CHAPTER 9

Microscopic Approach: from Boltzmann to Navier-Stokes

In the previous chapter we derived the closed Boltzmann equation:

$$\boxed{\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \{f, \mathcal{H}\} = I[f]}$$

where $I[f]$ is the collision integral, and we have used the shorthand notation f for the 1-particle DF $f^{(1)}$. In what follows we will adopt that notation throughout, and only use the superscript-notation whenever confusion might arise. The Boltzmann equation describes how the phase-space density around a particle (or fluid element) changes with time due to collisions.

Note that for a collisionless fluid, $I[f] = 0$, and the Boltzmann equation reduces to the collisionless Boltzmann equation (CBE):

$$\boxed{\frac{\mathrm{d}f}{\mathrm{d}t} = 0}$$

which expresses that the 6D phase-space density of a collisionless fluid is incompressible: the phase-space density around any given particle is **conserved**.

Adopting $q_i = x_i$, so that the conjugate momenta $p_i = mv_i = m\dot{x}_i$, we have that

$$\{f, \mathcal{H}\} = \frac{\partial f}{\partial x_i}\frac{\partial \mathcal{H}}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial \mathcal{H}}{\partial x_i} = \frac{p_i}{m}\frac{\partial f}{\partial x_i} + F_i\frac{\partial f}{\partial p_i}$$

where we have used the Hamiltonian equations of motion ($\partial\mathcal{H}/\partial q_i = -\dot{p}_i$ and $\partial\mathcal{H}/\partial p_i = \dot{q}_i$) and Newton's second law of motion ($\dot{p}_i = F_i$). Hence, if the force is gravity (as will typically be the case in astrophysical applications), then we can rewrite the Boltzmann equation, in vector-form, as

$$\boxed{\frac{\partial f}{\partial t} + \vec{v}\cdot\frac{\partial f}{\partial \vec{x}} - \nabla\Phi\cdot\frac{\partial f}{\partial \vec{v}} = I[f]}$$

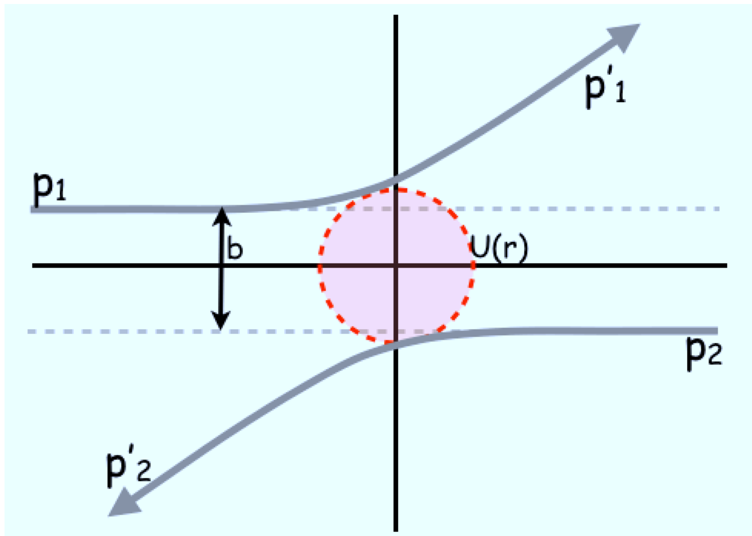

Figure 1: Illustration of 'collision' between two particles with momenta p_1 and p_2 due to interaction potential $U(r)$. The impact parameter of the collision is b.

