CHAPTER 26

Radiative Transfer

Consider an incoming signal of **specific intensity** $I_{\nu,0}$ passing through a cloud (i.e., any gaseous region). As the radiation transits a small path length $\mathrm{d}r$ through the cloud, its specific intensity changes by $\mathrm{d}I_\nu = \mathrm{d}I_{\nu,\mathrm{loss}} + \mathrm{d}I_{\nu,\mathrm{gain}}$. The loss-term describes the combined effect of scattering and absorption, which remove photons from the line-of-sight, while the gain-term describes all processes that add photons to the line-of-sight; these include all emission processes from the gas itself, as well as scattering of photons from any direction into the line-of-sight.

In what follows we ignore the contribution of scattering to $\mathrm{d}I_{\nu,\mathrm{gain}}$, as this term makes solving the equation of radiative transfer much more complicated. We will briefly comments on that below, but for now the only process that is assumed to contribute to $\mathrm{d}I_{\nu,\mathrm{gain}}$ are emission processes from the gas.

It is useful to define the following two coefficients:

• **Absorption coefficient**, $\alpha_\nu = n\,\sigma_\nu = \rho\,\kappa_\nu$, which has units $[\alpha_\nu] = \mathrm{cm}^{-1}$.

• **Emission coefficient**, j_ν, defined as the energy emitted per unit time, per unit volume, per unit frequency, per unit solid angle (i.e., $\mathrm{d}E = j_\nu\,\mathrm{d}t\,\mathrm{d}V\,\mathrm{d}\nu\,\mathrm{d}\Omega$, and thus $[j_\nu] = \mathrm{erg\,s^{-1}\,cm^{-3}\,Hz^{-1}\,sr^{-1}}$).

In terms of these two coefficients, the **equation of radiative transfer** can be written in either of the following forms

$$
\begin{aligned}
\frac{\mathrm{d}I_\nu}{\mathrm{d}r} &= -\alpha_\nu\, I_\nu + j_\nu \qquad \text{(form I)} \\
\frac{\mathrm{d}I_\nu}{\mathrm{d}\tau_\nu} &= -I_\nu + S_\nu \qquad \text{(form II)}
\end{aligned}
$$

where $S_\nu \equiv j_\nu/\alpha_\nu$ is called the **source function**, and has units of specific intensity (i.e., $[S_\nu] = \mathrm{erg\,s^{-1}\,cm^{-2}\,Hz^{-1}\,sr^{-1}}$). In order to derive form II from form I, recall that $\mathrm{d}\tau_\nu = \alpha_\nu\,\mathrm{d}r$ (see Chapter 24).

NOTE: we use the convention of τ_ν increasing *from* the source *towards* the observer. Some textbooks (e.g., Irwin) adopt the opposite convention, which results in some sign differences.

To get some insight, we will now consider a number of different cases:

$\boxed{\text{Case A}}$ **No Cloud**

In this case, there is no absorption ($\alpha_\nu = 0$) or emission ($j_\nu = 0$), other than the emission from the background source. Hence, we have that

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}r} = 0 \qquad \Rightarrow \qquad I_\nu = I_{\nu,0}$$

which expresses that intensity is a conserved quantity in vacuum.

$\boxed{\text{Case B}}$ **Absorption Only**

In this case, the cloud absorps background radiation, but does not emit anything ($j_\nu = S_\nu = 0$). Hence,

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}\tau_\nu} = -I_\nu$$

which is easily solved to yield

$$I_\nu = I_{\nu,0}\,\mathrm{e}^{-\tau_\nu}$$

which is the expected result (see Chapter 25).

$\boxed{\text{Case C}}$ **Emission Only**

If the cloud does not absorb ($\alpha_\nu = 0$) but does emit we have

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}r} = j_\nu \qquad \Rightarrow \qquad I_\nu = I_{\nu,0} + \int_0^l j_\nu(r)\,\mathrm{d}r$$

where l is the size of the cloud along the line-of-sight. This equation simply expresses that the *increase* of intensity is equal to the emission coefficient integrated along the line-of-sight.

