CHAPTER 7

The Navier-Stokes Equation

In Chapter 5 we ignored shear stresses, which resulted in the following momentum equations (in Lagrangian index form):

$$\frac{\mathrm{d}u_i}{\mathrm{d}t} = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} - \frac{\partial \Phi}{\partial x_i}$$

In the previous chapter, we showed that (for a Newtonian fluid) the stress tensor can be written as

$$\sigma_{ij} = -P\delta_{ij} + \mu\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\frac{\partial u_k}{\partial x_k}\right] + \eta\,\delta_{ij}\frac{\partial u_k}{\partial x_k}$$

We now incorporate this stress tensor in the momentum equations. Using that $\partial P/\partial x_i = \delta_{ij}\,\partial P/\partial x_j$ we can rewrite the above form as

$$\rho\frac{\mathrm{d}u_i}{\mathrm{d}t} = \frac{\partial(-P\delta_{ij})}{\partial x_j} - \rho\frac{\partial \Phi}{\partial x_i}$$

In order to take the shear into account, all we need to do now is to replace $-P\delta_{ij}$ with the stress tensor (effectively this means, adding a term that is the gradient of the viscous stress tensor, τ_{ij}). The result can be written as

$$\rho\frac{\mathrm{d}u_i}{\mathrm{d}t} = -\frac{\partial P}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} - \rho\frac{\partial \Phi}{\partial x_i}$$

These momentum equations are called the **Navier-Stokes equations**. It is more common, and more useful, to rewrite this by writing out the viscous stress tensor, which yields

$$\boxed{\rho\frac{\mathrm{d}u_i}{\mathrm{d}t} = -\frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\frac{\partial u_k}{\partial x_k}\right)\right] + \frac{\partial}{\partial x_i}\left(\eta\,\frac{\partial u_k}{\partial x_k}\right) - \rho\frac{\partial \Phi}{\partial x_i}}$$

These are the Navier-Stokes equations (in Lagragian index form) in all their glory, containing both the shear viscosity term and the bulk viscosity term (the latter is often ignored).

Note that μ and η are usually functions of density and temperature so that they have spatial variations. However, it is common to assume that these are suficiently small so that μ and η can be treated as constants, in which case they can be taken outside the differentials. In what follows we will make this assumption as well.

The Navier-Stokes equations in Lagrangian vector form are

$$\rho \frac{\mathrm{d}\vec{u}}{\mathrm{d}t} = -\nabla P + \mu \nabla^2 \vec{u} + \left(\eta + \frac{1}{3}\mu \right) \nabla(\nabla \cdot \vec{u}) - \rho \nabla \Phi$$

If we ignore the bulk viscosity ($\eta = 0$) then this reduces to

$$\boxed{\frac{\mathrm{d}\vec{u}}{\mathrm{d}t} = -\frac{\nabla P}{\rho} + \nu \left[\nabla^2 \vec{u} + \frac{1}{3} \nabla(\nabla \cdot \vec{u}) \right] - \nabla \Phi}$$

where we have introduced the kinetic viscosity $\nu \equiv \mu/\rho$. Note that these equations reduce to the **Euler equations** in the limit $\nu \to 0$.

As a final aside, it is often useful to use the vector calculus identity

$$\vec{u} \cdot \nabla \vec{u} = \nabla \left(\frac{\vec{u} \cdot \vec{u}}{2} \right) + (\nabla \times \vec{u}) \times \vec{u}$$

to write the Navier-Stokes equation in yet another form. Note that $\vec{u} \cdot \nabla \vec{u} = \frac{1}{2} \nabla u^2$, where $u \equiv |\vec{u}|$, for an **irrotational flow**.