Let us now take a closer look at the collision integral $I[f] = (\partial f/\partial t)_{\rm coll}$. Recall that we have made the assumption of a dilute gas, so that we only need to consider two-body interactions. In what follows, we make the additional assumption that all collisions are <u>elastic</u> [actually, this is sort of implied by the assumption of monoatoms]. An example is shown in Figure 1, where $\vec{p}_1 + \vec{p}_2 \rightarrow \vec{p}_1{}' + \vec{p}_2{}'$. Since we assumed a short-range, instantaneous and localized interaction, so that the external potential doesn't significantly vary over the interaction volume (the dashed circle in Fig. 1), we have

$$\begin{aligned} \textbf{momentum conservation:} \quad & \vec{p}_1 + \vec{p}_2 = \vec{p}_1{}' + \vec{p}_2{}' \\ \textbf{energy conservation:} \quad & |\vec{p}_1|^2 + |\vec{p}_2|^2 = |\vec{p}_1{}'|^2 + |\vec{p}_2{}'|^2 \end{aligned}$$

where we have assumed equal mass particles, which will be our assumption throughout.
In addition, we have <u>time-reversibility</u>, so that it is equally likely that the inverse process $(-\vec{p}_1{}' + -\vec{p}_2{}' \rightarrow -\vec{p}_1 + -\vec{p}_2)$ happens.

We can write the rate at which particles of momentum $\vec{p}_1$ at location $\vec{x}$

experience collisions $\vec{p}_1 + \vec{p}_2 \rightarrow \vec{p}_1{}' + \vec{p}_2{}'$ as

$$\mathcal{R} = \omega(\vec{p}_1, \vec{p}_2 | \vec{p}_1{}', \vec{p}_2{}')\, f^{(2)}(\vec{x}, \vec{x}, \vec{p}_1, \vec{p}_2)\, \mathrm{d}^3\vec{p}_2\, \mathrm{d}^3\vec{p}_1{}'\, \mathrm{d}^3\vec{p}_2{}'$$

Here $f^{(2)}(\vec{x}, \vec{x}, \vec{p}_1, \vec{p}_2)$ is the 2-particle DF, expressing the probability that at location $\vec{x}$, you encounter two particles with momenta $\vec{p}_1$ and $\vec{p}_2$, respectively. The function $\omega(\vec{p}_1, \vec{p}_2 | \vec{p}_1{}', \vec{p}_2{}')$ depends on the interaction potential $U(\vec{r})$ and can be calculated (using kinetic theory) via differential cross sections.

Using our assumption of molecular chaos, which states that the momenta of the interacting particles are independent, we have that

$$f^{(2)}(\vec{x}, \vec{x}, \vec{p}_1, \vec{p}_2) = f^{(1)}(\vec{x}, \vec{p}_1)\, f^{(1)}(\vec{x}, \vec{p}_2)$$

so that the collision integral can be written as

$$\boxed{I[f] = \int \mathrm{d}^3\vec{p}_2\, \mathrm{d}^3\vec{p}_1{}'\, \mathrm{d}^3\vec{p}_2{}'\, \omega(\vec{p}_1{}', \vec{p}_2{}' | \vec{p}_1, \vec{p}_2) \left[f(\vec{x}, \vec{p}_1{}')\, f(\vec{x}, \vec{p}_2{}') - f(\vec{x}, \vec{p}_1)\, f(\vec{x}, \vec{p}_2) \right]}$$

The first term within the square brackets describes the repleneshing collisions, in which particles at $(\vec{x}, \vec{p}_1{}')$ are scattered into $(\vec{x}, \vec{p}_1)$. The second term with the square brackets describes the depleting collisions, in which particles at $(\vec{x}, \vec{p}_1)$ are kicked out of their phase-space volume into $(\vec{x}, \vec{p}_1{}')$.

