CHAPTER 19

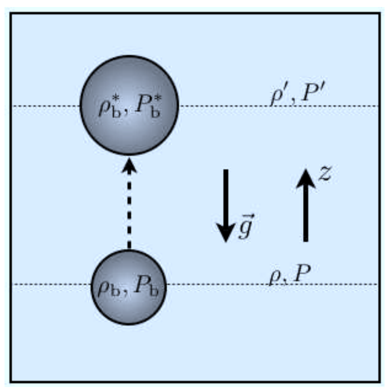
Fluid Instabilities

In this Chapter we discuss the following instabilities:

- convective instability (Schwarzschild criterion)
- interface instabilities (Rayleight Taylor & Kelvin-Helmholtz)
- gravitational instability (Jeans criterion)
- thermal instability (Field criterion)

Convective Instability: In astrophysics we often need to consider fluids heated from "below" (e.g., stars, Earth's atmosphere, where Sun heats surface, etc.)[1]. This results in a temperature gradient: hot at the base, colder further "up". Since warmer fluids are more buoyant ('lighter'), they like to be further up than colder ('heavier') fluids. The question we need to address is under what conditions this adverse temperature gradient becomes unstable, developing "overturning" motions known as thermal **convection**.

Consider a blob with density $\rho_{\rm b}$ and pressure $P_{\rm b}$ embedded in an ambient medium of density ρ and pressure P. Suppose the blob is displaced by a small distance δz upward. After the displacement the blob will have conditions $(\rho_{\rm b}^*, P_{\rm b}^*)$ and its new ambient medium is characterized by (ρ', P')c, where



$$\rho' = \rho + \frac{{\rm d}\rho}{{\rm d}z}\,\delta z \qquad\qquad P' = P + \frac{{\rm d}P}{{\rm d}z}\,\delta z$$

Initially the blob is assumed to be in **mechanical** and **thermal equilibrium** with its ambient medium, so that $\rho_{\rm b} = \rho$ and $P_{\rm b} = P$. After the displacement the blob needs to re-establish a new mechanical and thermal equilibrium. In general, the time scale on which it re-establishes mechanical

[1]Here and in what follows, 'up' refers to the direction opposite to that of gravity.

(pressure) equilibrium is the sound crossing time, $\tau_{\rm s}$, while re-establishing thermal equilibrium proceeds much slower, on the conduction time, $\tau_{\rm c}$. Given that $\tau_{\rm s} \ll \tau_{\rm c}$ we can assume that $P_{\rm b}^* = P'$, and treat the displacement as **adiabatic**. The latter implies that the process can be described by an adiabatic EoS: $P \propto \rho^\gamma$. Hence, we have that

$$\rho_{\rm b}^* = \rho_{\rm b} \left(\frac{P_{\rm b}^*}{P_{\rm b}}\right)^{1/\gamma} = \rho_{\rm b} \left(\frac{P'}{P}\right)^{1/\gamma} = \rho_{\rm b} \left[1 + \frac{1}{P}\frac{{\rm d}P}{{\rm d}z}\,\delta z\right]^{1/\gamma}$$

where the last step follows from eq. (1). In the limit of small displacements δz, we can use Taylor series expansion to show that, to first order,

$$\rho_{\rm b}^* = \rho + \frac{\rho}{\gamma\,P}\frac{{\rm d}P}{{\rm d}z}\,\delta z$$

where we have used that initially $\rho_{\rm b} = \rho$, and that the Taylor series expansion, $f(x) \simeq f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 + ...$, of $f(x) = [1+x]^{1/\gamma}$ is given by $f(x) \simeq 1 + \frac{1}{\gamma}x + ...$ Suppose we have a stratified medium in which ${\rm d}\rho/{\rm d}z < 0$ and ${\rm d}P/{\rm d}z < 0$. In that case, if $\rho_{\rm b}^* < \rho'$ then the displacement has made the blob more buoyant, resulting in **instability**. Hence, using that $\rho' = \rho + ({\rm d}\rho/{\rm d}z)\,\delta z$ we see that stability requires that

$$\frac{{\rm d}\rho}{{\rm d}z} < \frac{\rho}{\gamma\,P}\frac{{\rm d}P}{{\rm d}z}$$

This is called the **Schwarzschild criterion for convective stability**.

