CHAPTER 4

Basics of Fluid Dynamics

What is a fluid?

A fluid is a substance that can flow, has no fixed shape, and offers little resistance to an external stress

- In a fluid the constituent particles (atoms, ions, molecules, stars) can 'freely' move past one another.
- Fluids take on the shape of their container.
- A fluid changes its shape at a steady rate when acted upon by a stress force.

Different types of Fluids:

We distinguish collisional fluids (liquids and gases) from collisionless fluids (galaxies and dark matter halos). The latter are very important in astrophysics, but are rarely discussed in physics textbooks.

The main difference between liquids and gases is that the former are (to good approximation) incompressible, which means that a given mass of liquid occupies a given volume (i.e., $\mathrm{D}\rho/\mathrm{D}t = 0$). A gas, on the other hand, will completely fill the volume that is available to it.

Examples of Fluids in Astrophysics:

- **Stars:** stars are spheres of gas in hydrostatic equilibrium (i.e., gravitational force is balanced by pressure gradients). Densities and temperatures in a given star cover many orders of magnitude. To good approximation, its equation of state is that of an ideal gas.

- **Giant (gaseous) planets:** Similar to stars, gaseous planets are large spheres of gas, albeit with a rocky core. Contrary to stars, though, the gas is typically so dense and cold that it can no longer be described with the equation of state of an ideal gas.

- **Planet atmospheres:** The atmospheres of planets are stratified, gaseous fluids retained by the planet's gravity.

- **White Dwarfs & Neutron stars:** These objects (stellar remnants) can be described as fluids with a degenerate equation of state.

- **Proto-planetary disks:** the dense disks of gas and dust surrounding newly formed stars out of which planetary systems form.

- **Inter-Stellar Medium (ISM):** The gas in between the stars in a galaxy. The ISM is typically extremely complicated, and roughly has a three-phase structure: it consists of a dense, cold (~ 10K) molecular phase, a warm ($\sim 10^4$K) phase, and a dilute, hot ($\sim 10^6$K) phase. Stars form out of the dense molecular phase, while the hot phase is (shock) heated by supernova explosions. The reason for this three phase medium is associated with the various cooling mechanisms. At high temperature when all gas is ionized, the main cooling channel is Bremmstrahlung (acceleration of free electrons by positively charged ions). At low temperatures ($< 10^4$K), the main cooling channel is molecular cooling (or cooling through hyperfine transitions in metals).

- **Inter-Galactic Medium (IGM):** The gas in between galaxies. This gas is typically very, very dilute (low density). It is continuously 'exposed' to adiabatic cooling due to the expansion of the Universe, but also is heated by radiation from stars (galaxies) and AGN (active galactic nuclei). The latter, called 'reionization', assures that the typical temperature of the IGM is $\sim 10^4$K.

- **Intra-Cluster Medium (ICM):** The hot gas in clusters of galaxies. This is gas that has been shock heated when it fell into the cluster; typically gas passes through an accretion shock when it falls into a dark matter halo, converting its infall velocity into thermal motion.

- **Accretion disks:** Accretion disks are gaseous, viscous disks in which the viscosity (enhanced due to turbulence) causes a net rate of radial matter towards the center of the disk, while angular momentum is being transported outwards (accretion)
- **Galaxies (stellar component):** as we will see later, the stellar component of galaxies is a collisionless fluid; to very, very good approximation, two stars in a galaxy will never collide with other.
- **Dark matter halos:** Another example of a collisionless fluid (at least, we *assume* that dark matter is collisionless)...

Fluid Dynamics: The Microscopic Approach

In the microscopic approach a fluid is treated as a collection of a HUGE number of particles that interact via collisions. Using kinetic theory and statistical mechanics, one uses the Liouville equation to derive the Boltzmann equation via the BBGKY hierarchy. The zeroth, first and second moment equations of the Boltzmann equation ultimately give rise to the continuity equation, the momentum equations, and the energy equation, respectively. These are called the Navier-Stokes equations for a collisional fluid, and the Jeans equations for a collisionless fluid. If the viscosity and conductivity of the (collisional) fluid can be ignored, the Navier-Stokes equations reduce to the Euler equations. The derivation of the fluid equations based on this microscopic approach is presented in Chapters 8 and 9.