Because of the symmetries in $\omega(\vec{p}_1{}', \vec{p}_2{}' | \vec{p}_1, \vec{p}_2)$ (i.e., time-reversibility, and elasticity of collisions), it is straightforward to show that

$$\int \mathrm{d}^3\vec{p}\, A(\vec{x}, \vec{p}) \left(\frac{\partial f}{\partial t} \right)_{\mathrm{coll}} = 0$$

if

$$A(\vec{x}, \vec{p}_1) + A(\vec{x}, \vec{p}_2) = A(\vec{x}, \vec{p}_1{}') + A(\vec{x}, \vec{p}_2{}')$$

Quantities $A(\vec{x}, \vec{p})$ for which this is the case are called collisional invariants. There are three such quantities of interest to us

$A = 1$	**particle number conservations**
$A = \vec{p}$	**momentum conservation**
$A = \vec{p}^{\,2}/(2m)$	**energy conservation**

Thus far, we have derived the Boltzmann equation, and we have been able to write down an expression for the collision integral under the assumptions of (i) short-range, elastic collisions and (ii) molecular chaos. How do we proceed from here? Solving the actual Boltzmann equation, i.e. characterizing the evolution of f in 6D phase-space is extremely difficult, and provides little insight. Rather, we are interested what happens to our macroscopic quantities that describe the fluid (ρ, $\vec{u}$, P, ε, etc). We can use the Boltmann equation to describe the time-evolution of these macroscopic quantities by considering moment equations of the Boltzmann equation.

In mathematics, the $n^{\rm th}$-moment of a real-valued, continuous function $f(x)$ is

$$\mu_n = \int x^n f(x)\,\mathrm{d}x$$

If $f(x)$ is normalized, so that it can be interpreted as a probability function, then $\mu_n = \langle x^n \rangle$.

In our case, consider the scalar function $Q(\vec{v})$. The expectation value for Q at location $\vec{x}$ at time t is given by

$$\langle Q \rangle = \langle Q \rangle(\vec{x},t) = \frac{\int Q(\vec{v})\, f(\vec{x},\vec{v},t)\,\mathrm{d}^3\vec{v}}{\int f(\vec{x},\vec{v},t)\,\mathrm{d}^3\vec{v}}$$

Using that

$$n = n(\vec{x},t) = \int f(\vec{x},\vec{v},t)\,\mathrm{d}^3\vec{v}$$

we thus have that

$$\boxed{\int Q(\vec{v})\, f(\vec{x},\vec{v},t)\,\mathrm{d}^3\vec{v} = n\,\langle Q \rangle}$$

We will use this abundantly in what follows. In particular, define

$$g(\vec{x},t) = \int Q(\vec{v})\, f(\vec{x},\vec{v},t)\,\mathrm{d}^3\vec{v}$$

Then, in relation to fluid dynamics, there are a few functions $Q(\vec{v})$ that are of particular interest:

$$
\begin{array}{llrr}
Q(\vec{v}) = 1 & \Rightarrow & g(\vec{x},t) = n(\vec{x},t) & \text{number density} \\
Q(\vec{v}) = m & \Rightarrow & g(\vec{x},t) = \rho(\vec{x},t) & \text{mass density} \\
Q(\vec{v}) = m\,\vec{v} & \Rightarrow & g(\vec{x},t) = \rho(\vec{x},t)\,\vec{u}(\vec{x},t) & \text{momentum flux density} \\
Q(\vec{v}) = \frac{1}{2}m(\vec{v}-\vec{u})^2 & \Rightarrow & g(\vec{x},t) = \rho(\vec{x},t)\,\varepsilon(\vec{x},t) & \text{specific energy density}
\end{array}
$$

where we have used that $\langle\vec{v}\rangle = \vec{u}$, and $\langle(\vec{v}-\vec{u})^2/2\rangle = \varepsilon$.

This indicates that we can obtain dynamical equations for the macroscopic fluid quantities by multiplying the Boltzmann equation with appropriate functions, $Q(\vec{v})$, and integrating over all of velocity space.

