CHAPTER 3

Curvi-Linear Coordinate Systems

In astrophysics, one often works in **curvi-linear**, rather than **Cartesian** coordinate systems. The two most often encountered examples are the **cylindrical** (R, ϕ, z) and **spherical** (r, θ, ϕ) coordinate systems.

In this chapter we describe how to handle **vector calculus** in non-Cartesian coordinate systems (Euclidean spaces only). After giving the 'rules' for arbitrary coordinate systems, we apply them to cylindrical and spherical coordinate systems, respectively.

Vector Calculus in an Arbitrary Coordinate System:

Let (q_1, q_2, q_3) denote the coordinates of a point in an **arbitrary coordinate system**, defined by the **metric tensor** h_{ij}. The distance between (q_1, q_2, q_3) and $(q_1 + \mathrm{d}q_1, q_2 + \mathrm{d}q_2, q_3 + \mathrm{d}q_3)$ is

$$\mathrm{d}s^2 = h_{ij}\, \mathrm{d}q_i\, \mathrm{d}q_j$$

In what follows, we will only consider **orthogonal** systems for which $h_{ij} = 0$ if $i \neq j$, so that $\mathrm{d}s^2 = h_i^2\, \mathrm{d}q_i^2$ with

$$h_i \equiv \sqrt{h_{ii}} = \left| \frac{\partial \vec{x}}{\partial q_i} \right|$$

The **differential vector** is:

$$\mathrm{d}\vec{x} = \frac{\partial \vec{x}}{\partial q_1}\, \mathrm{d}q_1 + \frac{\partial \vec{x}}{\partial q_2}\, \mathrm{d}q_2 + \frac{\partial \vec{x}}{\partial q_3}\, \mathrm{d}q_3$$

The **unit directional vectors** are:

$$\vec{e}_i = \frac{\partial \vec{x}/\partial q_i}{|\partial \vec{x}/\partial q_i|} = \frac{1}{h_i} \frac{\partial \vec{x}}{\partial q_i}$$

so that $\mathrm{d}\vec{x} = \sum_i h_i\, \mathrm{d}q_i\, \vec{e}_i$ and $\mathrm{d}^3\vec{x} = h_1\, h_2\, h_3\, \mathrm{d}q_1\, \mathrm{d}q_2\, \mathrm{d}q_3$.

Let $\mathcal{B} = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ be the **basis** in our (q_1, q_2, q_3) coordinate system, and let $[\vec{a}]_\mathcal{B}$ denote the vector $\vec{a}$ in basis $\mathcal{B}$. Similarly, $[\vec{a}]_\mathcal{C}$ denotes $\vec{a}$ in the standard Cartesian basis $\mathcal{C} = \{\vec{e}_x, \vec{e}_y, \vec{e}_z\}$. The relation between $[\vec{a}]_\mathcal{B}$ and $[\vec{a}]_\mathcal{C}$ is given by

$$[\vec{a}]_\mathcal{C} = \mathbf{T}\,[\vec{a}]_\mathcal{B}\,, \qquad\qquad [\vec{a}]_\mathcal{B} = \mathbf{T}^{-1}\,[\vec{a}]_\mathcal{C}$$

Here $\mathbf{T}$ is the <u>transformation of basis matrix</u>, whose columns are the unit-direction vectors $\vec{e}_i$, i.e., $T_{ij} = e_{ij}$. Since the columns of $\mathbf{T}$ are unit vectors that are orthogonal to each other, the matric $\mathbf{T}$ is said to be <u>orthogonal</u>, which implies that $\mathbf{T}^{-1} = \mathbf{T}^\mathrm{T}$ (the inverse is equal to the transpose), and $\det(T) = \pm 1$.

Using the above, we have that the position and velocity vectors are given by

$$[\vec{x}]_\mathcal{B} = \sum_i \frac{1}{h_i}\left(\frac{\partial \vec{x}}{\partial q_i}\cdot\vec{x}\right)\vec{e}_i$$

and

$$[\vec{v}]_\mathcal{B} = \sum_i h_i\,\dot{q}_i\,\vec{e}_i$$

with $\vec{x} = (x, y, z)$ the position vector in Cartesian coordinates, and $\dot{q}_i = \mathrm{d}q_i/\mathrm{d}t$.

