CHAPTER 6

Viscosity & The Stress Tensor

In deriving the momentum equation, in the previous chapter, we made the simplifying assumption that the force acting on a surface of fluid element is a pure normal force (a force acting along the normal to the surface). However, in general, this force per unit area, called the **stress**, can have any angle wrt the normal. It is useful to decompose the stress in a normal stress, which is the component of the stress along the normal to the surface, and a shear stress, which is the component along the tangent to the surface.

Sign Convention: The stress $\vec{\Sigma}(\vec{x}, \hat{n})$ acting at location $\vec{x}$ on a surface with normal $\hat{n}$, is excerted *by* the fluid on the side of the surface to which the normal points, *on* the fluid from which the normal points. Hence, in the case of pure, normal pressure, we have that $\Sigma = -P$.

Stress Tensor: The stress tensor σ_{ij} is defined such that $\Sigma_i(\hat{n}) = \sigma_{ij}\, n_j$. Here $\Sigma_i(\hat{n})$ is the i-component of the stress acting on a surface with normal $\hat{n}$, whose j-component is given by n_j. Note that if $\sigma_{ij} = \sigma\delta_{ij}$ then there are only normal stresses (shear stresses vanish).

Viscosity: a measure of a fluid's resistance to deformation by shear stress. For liquids, viscosity corresponds to the informal concept of "thickness". A fluid with zero viscosity is called inviscid.

Microscopic Origin of Viscosity: In the presence of a velocity gradient, $\partial u_x/\partial y$, their will be momentum transfer in the y-direction because of the microscopic motion of the fluid particles. In general, momentum will be transferred from the faster moving layers to the slower moving layers. This net transfer of momentum acts as a friction force in the direction of u (i.e., a resistance against shear) and gives rise to the concept of viscosity. From this, it is clear that an inviscid fluid needs to have zero mean-free path. Such a fluid is called an ideal fluid.

Velocity of fluid particles: We can split the velocity, $\vec{v}$, of a fluid particle in a streaming velocity, $\vec{u}$ and a 'random' velocity, $\vec{w}$:

$$\vec{v} = \vec{u} + \vec{w}$$

where $\langle \vec{v} \rangle = \vec{u}$, $\langle \vec{w} \rangle = 0$ and $\langle . \rangle$ indicates the average over a fluid *element*. If we define v_i as the velocity in the i-direction, we have that

$$\langle v_i \, v_j \rangle = u_i \, u_j + \langle w_i \, w_j \rangle$$

Using these definitions of velocities, we can define a number of important tensors:

Total Stress Tensor:	$\sigma_{ij} \equiv -\rho \langle w_i \, w_j \rangle$
Momentum Flux Density Tensor:	$\Pi_{ij} \equiv +\rho \langle v_i \, v_j \rangle$
Ram Pressure Tensor:	$\Sigma_{ij} \equiv +\rho u_i u_j$
Viscous Stress Tensor:	$\tau_{ij} \equiv \sigma_{ij} + P \, \delta_{ij}$

Note: the viscous stress tensor is also known as the deviatoric stress tensor. It is the component of the stress tensor, σ_{ij}, that is responsible for shear, which in turn gives rise to viscosity.

The following relations hold:

$$\begin{aligned} \Pi_{ij} &= \rho u_i u_j + P \delta_{ij} - \tau_{ij} \\ \\ \sigma_{ij} &= -P \delta_{ij} + \tau_{ij} \\ &= \rho u_i u_j - \Pi_{ij} \end{aligned}$$

As is manifest from $\sigma_{ij} = -\rho \langle w_i w_j \rangle$, the stress tensor is symmetric ($\sigma_{ij} = \sigma_{ji}$), and has therefore 6 independent elements. By replacing the simple isotropic pressure, P, with the more general stress tensor, σ_{ij}, we have moved from having one unknown to having 6 unknowns. This has a dramatic impact on our ability to close our set of equations. We will need to find some constitutive relations for the stress tensor. In what follows we do so, by demonstrating that we can relate σ_{ij} to the state of the fluid flow.