Hence, we seek to solve equations of the form

$$\int Q(\vec{v})\,\left[\frac{\partial f}{\partial t} + \vec{v}\cdot\nabla f - \nabla\Phi\cdot\frac{\partial f}{\partial\vec{v}}\right]\,\mathrm{d}^3\vec{v} = \int Q(\vec{v})\,\left(\frac{\partial f}{\partial t}\right)_{\rm coll}\,\mathrm{d}^3\vec{v}$$

In what follows, we restrict ourselves to $Q(\vec{v})$ that are <u>collisional invariants</u> so that the integral on the right-hand side vanishes, and we are left with

$$\int Q(\vec{v})\,\frac{\partial f}{\partial t}\,\mathrm{d}^3\vec{v} + \int Q(\vec{v})\,\vec{v}\cdot\nabla f\,\mathrm{d}^3\vec{v} - \int Q(\vec{v})\,\nabla\Phi\cdot\frac{\partial f}{\partial\vec{v}}\,\mathrm{d}^3\vec{v} = 0$$

Since mass, momentum and energy are all conserved in elastic, short-range collisions we have that the momentum integral over the collision integral will be zero for the zeroth, first and second order moment equations! In other words, *although collisional and collisionless systems solve different Boltzmann equations, their zeroth, first and second moment equations are identical!*

We now split the above equation in three terms:

$$
\begin{array}{rl}
\mathbf{I} & \displaystyle\int Q(\vec{v})\,\frac{\partial f}{\partial t}\,\mathrm{d}^3\vec{v} \\
\mathbf{II} & \displaystyle\int Q(\vec{v})\,v_i\frac{\partial f}{\partial x_i}\,\mathrm{d}^3\vec{v} \\
\mathbf{III} & \displaystyle\int Q(\vec{v})\,\frac{\partial\Phi}{\partial x_i}\,\frac{\partial f}{\partial v_i}\,\mathrm{d}^3\vec{v}
\end{array}
$$

where we have that $\mathbf{I} + \mathbf{II} - \mathbf{III} = 0$, as long as Q is a collisional invariant.

We now proceed to rewrite each of these three integrals in turn.

Integral I

The first integral can be written as

$$\int Q(\vec{v})\,\frac{\partial f}{\partial t}\,\mathrm{d}^3\vec{v} = \int \frac{\partial Q f}{\partial t}\,\mathrm{d}^3\vec{v} = \frac{\partial}{\partial t}\int Q f\,\mathrm{d}^3\vec{v} = \frac{\partial}{\partial t} n\langle Q\rangle$$

where we have used that both $Q(\vec{v})$ and the integration volume are independent of time.

Integral II

Using similar logic, the second integral can be written as

$$\int Q(\vec{v})\,v_i\frac{\partial f}{\partial x_i}\,\mathrm{d}^3\vec{v} = \int \frac{\partial Q\,v_i\,f}{\partial x_i}\,\mathrm{d}^3\vec{v} = \frac{\partial}{\partial x_i}\int Q\,v_i\,f\,\mathrm{d}^3\vec{v} = \frac{\partial}{\partial x_i}\Big[n\,\langle Q\,v_i\rangle\Big]$$

Here we have used that

$$Q\,v_i\,\frac{\partial f}{\partial x_i} = \frac{\partial (Q\,v_i\,f)}{\partial x_i} - f\,\frac{\partial Q\,v_i}{\partial x_i} = \frac{\partial (Q\,v_i\,f)}{\partial x_i}$$

where the last step follows from the fact that neither v_i nor Q depend on x_i.

Integral III

For the third, and last integral, we are going to define $\vec{F} = \nabla\Phi$ and $\nabla_v \equiv (\partial/\partial v_x, \partial/\partial v_y, \partial/\partial v_z)$, i.e., ∇_v is the equivalent of ∇ but in velocity space. This allows us to write