Next we write out the gradient, the divergence, the curl and the Laplacian:

The gradient:

$$\nabla\psi = \frac{1}{h_i}\frac{\partial\psi}{\partial q_i}\vec{e}_i$$

The divergence:

$$\nabla \cdot \vec{A} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}(h_2 h_3 A_1) + \frac{\partial}{\partial q_2}(h_3 h_1 A_2) + \frac{\partial}{\partial q_3}(h_1 h_2 A_3)\right]$$

The curl (only one component shown)**:**

$$(\nabla \times \vec{A})_3 = \frac{1}{h_1 h_2}\left[\frac{\partial}{\partial q_1}(h_2 A_2) - \frac{\partial}{\partial q_2}(h_1 A_1)\right]$$

The Laplacian:

$$\nabla^2\psi = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial\psi}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial\psi}{\partial q_2}\right) + \frac{\partial}{\partial q_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial\psi}{\partial q_3}\right)\right]$$

Vector Calculus in Cylindrical Coordinates:

For cylindrical coordinates (R, ϕ, z) we have that

$$x = R\cos\phi \qquad y = R\sin\phi \qquad z = z$$

The **scale factors** of the **metric** therefore are:

$$h_R = 1 \qquad h_\phi = R \qquad h_z = 1$$

and the position vector is $\vec{x} = R\vec{e}_R + z\vec{e}_z$.

Let $\vec{A} = A_R\vec{e}_R + A_\phi\vec{e}_\phi + A_z\vec{e}_z$ an arbitrary vector, then

$$\begin{aligned} A_R &= A_x\cos\phi - A_y\sin\phi \\ A_\phi &= -A_x\sin\phi + A_y\cos\phi \\ A_z &= A_z \end{aligned}$$

In cylindrical coordinates the **velocity vector** becomes:

$$\begin{aligned} \vec{v} &= \dot{R}\,\vec{e}_R + R\,\dot{\vec{e}}_R + \dot{z}\,\vec{e}_z \\ &= \dot{R}\,\vec{e}_R + R\,\dot{\phi}\,\vec{e}_\phi + \dot{z}\,\vec{e}_z \end{aligned}$$

The Gradient:

$$\nabla\cdot\vec{A} = \frac{1}{R}\frac{\partial}{\partial R}(RA_R) + \frac{1}{R}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z}$$

The Laplacian:

$$\nabla^2\psi = \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial\psi}{\partial R}\right) + \frac{1}{R^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}$$

Vector Calculus in Spherical Coordinates:

For spherical coordinates (r, θ, ϕ) we have that

$$x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi \qquad z = r\cos\theta$$

The **scale factors** of the **metric** therefore are:

$$h_r = 1 \qquad h_\theta = r \qquad h_\phi = r\sin\theta$$

and the position vector is $\vec{x} = r\vec{e}_r$.

Let $\vec{A} = A_r\vec{e}_r + A_\theta\vec{e}_\theta + A_\phi\vec{e}_\phi$ an arbitrary vector, then

$$\begin{aligned} A_r &= A_x\sin\theta\cos\phi + A_y\sin\theta\sin\phi + A_z\cos\theta \\ A_\theta &= A_x\cos\theta\cos\phi + A_y\cos\theta\sin\phi - A_z\sin\theta \\ A_\phi &= -A_x\sin\phi + A_y\cos\phi \end{aligned}$$

In spherical coordinates the **velocity vector** becomes:

$$\begin{aligned} \vec{v} &= \dot{r}\,\vec{e}_r + r\,\dot{\vec{e}}_r \\ &= \dot{r}\,\vec{e}_r + r\,\dot{\theta}\,\vec{e}_\theta + r\,\sin\theta\,\dot{\phi}\,\vec{e}_\phi \end{aligned}$$

The Gradient:

$$\nabla\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

The Laplacian:

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\psi^2}$$