Fluid Dynamics: The Macroscopic Approach:

In the macroscopic approach, the fluid is treated as a continuum, that is 'made up' of fluid elements (FE). These are small fluid volumes that nevertheless contain many particles, that are significantly larger than the mean-free path of the particles, and for which one can define local hydro-dynamical variables such as density, pressure and temperature. The requirements are:

1. the FE needs to be much smaller than the characteristic scale in the problem, which is the scale over which the hydrodynamical quantities

Q change by an order of magnitude, i.e.

$$l_{\rm FE} \ll l_{\rm scale} \sim \frac{Q}{\nabla Q}$$

2. the FE needs to be sufficiently large that fluctuations due to the finite number of particles ('discreteness noise') can be neglected, i.e.,

$$n\, l_{\rm FE}^3 \gg 1$$

where n is the number density of particles.

3. the FE needs to be sufficiently large that it 'knows' about the local conditions through collisions among the constituent particles, i.e.,

$$l_{\rm FE} \gg \lambda$$

where λ is the mean-free path of the fluid particles.

Note that no fluid element can be defined for a collisionless fluid. The implications are that one cannot use the macroscopic approach to derive the equations that govern a collisionless fluid.

Fluid Dynamics: closure:
In general, a fluid element is characterized by the following six hydro-dynamical variables:

mass density	ρ	[g/cm^3]	
fluid velocity	$\vec{u}$	[cm/s]	(3 components)
pressure	P	[erg/cm^3]	
specific internal energy	ε	[erg/g]	

Note that $\vec{u}$ is the velocity of the fluid element, not to be confused with the velocity $\vec{v}$ of individual fluid particles, used in the Boltzmann distribution function. Rather, $\vec{u}$ is (roughly) a vector sum of all particles velocities $\vec{v}$ that make up the fluid element.

If we can ignore viscosity and conductivity then these variable are related via the Euler equations:

1 continuum equation	relating ρ and $\vec{u}$
3 momentum equations	relating ρ, $\vec{u}$ and P
1 energy equation	relating ρ, $\vec{u}$, P and ε

Thus we have a total of 5 equations for 6 unknowns. One can solve the set ('close it') by using a constitutive relation. In almost all cases, this is the equation of state (EoS) $P = P(\rho, \varepsilon)$.

• Sometimes the EoS is expressed as $P = P(\rho, T)$. In that case another constitution relation is needed, typically $\varepsilon = \varepsilon(\rho, T)$.

• If the EoS is barotropic, i.e., if $P = P(\rho)$, then the energy equation is not needed to close the set of equations. There are two barotropic EoS that are encountered frequently in astrophysics: the isothermal EoS, which describes a fluid for which cooling and heating always balance each other to maintain a constant temperature, and the adiabatic EoS, in which there is no net heating or cooling (other than adiabatic heating or cooling due to the compression or expansion of volume, i.e., the $P\,\mathrm{d}V$ work). We will discuss these cases in more detail later in the course.

• No EoS exists for a collisionless fluid. Consequently, for a collisionless fluid one can never close the set of fluid equations, unless one makes a number of simplifying assumptions (i.e., one postulates various symmetries)

• In the case the fluid is exposed to an external force (i.e., a gravitational or electrical field), the momentum and energy equations contain an extra force term.

• In the case the fluid is self-gravitating (i.e., in the case of stars or galaxies) there is an additional unknown, the gravitational potential Φ. However, there is also an additional equation, the Poison equation relating Φ to ρ, so that the set of equations remains closed.

Fluid Dynamics: Eulerian *vs.* Lagrangian Formalism:

One distinguishes two different formalisms for treating fluid dynamics:

- **Eulerian Formalism:** in this formalism one solves the fluid equations 'at fixed positions': the evolution of a quantity Q is described by the local (or partial, or Eulerian) derivative $\partial Q/\partial t$. An Eulerian hydrodynamics code is a 'grid-based code', which solves the hydro equations on a fixed grid, or using an adaptive grid, which refines resolution where needed. The latter is called Adaptive Mesh Refinement (AMR).