$$\begin{aligned}
\int Q\,\vec{F}\cdot\nabla_v f\,\mathrm{d}^3\vec{v} &= \int \nabla_v\cdot(Qf\vec{F})\mathrm{d}^3\vec{v} - \int f\,\nabla_v\cdot(Q\vec{F})\,\mathrm{d}^3\vec{v} \\
&= \int Qf\vec{F}\mathrm{d}^2S_v - \int f\,\frac{\partial QF_i}{\partial v_i}\,\mathrm{d}^3\vec{v} \\
&= -\int fQ\frac{\partial F_i}{\partial v_i}\,\mathrm{d}^3\vec{v} - \int fF_i\,\frac{\partial Q}{\partial v_i}\,\mathrm{d}^3\vec{v} \\
&= -\int f\frac{\partial \Phi}{\partial x_i}\,\frac{\partial Q}{\partial v_i}\,\mathrm{d}^3\vec{v} = -\frac{\partial\Phi}{\partial x_i} n\left\langle\frac{\partial Q}{\partial v_i}\right\rangle
\end{aligned}$$

Here we have used Gauss' divergence theorem, and the fact that the integral of $Qf\vec{F}$ over the surface S_v (which is a sphere with radius $|\vec{v}| = \infty$) is equal to zero. This follows from the 'normalization' requirement that $\int f\,\mathrm{d}^3\vec{v} = n$. We have also used that $F_i = \partial\Phi/\partial x_i$ is independent of v_i.

Combining the above expressions for **I**, **II**, and **III**, we obtain that

$$\boxed{\frac{\partial}{\partial t} n\langle Q\rangle + \frac{\partial}{\partial x_i}\left[n\langle Qv_i\rangle\right] + \frac{\partial\Phi}{\partial x_i} n \left\langle \frac{\partial Q}{\partial v_i}\right\rangle = 0}$$

In what follows we refer to this as the 'master-moment-equation'.

Now let us consider $Q = m$, which is indeed a collisional invariant, as required. Substitution in the master-moment equation, and using that $\langle m\rangle = m$, that $mn = \rho$ and that $\langle mv_i\rangle = m\langle v_i\rangle = mu_i$, we obtain

$$\boxed{\frac{\partial\rho}{\partial t} + \frac{\partial\rho u_i}{\partial x_i} = 0}$$

which we recognize as the continuity equation in Eulerian index form.

Nex we consider $Q = mv_j$, which is also a collisional invariant. Using that $n\langle mv_j v_i\rangle = \rho\langle v_i v_j\rangle$ and that

$$\frac{\partial\Phi}{\partial x_i} n \left\langle \frac{\partial mv_j}{\partial v_i}\right\rangle = \frac{\partial\Phi}{\partial x_i} \rho \left\langle \frac{\partial v_j}{\partial v_i}\right\rangle = \frac{\partial\Phi}{\partial x_i}\rho\delta_{ij} = \rho\frac{\partial\Phi}{\partial x_j}$$

substitution of $Q = mv_j$ in the master-moment equation yields

$$\frac{\partial\rho u_j}{\partial t} + \frac{\partial\rho\langle v_i v_j\rangle}{\partial x_i} + \rho\frac{\partial\Phi}{\partial x_j} = 0$$

Next we use that

$$\frac{\partial\rho u_j}{\partial t} = \rho\frac{\partial u_j}{\partial t} + u_j\frac{\partial\rho}{\partial t} = \rho\frac{\partial u_j}{\partial t} - u_j\,\frac{\partial\rho u_k}{\partial x_k}$$

where, in the last step, we have used the continuity equation. Substitution in the above equation, and using the k is a mere dummy variable (which can therefore be replaced by i), we obtain that