It is often convenient to rewrite this criterion in a form that contains the temperature. Using that

$$\rho = \rho(P,T) = \frac{\mu\, m_{\rm p}}{k_{\rm B}T}P$$

it is straightforward to show that

$$\frac{{\rm d}\rho}{{\rm d}z} = \frac{\rho}{P}\frac{{\rm d}P}{{\rm d}z} - \frac{\rho}{T}\frac{{\rm d}T}{{\rm d}z}$$

Substitution in $\rho' = \rho + ({\rm d}\rho/{\rm d}z)\,\delta z$ then yields that

$$\rho_{\rm b}^* - \rho' = \left[-(1-\frac{1}{\gamma})\frac{\rho}{P}\frac{{\rm d}P}{{\rm d}z} + \frac{\rho}{T}\frac{{\rm d}T}{{\rm d}z}\right]\,\delta z$$

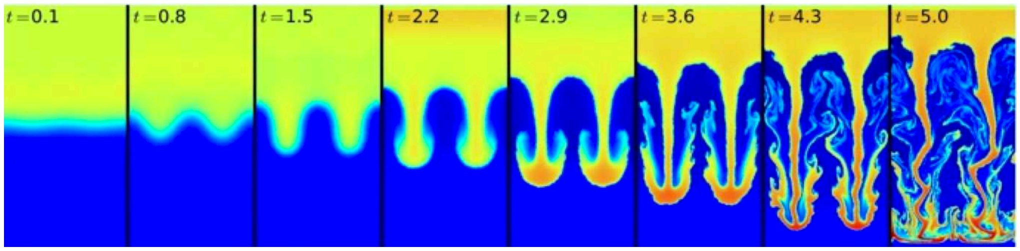


Figure 11: Example of Rayleigh-Taylor instability in a hydro-dynamical simulation.

Since stability requires that $\rho_{\rm b}^{*} - \rho' > 0$, and using that $\delta z > 0$, $\mathrm{d}P/\mathrm{d}z < 0$ and $\mathrm{d}T/\mathrm{d}z < 0$ we can rewrite the above Schwarzschild criterion for stability as

$$\left|\frac{\mathrm{d}T}{\mathrm{d}z}\right| < \left(1 - \frac{1}{\gamma}\right) \frac{T}{P} \left|\frac{\mathrm{d}P}{\mathrm{d}z}\right|$$

This shows that if the temperature gradient becomes too large the system becomes convectively unstable: blobs will rise up until they start to loose their thermal energy to the ambient medium, resulting in convective energy transport that tries to "overturn" the hot (high entropy) and cold (low entropy) material. In fact, without any proof we mention that in terms of the specific entropy, s, one can also write the Schwarzschild criterion for convective stability as $\mathrm{d}s/\mathrm{d}z > 0$.

Rayleigh-Taylor Instability: The Rayleigh-Taylor (RT) instability is an instability of an interface between two fluids of different densities that occurs when one of the fluids is accelerated into the other. Examples include supernova explosions in which expanding core gas is accelerated into denser shell gas and the common terrestrial example of a denser fluid such as water suspended above a lighter fluid such as oil in the Earth's gravitational field.