$\boxed{\text{Case D}}$ **Cloud in Thermal Equilibrium w/o Background Source**

Consider a cloud in TE, i.e., specified by a single temperature T (kinetic temperature is equal to radiation temperature). Since in a system in TE there can be no net transport of energy, we have that

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}r} = -\alpha_\nu\, I_\nu + j_\nu = 0 \qquad \Rightarrow \qquad I_\nu = \frac{j_\nu}{\alpha_\nu} = S_\nu$$

Since the observer must see a black body of temperature T, we also have that $I_\nu = B_\nu(T)$ (i.e., the intensity is given by a Planck curve corresponding to the temperature of the cloud), and we thus have that

$$\begin{aligned} I_\nu &= S_\nu = B_\nu(T) \\ j_\nu &= \alpha_\nu\, B_\nu(T) \end{aligned}$$

The latter of these equivalent relations is sometimes called **Kirchoff's law**, and simply expresses that a black body needs to establish a balance between emission and absorption (i.e., $B_\nu(T) = j_\nu/\alpha_\nu$).

$\boxed{\text{Case E}}$ **Emission & Absorption (formal solution)**

Consider the general case with both emission and absorption (but where we ignore the fact that scattering can scatter photons into my line of sight). Starting from form II of the equation of radiative transfer, multiplying both sides with e^{τ_ν}, we obtain that

$$\frac{\mathrm{d}\tilde{I}_\nu}{\mathrm{d}\tau_\nu} = -\tilde{S}_\nu$$

where $\tilde{I}_\nu \equiv I_\nu\, \mathrm{e}^{\tau_\nu}$ and $\tilde{S}_\nu \equiv S_\nu\, \mathrm{e}^{\tau_\nu}$. We can rewrite the above differential equation as

$$\int_{\tilde{I}_{\nu,0}}^{I_\nu} \mathrm{d}\tilde{I}_\nu = \int_0^{\tau_\nu} \tilde{S}_\nu\, \mathrm{d}\tau_\nu$$

Using that $\tilde{I}_{\nu,0} = I_{\nu,0}\,\mathrm{e}^0 = I_{\nu,0}$ the solution to this simple integral equation is

$$I_\nu = I_{\nu,0}\,\mathrm{e}^{-\tau_\nu} + \int_0^{\tau_\nu} S_\nu(\tau'_\nu)\,\mathrm{e}^{-(\tau_\nu - \tau'_\nu)}\,\mathrm{d}\tau'_\nu$$

where τ_ν is the total optical depth along the line of sight (i.e., through the cloud). The above is the formal solution, which, under the simplifying assumption that the **source function** is constant along the line of sight reduces to

$$I_\nu = I_{\nu,0}\,\mathrm{e}^{-\tau_\nu} + S_\nu\,\left(1 - \mathrm{e}^{-\tau_\nu}\right)$$

The first term expresses the attenuation of the background signal, the second term expresses the added signal due to the emission from the cloud, while the third term describes the cloud's self-absorption.

Using the above formal solution to the equation of radiative transfer, we have the following two extremes:

$$\begin{array}{lcl} \tau_\nu \gg 1 & \Rightarrow & I_\nu = S_\nu \\ \tau_\nu \ll 1 & \Rightarrow & I_\nu = I_{\nu,0}\,(1 - \tau_\nu) + S_\nu\,\tau_\nu \end{array}$$

where, for the latter case, we have used the Taylor series expansion for the exponential. In the high optical depth case, the observer just 'sees' the outer layers of the cloud, and therefore the observed intensity is simply the source function of the cloud (the observed signal contains no contribution from the background source). In the small optical depth limit, the contribution from the cloud is suppressed by a factor τ_ν, while that from the background source is attenuated by a factor $(1 - \tau_\nu)$.