$$
\begin{aligned}
& \rho\frac{\partial u_j}{\partial t} - u_j\frac{\partial \rho u_i}{\partial x_i} + \frac{\partial \rho\langle v_i v_j\rangle}{\partial x_i} + \rho\frac{\partial \Phi}{\partial x_j} = 0 \\
\Leftrightarrow \quad & \rho\frac{\partial u_j}{\partial t} - \left[\frac{\partial \rho u_i u_j}{\partial x_i} - \rho u_i\frac{\partial u_j}{\partial x_i}\right] + \frac{\partial \rho\langle v_i v_j\rangle}{\partial x_i} + \rho\frac{\partial \Phi}{\partial x_j} = 0 \\
\Leftrightarrow \quad & \rho\frac{\partial u_j}{\partial t} + \rho u_i\frac{\partial u_j}{\partial x_i} + \frac{\partial\left[\rho\langle v_i v_j\rangle - \rho u_i u_j\right]}{\partial x_i} + \rho\frac{\partial \Phi}{\partial x_j} = 0
\end{aligned}
$$

If we now restrict ourselves to collisional fluids, and use that the stress tensor can be written as

$$\sigma_{ij} = -\rho\langle w_i w_j\rangle = -\rho\langle v_i v_j\rangle + \rho u_i u_j = -P\delta_{ij} + \tau_{ij}$$

then the equation above can be rewritten as

$$\boxed{\frac{\partial u_j}{\partial t} + u_i\frac{\partial u_j}{\partial x_i} = -\frac{1}{\rho}\frac{\partial P}{\partial x_j} + \frac{1}{\rho}\frac{\partial \tau_{ij}}{\partial x_i} - \frac{\partial \Phi}{\partial x_j}}$$

which we recognize as the momentum equations (**Navier-Stokes**) in Eulerian index form. As we have seen in Chapter 6, as long as the fluid is Newtonian, the viscous stress tensor, τ_{ij}, can be described by two parameters only: the coefficient of shear viscosity, μ, and the coefficient of bulk viscosity, η (which can typically be ignored).

If instead we assume a collisionless fluid, then

$$\sigma_{ij} = -\rho\left\langle w_i w_j\right\rangle = -\rho\langle v_i v_j\rangle + \rho u_i u_j$$

and the momentum equations (now called the **Jeans equations**) reduce to

$$\boxed{\frac{\partial u_j}{\partial t} + u_i\frac{\partial u_j}{\partial x_i} = \frac{1}{\rho}\frac{\partial \sigma_{ij}}{\partial x_i} - \frac{\partial \Phi}{\partial x_j}}$$

In this case, we have no constraints on σ_{ij} other than that it is **manifest symmetric**; *for a collisionless fluid the stress tensor $\sigma_{ij} = \sigma_{ij}(\vec{x}, t)$ has a total of 6 unknowns.* The Jeans equations form the basis for building dynamical models of galaxies. However, since they contain many more unknowns than the number of equations, they can in general not be solved unless one makes a number of highly oversimplified assumptions (i.e., the system is spherically symmetric, the velocity structure is isotropic, etc.). This is the topic of **Galactic Dynamics**. Note that adding higher order moment equations ($Q(v) \propto v^a$ with $a \geq 3$) doesn't help in achieving closure since the new equations also add new unknowns, such as $\langle v_i v_j v_k \rangle$, etc. Ultimately, the problem is that collisionless fluids do not have constitutive equations such as the **equation of state** for a collisionless fluid.

Summary

For collisionless systems we have:

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} + \frac{\partial(\rho \langle v_i \rangle)}{\partial x_i} &= 0 \\
\frac{\partial \langle v_j \rangle}{\partial t} + \langle v_i \rangle \frac{\partial \langle v_j \rangle}{\partial x_i} &= -\frac{\partial \Phi}{\partial x_j} + \frac{1}{\rho} \frac{\partial \sigma_{ij}}{\partial x_i} \\
\nabla^2 \Phi &= 4\pi \, G \, \rho
\end{aligned}
$$

This is a set of 5 equations with 11 unknowns (ρ, Φ, $\langle v_i \rangle$ [3] and σ_{ij} [6])! Closure can only be achieved by making a number of simplifying assumptions.

