CHAPTER 2

Conservative Vector Fields & Integral Theorems

Line Integral of a Conservative Vector Field: Consider a curve γ running from location $\vec{x}_0$ to $\vec{x}_1$. Let $\mathrm{d}\vec{l}$ be the directional element of length along γ (i.e., with direction equal to that of the tangent vector to γ), then, for any scalar field $\Phi(\vec{x})$,

$$\int_{\vec{x}_0}^{\vec{x}_1} \nabla\Phi \cdot \mathrm{d}\vec{l} = \int_{\vec{x}_0}^{\vec{x}_1} \mathrm{d}\Phi = \Phi(\vec{x}_1) - \Phi(\vec{x}_0)$$

This implies that the line integral is independent of γ, and hence

$$\oint_c \nabla\Phi \cdot \mathrm{d}\vec{l} = 0$$

where c is a closed curve, and the integral is to be performed in the counter-clockwise direction.

Conservative Vector Fields:
A conservative vector field $\vec{F}$ has the following properties:

- $\vec{F}(\vec{x})$ is a gradient field, which means that there is a scalar field $\Phi(\vec{x})$ so that $\vec{F} = \nabla\Phi$
- Path independence: $\oint_c \vec{F} \cdot \mathrm{d}\vec{l} = 0$
- Irrotational = curl-free: $\nabla \times \vec{F} = 0$

Green's Theorem: Consider a 2D vector field $\vec{F} = F_x\hat{\imath} + F_y\hat{\jmath}$

$$\oint \vec{F} \cdot \mathrm{d}\vec{l} = \int\int_A \nabla \times \vec{F}\, \mathrm{d}A$$

$$\oint \vec{F} \cdot \hat{n}\, \mathrm{d}l = \int\int_A \nabla \cdot \vec{F}\, \mathrm{d}A$$

Gauss' Divergence Theorem: Consider a 3D vector field $\vec{F} = (F_x, F_y, F_z)$
If S is a closed surface bounding a region D with normal pointing outwards, and $\vec{F}$ is a vector field defined and differentiable over all of D, then

$$\int\int_S \vec{F} \cdot \mathrm{d}\vec{S} = \int\int\int_D \nabla \cdot \vec{F}\, \mathrm{d}V$$

Stokes' Curl Theorem: Consider a 3D vector field $\vec{F} = (F_x, F_y, F_z)$
If C is a closed curve, and S is *any* surface bounded by C, then

$$\oint_c \vec{F} \cdot \mathrm{d}\vec{l} = \int\int_S (\nabla \times \vec{F}) \cdot \hat{n}\, \mathrm{d}S$$

NOTE: The curve of the line intergral must have positive orientation, meaning that $\mathrm{d}\vec{l}$ points counterclockwise when the normal of the surface points towards the viewer.