In order to get insight into the form of τ_{ij}, recall it describes the correlations between random motions of particles in different directions. Such correlations arise because of velocity shear, which causes a transport of momentum across layers. Hence, τ_{ij}, and thus also the stress tensor, σ_{ij}, must be related to the **deformation tensor** $\mathcal{T}_{ij} = \partial u_i/\partial x_j$. It is useful to split the deformation tensor in its symmetric and anti-symetric components:

$$\frac{\partial u_i}{\partial x_j} = e_{ij} + \xi_{ij}$$

where

$$\begin{aligned} e_{ij} &= \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right] \\ \xi_{ij} &= \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right] \end{aligned}$$

The symmetric part of the deformation tensor, e_{ij}, is called the **rate of strain tensor**, while the anti-symmetric part, ξ_{ij}, expresses the **vorticity** $\vec{w} \equiv \nabla \times \vec{u}$ in the velocity field, i.e., $\xi_{ij} = -\frac{1}{2}\,\varepsilon_{ijk}\,w_k$. Note that

$$e_{kk} = \frac{1}{2}\left[\frac{\partial u_k}{\partial x_k} + \frac{\partial u_k}{\partial x_k}\right] = \frac{\partial u_k}{\partial x_k} = \nabla \cdot \vec{u} = \mathrm{div}\,\vec{u}$$

from which it is clear that the diagonal elements of the rate of strain tensor describes the rate of stretching (called the strain) along the corresponding axes. The off-diagonal elements represent the rate of shearing strain of the fluid element. Note that one can always find a coordinate system for which e_{ij} is diagonal. The axes of that coordinate frame indicate the eigendirections of the strain (compression of stretching) on the fluid element. It also implies that *the motion of a fluid element consists of three, and only three, basic parts: translation* (expressed by $\vec{u} = \langle \vec{v} \rangle$), *pure strain* (expressed by e_{ij}), *and rotation* (expressed by ξ_{ij}).

In terms of the relation between the viscous stress tensor, τ_{ij}, and the deformation tensor, $\mathcal{T}_{ij}$, there are a number of properties that are important.

- **Locality:** the $\tau_{ij} - \mathcal{T}_{ij}$-relation is said to be local if the stress tensor is only a function of the deformation tensor and thermodynamic state functions like temperature.
- **Homogeneity:** the $\tau_{ij} - \mathcal{T}_{ij}$-relation is said to be homogeneous if is everywhere the same. The viscous stress tensor may depend on location $\vec{x}$ only insofar as $\mathcal{T}_{ij}$ or the thermodynamic state functions depend on $\vec{x}$. This distinguishes a fluid from a solid, in which the stress tensor depends on the stress itself.
- **Isotropy:** the $\tau_{ij} - \mathcal{T}_{ij}$-relation is said to be isotropic if it has no preferred direction.
- **Linearity:** the $\tau_{ij} - \mathcal{T}_{ij}$-relation is said to be linear if the relation between the stress and rate-of-strain if linear. This is equivalent to saying that τ_{ij} does not depend on $\nabla^2\vec{u}$ or higher-order derivatives.

A fluid that is local, homogeneous and isotropic is called a **Stokesian** fluid. A Stokesian fluid that is linear is called a **Newtonian** fluid. Experiments have shown that most fluids are Newtonian to good approximation. Hence, in what follows we will assume that our fluids are Newtonian, unless specifically stated otherwise. For a Newtonian fluid one can write

$$\tau_{ij} = A_{ijkl} \frac{\partial u_k}{\partial x_l}$$

where A_{ijkl} is a fourth-order proportionality tensor. Because of the requirements for homogeneity and isotropy, it can be shown that the most general form of A_{ijkl} is

$$A_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu \left(\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} \right)$$

Hence, the most general form for the viscous stress tensor is

$$\tau_{ij} = 2\mu e_{ij} + \lambda e_{kk} \delta_{ij}$$

where μ is called the coefficient of shear viscosity, λ is a scalar, δ_{ij} is the Kronecker delta function, and $e_{kk} = \mathrm{Tr}(e) = \partial u_k / \partial x_k = \nabla \cdot \vec{u}$.

