^2 = k\,[\langle v_\phi^2 \rangle - \langle v_R^2 \rangle]$ with k a free parameter.

Once you pick a value for k, the full dynamics are specified: next step is to project on the sky and compare to the observed $v_{rot}(\mathbf{x},\mathbf{y})$ and $\sigma_{los}(\mathbf{x},\mathbf{y})$. Use goodness-of-fit to find best fit values for inclination angle, mass-to-light ratio, M_{BH}, and dark matter halo…

Example: NGC 4342

van den Bosch et al. 1998,
Cretton & van den Bosch 1999





van den Bosch et al. 1998,
Cretton & van den Bosch 1999

Jeans Modelling:

k=1

$i = 90^o$

Non-zero BH seems to be required, but rms-velocity poorly fit.

Too many assumptions made….

Even if a Jeans model exists that can fit the data, there is not guarantee that f≥0…

Robust `detection' of BH mass requires more general, Schwarzschild orbit modelling

van den Bosch et al. 1998,
Cretton & van den Bosch 1999