It is easy to see where the RT instability comes from. Consider a fluid of density ρ_2 sitting on top of a fluid of density $\rho_1 < \rho_2$ in a gravitational field that is pointing in the downward direction. Consider a small perturbation in which the initially horizontal interface takes on a small amplitude, sinusoidal deformation. Since this implies moving a certain volume of denser material down, and an equally large volume of the lighter material up, it is

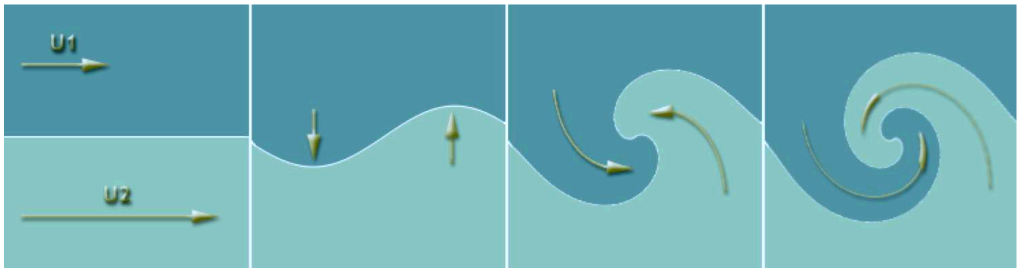


Figure 12: Illustration of onset of Kelvin-Helmholtz instability

immediately clear that the potential energy of this 'perturbed' configuration is lower than that of the initial state, and therefore energetically favorable. Simply put, the initial configuration is unstable to small deformations of the interface.

Stability analysis (i.e., perturbation analysis of the fluid equations) shows that the **dispersion relation** corresponding to the RT instability is given by

$$\omega = \pm i\, k \sqrt{\frac{g}{k}\,\frac{\rho_2 - \rho_1}{\rho_2 + \rho_1}}$$

where g is the gravitational acceleration, and the factor $(\rho_2 - \rho_1)/(\rho_2 + \rho_1)$ is called the **Atwood number**. Since the wavenumber of the perturbation $k > 0$ we see that ω is imaginary, which implies that the perturbations will grow exponentially (i.e., the system is unstable).

Kelvin-Helmholtz Instability: the Kelvin-Helmholtz (KH) instability is an interface instability that arises when two fluids with different densities have a velocity difference across their interface. Similar to the RT instability, the KH instability manifests itself as a small wavy pattern in the interface which develops into turbulence and which causes **mixing**. Examples where KH instability plays a role are wind blowing over water, (astrophysical) jets, the cloud bands on Jupiter (in particular the famous red spot), and clouds of denser gas falling through the hot, low density intra-cluster medium (ICM).

If **surface tension** is negligble than in principle any velocity difference across an interface is KH unstable. However, surface tension stabilizes the short wavelength modes so that typically **KH instability** kicks in above some velocity treshold.

Stability analysis (i.e., perturbation analysis of the fluid equations) shows that the **dispersion relation** corresponding to the KH instability is given by

$$\omega = \frac{(\rho_{\rm h}\,\rho_{\rm c})^{1/2}\,v}{\rho_{\rm h}+\rho_{\rm c}}\,k$$

where $\rho_{\rm h}$ and $\rho_{\rm c}$ are the densities of the hot and cold media respectively (with $\rho_{\rm h} < \rho_{\rm c}$), and v is the interface velocity.

Consider a cold cloud of radius $R_{\rm c}$ falling into a cluster. If the cloud started out at a large distance from the cluster with zero velocity, than at infall it has a velocity $v \sim v_{\rm esc} \sim c_{\rm s,h}$, where the latter is the sound speed of the hot ICM, which is assumed to be in hydrostatic equilibrium. Defining the cloud's overdensity $\delta = \rho_{\rm c}/\rho_{\rm h} - 1$, we can write the dispersion relation as

$$\omega = \frac{\rho_{\rm h}\,(\rho_{\rm c}/\rho_{\rm h})^{1/2}}{\rho_{\rm h}[1+(\rho_{\rm c}/\rho_{\rm h})]}\,c_{\rm s,h}\,k = \frac{(\delta+1)^{1/2}}{\delta+2}\,c_{\rm s,h}\,k$$

The mode that will destroy the cloud has $k \sim 1/R_{\rm c}$, so that the time-scale for cloud destruction is

$$\tau_{\rm KH} \simeq \frac{1}{\omega} \simeq \frac{R_{\rm c}}{c_{\rm s,h}}\frac{\delta+2}{(\delta+1)^{1/2}}$$