For collisional systems we have:

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} + \frac{\partial(\rho \langle v_i \rangle)}{\partial x_i} &= 0 \\
\frac{\partial \langle v_j \rangle}{\partial t} + \langle v_i \rangle \frac{\partial \langle v_j \rangle}{\partial x_i} &= -\frac{\partial \Phi}{\partial x_j} + \frac{1}{\rho} \frac{\partial \sigma_{ij}}{\partial x_i} \\
\nabla^2 \Phi &= 4\pi G \rho \\
P &= P(\rho, T)
\end{aligned}
$$

Using that the **stress tensor** is entirely described by the pressure P, and the coefficients of shear and bulk viscosity, μ and η, this constitutes a set of 6 equations with 9 unknowns: (ρ, Φ, $\langle v_i \rangle$ [3], P, μ, η, T). If the fluid is **inviscid** then $\mu = \eta = 0$, which reduces the number of unknowns to 7. Closure can then be achieved by either assuming a **barotropic EoS**, $P = P(\rho)$, or by including one extra equation (the **energy equation**; see Chapter 14).

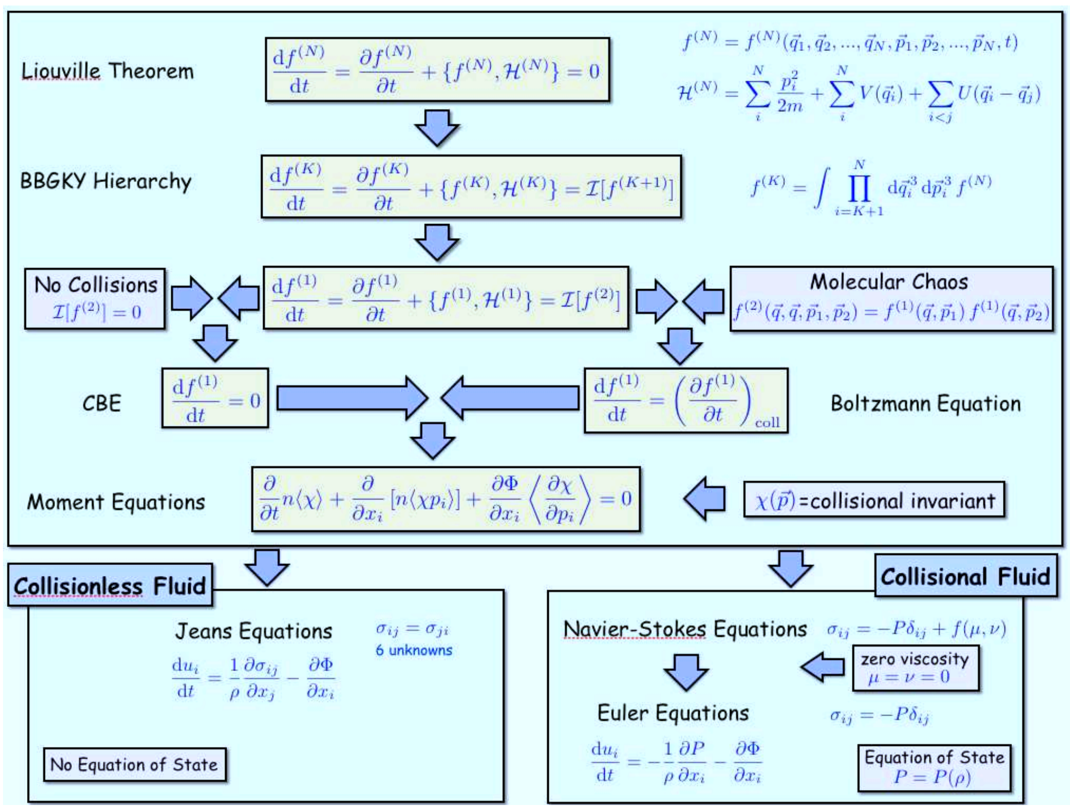

Figure 2: Flowchart of the origin of the dynamical equations describing fluids.