To get some further insight into the source function and radiative transfer in general, consider form II of the radiate transfer equation. If $I_\nu > S_\nu$ then $\mathrm{d}I_\nu/\mathrm{d}\tau_\nu < 0$, so that the specific intensity *decreases* along the line of sight. If, on the other hand, $I_\nu < S_\nu$ then $\mathrm{d}I_\nu/\mathrm{d}\tau_\nu > 0$, indicating that the specific intensity *increases* along the line of sight. Hence, I_ν tends towards S_ν. If the optical depth of the cloud is sufficiently large than this 'tendency' will succeed, and $I_\nu = S_\nu$.

An important special case of the general solution derived above is if the cloud is in local thermal equilibrium (LTE). This is very often the case, since over the mean free path of the photons, every system will tend to be in LTE, unless it was recently disturbed and has yet been able to equilibrate. In the case of LTE, we have that, over a patch smaller than or equal to the mean free path of the photons, we have that $S_\nu \equiv j_\nu/\alpha_\nu = B_\nu(T)$, where T is the kinetic temperature (= radiation temperature) of the patch.

The solution to the **equation of radiative transfer** now is

$$I_\nu = I_{\nu,0}\,\mathrm{e}^{-\tau_\nu} + B_\nu(T)\,\left[1 - \mathrm{e}^{-\tau_\nu}\right]$$

Note that I_ν is **not** constant throughout the cloud, as was the case for a cloud in TE. In the case of LTE, however, there can be a non-zero gradient $\mathrm{d}I_\nu/\mathrm{d}r$.

Before we interpret this result in detail, it is important to distinguish

Blackbody Radiation: $I_\nu = B_\nu(T)$
Thermal Radiation: $S_\nu = B_\nu(T)$

`NOTE:` thermal radiation is radiation emitted by matter in thermal equilibrium.

Keeping this difference in mind, we now look at the solution to our equation of radiative transfer for a cloud in LTE at its two extremes:

$$\begin{array}{lcl} \tau_\nu \gg 1 & \Rightarrow & I_\nu = B_\nu(T) \\ \tau_\nu \ll 1 & \Rightarrow & I_\nu = I_{\nu,0}\,(1 - \tau_\nu) + B_\nu(T)\,\tau_\nu \end{array}$$

The former expresses that *an optically thick cloud in LTE emits black body radiation.* This is characterized by the fact that (i) if there is a background source, you can't see it, (ii) you can look into the source only for about one mean free path of the photons (which is much smaller than the size of the source), and (iii) the only information available to an observer is the temperature of the cloud (the observed intensity is a Planck curve of temperature T).

A good example of gas clouds in LTE are stars!

In the optically thin limit, the observed intensity depends on the background source (if present), and depends on *both* the temperature (sets source function) and density (sets optical depth) of the cloud (recall that $\tau_\nu \propto \kappa_\nu \, \rho \, l$).

In the case without background source we have that

$$I_\nu = \begin{cases} B_\nu(T) & \text{if } \tau_\nu \gg 1 \\ \tau_\nu \, B_\nu(T) & \text{if } \tau_\nu \ll 1 \end{cases}$$

Note that this is different from `case D`, in which we considered a cloud in TE without background source. In that case we obtained that $I_\nu = B_\nu(T)$ *independent of* τ_ν. In the case of LTE, however, there are radial gradients, which are responsible for diminishing the intensity by the optical depth in the case where $\tau_\nu \ll 1$. This may seem somewhat 'counter-intuitive', as it indicates that a cloud of larger optical depth is more intense!!!

Based on the above, we have that, in the case of a cloud in LTE without background source, $I_\nu \leq B_\nu(T)$, where T is the temperature of the cloud. If we express the intensity in terms of the **brightness temperature** we have that $T_{\mathrm{B},\nu} \leq T$. Hence, for a cloud in LTE without background source the observed brightness temperature is a lower limit on the kinetic temperature of the cloud.