Note that (in a Newtonian fluid) the viscous stress tensor depends only on the symmetric component of the deformation tensor (the rate-of-strain tensor e_{ij}), but not on the antisymmetric component which describes vorticity. You can understand the fact that viscosity and vorticity are unrelated by considering a fluid disk in solid body rotation (i.e., $\nabla \cdot \vec{u} = 0$ and $\nabla \times \vec{u} = \vec{w} \neq 0$). In such a fluid there is no "slippage", hence no shear, and therefore no manifestation of viscosity.

From the expression for the viscous stress tensor, it is also clear that τ_{ij} vanishes for a fluid at rest (or with a homogeneous and steady velocity field). Hence, a fluid at rest may be treated as inviscid.

Thus far we have derived that the stress tensor, σ_{ij}, which in principle has 6 unknowns, can be reduced to a function of three unknowns only (P, μ, λ) as long as the fluid is Newtonian. Note that these three scalars, in general, are functions of temperature and density. We now focus on these three scalars in more detail, starting with the pressure P. To be exact, P is the thermodynamic equilibrium pressure, and is normally computed thermodynamically from some equation of state, $P = P(\rho, T)$. It is related to the translational kinetic energy of the particles when the fluid, in equilibrium, has reached equipartition of energy among all its degrees of freedom, including (in the case of molecules) rotational and vibrations degrees of freedom.

In addition to the thermodynamic equilibrium pressure, P, we can also define a mechanical pressure, $P_{\rm m}$, which is purely related to the translational motion of the particles, independent of whether the system has reached full equipartition of energy. The mechanical pressure is simply the average normal stress and therefore follows from the stress tensor according to

$$P_{\rm m} = -\frac{1}{3}\,{\rm Tr}(\sigma_{ij}) = -\frac{1}{3}\,(\sigma_{11} + \sigma_{22} + \sigma_{33})$$

Using that

$$\sigma_{ij} = -P\,\delta_{ij} + 2\,\mu\,e_{ij} + \lambda\,e_{kk}\,\delta_{ij}$$

we thus obtain the following relation between the two pressures:

$$P_{\rm m} = P - \eta\,\nabla\cdot\vec{u}$$

where

$$\eta = \frac{2}{3}\mu + \lambda = \frac{P - P_{\rm m}}{\nabla\cdot\vec{u}}$$

is called the coefficient of bulk viscosity (aka the 'second viscosity') We can now write the stress tensor as

$$\boxed{\sigma_{ij} = -P\delta_{ij} + \mu\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\frac{\partial u_k}{\partial x_k}\right] + \eta\,\delta_{ij}\frac{\partial u_k}{\partial x_k}}$$

This is the full expression for the stress tensor in terms of the coefficients of shear viscosity, μ, and bulk viscosity, η.

The bulk viscosity, η, is only non-zero if $P \neq P_{\rm m}$. This can only happen if the constituent particles of the fluid have degrees of freedom beyond position and momentum (i.e., when they are molecules with rotational or vibrational degrees of freedom). Hence, for a fluid of monoatoms, $\eta = 0$ and $\lambda = -2\mu/3$. From the fact that $P = P_{\rm m} + \eta\nabla\cdot\vec{u}$ it is clear that for an incompressible fluid $P = P_{\rm m}$ and the value of η is irrelevant; *bulk viscosity plays no role in incompressible fluids*. The only time when $P_{\rm m} \neq P$ is when a fluid consisting of molecules has just undergone a large volumetric change (i.e., during a shock). In that case there may be a lag between the time the translational motions reach equilibrium and the time when the system reaches full equipartition in energy among all degrees of freedom. In astrophysics, bulk viscosity can generally be ignored, but be aware that is may be important in shocks.