Assuming pressure equilibrium between cloud and ICM, and adopting the EoS of an ideal gas, implies that $\rho_{\rm h}\,T_{\rm h} = \rho_{\rm c}\,T_{\rm c}$, so that

$$\frac{c_{\rm s,h}}{c_{\rm s,c}} = \frac{T_{\rm h}^{1/2}}{T_{\rm c}^{1/2}} = \frac{\rho_{\rm c}^{1/2}}{\rho_{\rm h}^{1/2}} = (\delta+1)^{1/2}$$

Hence, one finds that **Kelvin-Helmholtz time** for cloud destruction is

$$\tau_{\rm KH} \simeq \frac{1}{\omega} \simeq \frac{R_{\rm c}}{c_{\rm s,c}}\,\frac{\delta+2}{\delta+1}$$

Note that $\tau_{\rm KH} \sim \zeta(R_{\rm c}/c_{\rm s,c}) = \zeta\tau_{\rm s}$, with $\zeta = 1(2)$ for $\delta \gg 1 (\ll 1)$. Hence, the **Kelvin-Helmholtz instability** will typically destroy clouds falling into a hot "atmosphere" on a time scale between one and two **sound crossing times**, $\tau_{\rm s}$, of the cloud. Note, though, that magnetic fields and/or radiative cooling at the interface may stabilize the clouds.

Gravitational Instability: In our discussion of sound waves we used perturbation analysis to derive a dispersion relation $\omega^2 = k^2\, c_{\rm s}^2$. In deriving that equation we ignored gravity by setting $\nabla\Phi = 0$ (see Chapter 17). If you do not ignore gravity, then you add one more perturbed quantity; $\Phi = \Phi_0 + \Phi_1$ and one more equation, namely the **Poisson equation** $\nabla^2\Phi = 4\pi G\rho$.
It is straightforward to show that this results in a modified **dispersion relation**:

$$\omega^2 = k^2\, c_{\rm s}^2 - 4\pi G\rho_0 = c_{\rm s}^2\, \left(k^2 - k_{\rm J}^2\right)$$

where we have introduced the **Jeans wavenumber**

$$k_{\rm J} = \frac{\sqrt{4\pi G\rho_0}}{c_{\rm s}}$$

to which we can also associate a **Jeans length**

$$\lambda_{\rm J} \equiv \frac{2\pi}{k_{\rm J}} = \sqrt{\frac{\pi}{G\rho_0}}\, c_{\rm s}$$

and a **Jeans mass**

$$M_{\rm J} = \frac{4}{3}\pi\rho_0 \left(\frac{\lambda_{\rm J}}{2}\right)^3 = \frac{\pi}{6}\,\rho_0\,\lambda_{\rm J}^3$$

From the dispersion relation one immediately sees that the system is **unstable** (i.e., ω is imaginary) if $k < k_{\rm J}$ (or, equivalently, $\lambda > \lambda_{\rm J}$ or $M > M_{\rm J}$). This is called the **Jeans criterion for gravitational instability**. It expresses when pressure forces (which try to disperse matter) are no longer able to overcome gravity (which tries to make matter collapse), resulting in exponential gravitational collapse on a time scale

$$\tau_{\rm ff} = \sqrt{\frac{3\,\pi}{32\,G\,\rho}}$$

known as the **free-fall time** for gravitational collapse.

The Jeans stability criterion is of utmost importance in astrophysics. It is used to describes the formation of galaxies and large scale structure in an expanding space-time (in this case the growth-rate is not exponential, but only power-law), to describe the formation of stars in molecular clouds within galaxies, and it may even play an important role in the formation of planets in protoplanetary disks.