- **Lagrangian Formalism:** in this formalism one solves the fluid equations 'comoving with the fluid', i.e., either at a fixed particle (collisionless fluid) or at a fixed fluid element (collisional fluid). The evolution of a quantity Q is described by the substantial (or Lagrangian) derivative $\mathrm{d}Q/\mathrm{d}t$ (sometimes written as $\mathrm{D}Q/\mathrm{D}t$). A Lagrangian hydrodynamics code is a 'particle-based code', which solves the hydro equations per simulation particle. Since it needs to smooth over neighboring particles in order to compute quantities such as the fluid density, it is called Smoothed Particle Hydrodynamics (SPH).

In order to derive an expression for the substantial derivative $\mathrm{d}Q/\mathrm{d}t$, realize that $Q = Q(t, x, y, z)$. When the fluid element moves, the scalar quantity Q experiences a change

$$\mathrm{d}Q = \frac{\partial Q}{\partial t}\,\mathrm{d}t + \frac{\partial Q}{\partial x}\,\mathrm{d}x + \frac{\partial Q}{\partial y}\,\mathrm{d}y + \frac{\partial Q}{\partial z}\,\mathrm{d}z$$

Dividing by $\mathrm{d}t$ yields

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \frac{\partial Q}{\partial t} + \frac{\partial Q}{\partial x}\,u_x + \frac{\partial Q}{\partial y}\,u_y + \frac{\partial Q}{\partial z}\,u_z$$

where we have used that $\mathrm{d}x/\mathrm{d}t = u_x$, which is the x-component of the fluid velocity $\vec{u}$, etc. Hence we have that

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \frac{\partial Q}{\partial t} + \vec{u} \cdot \nabla Q$$

Using a similar derivation, but now for a vector quantity $\vec{A}(\vec{x},t)$, it is straightforward to show that

$$\frac{\mathrm{d}\vec{A}}{\mathrm{d}t} = \frac{\partial \vec{A}}{\partial t} + (\vec{u}\cdot\nabla)\,\vec{A}$$

which, in index-notation, is written as

$$\frac{\mathrm{d}A_i}{\mathrm{d}t} = \frac{\partial A_i}{\partial t} + u_j\,\frac{\partial A_i}{\partial x_j}$$

Another way to derive the above relation between the Eulerian and Lagrangian derivatives, is to think of $\mathrm{d}Q/\mathrm{d}t$ as

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \lim_{\delta t\to 0}\left[\frac{Q(\vec{x}+\delta\vec{x},t+\delta t) - Q(\vec{x},t)}{\delta t}\right]$$

Using that

$$\vec{u} = \lim_{\delta t\to 0}\left[\frac{\vec{x}(t+\delta t) - \vec{x}(t)}{\delta t}\right] = \frac{\delta\vec{x}}{\delta t}$$

and

$$\nabla Q = \lim_{\delta\vec{x}\to 0}\left[\frac{Q(\vec{x}+\delta\vec{x},t) - Q(\vec{x},t)}{\delta\vec{x}}\right]$$

it is straightforward to show that this results in the same expression for the substantial derivative as above.

Kinematic Concepts: Streamlines, Streaklines and Particle Paths:
In fluid dynamics it is often useful to distinguish the following kinematic constructs:

- **Streamlines:** curves that are instantaneously tangent to the velocity vector of the flow. Streamlines show the direction a massless fluid element will travel in at any point in time.

- **Streaklines:** the locus of points of all the fluid particles that have passed continuously through a particular spatial point in the past. Dye steadily injected into the fluid at a fixed point extends along a streakline.

- **Particle paths:** (aka pathlines) are the trajectories that individual fluid elements follow. The direction the path takes is determined by the streamlines of the fluid at each moment in time.

Only if the flow is steady, which means that all partial time derivatives vanish (i.e., $\partial\vec{u}/\partial t = \partial\rho/\partial t = \partial P/\partial t$), will streamlines be identical to streaklines be identical to particle paths. For a non-steady flow, they will differ from each other.