What about scattering? In the most general case, any element in the cloud receives radiation coming from all 4π sterradian, and a certain fraction of that radiation will be scattered into the line-of-sight of an observer.

In general, the scattering can (will) be **non-isotropic** (e.g., Thomson scattering) and **incoherent** (e.g., Compton scattering or resonant scattering), and the final equation of radiative transfer can only be solved numerically.

In the simplified case of **isotropic, coherent** scattering the corresponding emission coeffient can be found by simply equating the power absorbed per unit volume to that emitted (for each frequency);

$$j_{\nu,\mathrm{scat}} = \alpha_{\nu,\mathrm{scat}}\, J_\nu$$

where $\alpha_{\nu,\mathrm{scat}}$ is the **absorption coefficient** of the scattering processes, while

$$J_\nu = \frac{1}{4\pi} \int I_\nu \,\mathrm{d}\Omega$$

is the **mean intensity**, averaged over all 4π sterradian.

The **source function** due to scattering is then simply

$$S_\nu \equiv \frac{j_{\nu,\mathrm{scat}}}{\alpha_{\nu,\mathrm{scat}}} = J_\nu = \frac{1}{4\pi} \int I_\nu \,\mathrm{d}\Omega$$

Hence, the source function due to **isotropic, coherent scattering** is simply the mean intensity.

The radiative transfer equation for pure scattering (no background source, and no emission) is

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}r} = -\alpha_{\nu,\mathrm{scat}}\, (I_\nu - J_\nu)$$

Even this oversimplified case of pure isotropic, coherent scattering is not easily solved. Since J_ν involves an integration (over all 4π sterradian), the above equation is an integro-differential equation, which are extremely difficult to solve in general; one typically has to resort to numerical methods (see `Rybicki & Lightmann 1979` for more details).

Observability of Emission & Absorption Lines: Consider a cloud in front of some background source. Assume the cloud is in LTE at temperature T. Assume that α_ν is only non-zero at a specific frequency, ν_1, corresponding to some electron transition. Given that resonant scattering is typically orders of magnitude more efficient than other scattering mechanisms, this is a reasonable approximation. The intensity observed is

$$I_\nu = I_{\nu,0}\, \mathrm{e}^{-\tau_\nu} + B_\nu(T)\, \left[1 - \mathrm{e}^{-\tau_\nu}\right]$$

At all frequencies other than ν_1 we have $\tau_\nu = 0$, and thus $I_\nu = I_{\nu,0}$. Now assume that the observer sees an **absorption line** at $\nu = \nu_1$. This implies that

$$I_{\nu_1} = I_{\nu_1,0}\,\mathrm{e}^{-\tau_{\nu_1}} + B_{\nu_1}(T)\,\left[1 - \mathrm{e}^{-\tau_{\nu_1}}\right] < I_{\nu_1,0}$$

while in the case of an **emission line**

$$I_{\nu_1} = I_{\nu_1,0}\,\mathrm{e}^{-\tau_{\nu_1}} + B_{\nu_1}(T)\,\left[1 - \mathrm{e}^{-\tau_{\nu_1}}\right] > I_{\nu_1,0}$$

Rearranging, we then have that

Absorption Line:	$B_\nu(T) < I_{\nu,0}$	$T < T_{\mathrm{B},\nu}$
Emission Line:	$B_\nu(T) > I_{\nu,0}$	$T > T_{\mathrm{B},\nu}$

where T is the (kinetic) temperature of the cloud, and $T_{\mathrm{B},\nu}$ is the **brightness temperature** of the background source, at the frequency of the line. Hence, if the cloud is colder (hotter) than the source, an absorption (emission) line will arise. In the case of no background source, we effectively have that $T_{\mathrm{B},\nu} = 0$, and the cloud will thus reveal an emission line. In the case where $T = T_{\mathrm{B},\nu}$ no line will be visible, independent of the optical depth of the cloud!