In deriving the Jeans Stability criterion you will encounter a somewhat puzzling issue. Consider the **Poisson equation** for the unperturbed medium (which has density ρ_0 and gravitational potential Φ_0):

$$\nabla^2\Phi_0 = 4\pi G\rho_0$$

Since the initial, unperturbed medium is supposed to be homogeneous there can be no gravitational force; hence $\nabla\Phi_0 = 0$ everywhere. The above Poisson equation then implies that $\rho_0 = 0$. In other words, an unperturbed, homogeneous density field of non-zero density does not seem to exist. Sir James Jeans 'ignored' this 'nuisance' in his derivation, which has since become known as the **Jeans swindle**. The problem arises because Newtonian physics is not equipped to deal with systems of inifinite extent (a requirement for a perfectly homogeneous density distribution). See Kiessling (1999; arXiv:9910247) for a detailed discussion, including an elegant demonstration that the Jeans swindle is actually vindicated!

Thermal Instability: Let $\mathcal{L} = \mathcal{L}(\rho, T) = \mathcal{C} - \mathcal{H}$ be the net cooling rate. If $\mathcal{L} = 0$ the system is said to be in **thermal equilibrium** (TE), while $\mathcal{L} > 0$ and $\mathcal{L} < 0$ correspond to cooling and heating, respectively.

The condition $\mathcal{L}(\rho, T) = 0$ corresponds to a curve in the (ρ, T)-plane with a shape similar to that shown in Fig. 11. It has flat parts at $T \sim 10^6$K, at $T \sim 10^4$K, at $T \sim 10 - 100$K. This can be understood from simple atomic physics, and will be discussed in some detail in the lectures on radiative processes (see § 8.5.1 of Mo, van den Bosch & White, 2010 for a detailed discussion). Above the TE curve we have that $\mathcal{L} > 0$ (net cooling), while below it $\mathcal{L} < 0$ (net heating). The dotted curve indicates a line of constant pressure ($T \propto \rho^{-1}$). Consider a blob in thermal and mechanical (pressure) equilibrium with its ambient medium, and with a pressure indicated by the

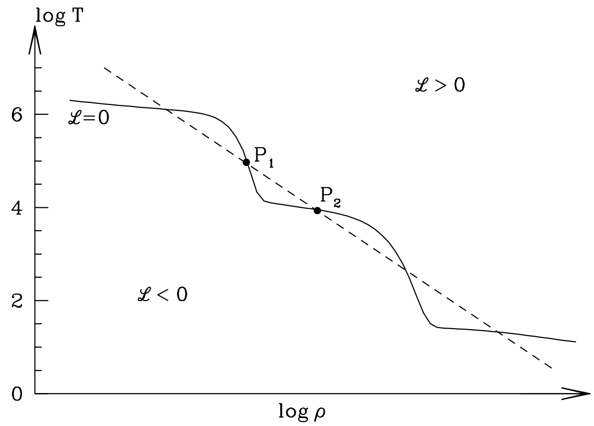


Figure 13: The locus of thermal equilibrium ($\mathcal{L} = 0$) in the (ρ, T) plane, illustrating the principle of thermal instability. The dashed line indicates a line of constant pressure.

dashed line. There are five possible solutions for the density and temperature of the blob, two of which are indicated by P_1 and P_2; here confusingly the P refers to 'point' rather than 'pressure'. Suppose I have a blob located at point P_2. If I heat the blob, displacing it from TE along the constant pressure curve (i.e., the blob is assumed small enough that the sound crossing time, on which the blob re-established mechanical equilibrium, is short). The blob now finds itself in the region where $\mathcal{L} > 0$ (i.e, net cooling), so that it will cool back to its original location on the TE-curve; the blob is **stable**. For similar reasons, it is easy to see that a blob located at point P_1 is **unstable**. This instability is called **thermal instability**, and it explains why the ISM is a **three-phase** medium, with gas of three different temperatures ($T \sim 10^6\,\mathrm{K}, 10^4\,\mathrm{K}$, and $\sim 10 - 100\,\mathrm{K}$) coexisting in pressure equilibrium. Gas at any other temperature but in pressure equilibrium is thermall unstable.

It is easy to see that the requirement for **thermal instability** translates into

$$\left(\frac{\partial \mathcal{L}}{\partial T}\right)_P < 0$$

which is known as the **Field criterion for thermal instability** (after American astrophysicist George B. Field).