CHAPTER 18

Shocks

When discussing sound waves (see Chapter 17), we considered small (linear) perturbations. In this Chapter we consider the case in which the perturbations are large (non-linear). Typically, a large disturbance results in an abrupt **discontinuity** in the fluid, called a **shock**. Note: not all discontinuities are shocks, but all shocks are discontinuities.

Mach Number: if v is the flow speed of the fluid, and $c_{\rm s}$ is the sound speed, then the Mach number of the flow is defined as

$$\mathcal{M} = \frac{v}{c_{\rm s}}$$

Note: *simply accelerating a flow to supersonic speeds does* **not** *necessarily generate a shock*. Shocks only arise when an obstruction in the flow causes a deceleration of fluid moving at supersonic speeds. The reason is that disturbances cannot propagate upstream, so that the flow cannot 'adjust itself' to the obstacle because there is no way of propagating a signal (which always goes at the sound speed) in the upstream direction. Consequently, the flow remains undisturbed until it hits the obstacle, resulting in a discontinuous change in flow properties; a shock.

Structure of a Shock: Fig. 10 shows the structure of a planar shock. The shock has a finite, non-zero width (typically a few mean-free paths of the fluid particles), and separates the 'up-stream', pre-shocked gas, from the 'down-stream', shocked gas.

For reasons that will become clear in what follows, it is useful to split the downstream region in two sub-regions; one in which the fluid is out of thermal equilibrium, with net cooling $\mathcal{L} > 0$, and, further away from the shock, a region where the downstream gas is (once again) in thermal equilibrium (i.e.,

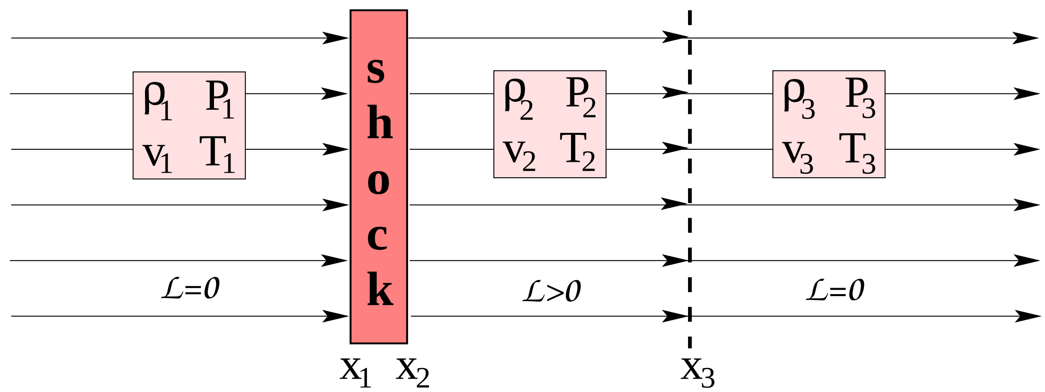

Figure 10: Structure of a planar shock.

$\mathcal{L} = 0$). If the transition between these two sub-regions falls well outside the shock (i.e., if $x_3 \gg x_2$) the shock is said to be **adiabatic**. In that case, we can derive a relation between the upstream (pre-shocked) properties (ρ_1, P_1, T_1, u_1) and the downstream (post-shocked) properties (ρ_2, P_2, T_2, u_2); these relations are called the Rankine-Hugoniot jump conditions. Linking the properties in region three (ρ_3, P_3, T_3, u_3) to those in the pre-shocked gas is in general not possible, except in the case where $T_3 = T_1$. In this case one may consider the shock to be **isothermal**.

Rankine-Hugoniot jump conditions: We now derive the relations between the up- and down-stream quantities, under the assumption that the shock is adiabatic.

Consider a rectangular volume V that encloses part of the shock; it has a thickness $\mathrm{d}x > (x_2 - x_1)$ and is centered in the x-direction on the middle of shock. At fixed x the volume is bounded by an area A. If we ignore variations in ρ and $\vec{u}$ in the y- and z-directions, the **continuity equation** becomes

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho\, u_x) = 0$$

If we itegrate this equation over our volume V we obtain

$$\begin{aligned} &\int\int\int \frac{\partial \rho}{\partial t}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z + \int\int\int \frac{\partial}{\partial x}(\rho u_x)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = 0 \\ \Leftrightarrow\quad & A\int \frac{\partial \rho}{\partial t}\,\mathrm{d}x + A\int \frac{\partial}{\partial x}(\rho u_x)\,\mathrm{d}x = 0 \\ \Leftrightarrow\quad & \frac{\partial}{\partial t}\int \rho\,\mathrm{d}x + \int \mathrm{d}(\rho u_x) = 0 \end{aligned}$$

Since $\frac{\partial}{\partial t}\int \rho\,\mathrm{d}V = 0$ (there is no mass accumulation in the shock), we have that

$$\rho u_x|_{+\mathrm{d}x/2} = \rho u_x|_{-\mathrm{d}x/2}$$

In terms of the upstream (index 1) and downstream (index 2) quantities:

$$\boxed{\rho_1\,u_1 = \rho_2\,u_2}$$

This equation describes **mass conservation** across shock.

The momentum equation in the x-direction, ignoring viscosity, is given by

$$\frac{\partial}{\partial t}(\rho\,u_x) = -\frac{\partial}{\partial x}(\rho\,u_x\,u_x + P) - \rho\frac{\partial \Phi}{\partial x}$$

Integrating this equation over V and ignoring any gradient in Φ across the shock, we obtain

$$\boxed{\rho_1\,u_1^2 + P_1 = \rho_2\,u_2^2 + P_2}$$

This equation describes how the shock **converts ram pressure into thermal pressure**.

Finally, applying the same to the **energy equation** under the assumption that the shock is adiabatic (i.e., $\mathrm{d}Q/\mathrm{d}t = 0$), one finds that $(E + P)u$ has to be the same on both sides of the shock, i.e.,

$$\left[\frac{1}{2}u^2 + \Phi + \varepsilon + \frac{P}{\rho}\right]\rho\,u = \text{constant}$$

We have already seen that $\rho\, u$ is constant. Hence, if we once more ignore gradients in Φ across the shock, we obtain that

$$\boxed{\tfrac{1}{2}u_1^2 + \varepsilon_1 + P_1/\rho_1 = \tfrac{1}{2}u_2^2 + \varepsilon_2 + P_2/\rho_2}$$

This equation describes how the shock **converts kinetic energy into enthalpy**. Qualitatively, a shock converts an ordered flow upstream into a disordered (hot) flow downstream.

The three equations in the rectangular boxes are known as the **Rankine-Hugoniot (RH) jump conditions for an adiabatic shock**. Using straightforward but tedious algebra, these RH jump conditions can be written in a more useful form using the **Mach number** $\mathcal{M}_1$ of the upstream gas:

$$\begin{aligned}
\frac{\rho_2}{\rho_1} &= \frac{u_1}{u_2} = \left[\frac{1}{\mathcal{M}_1^2} + \frac{\gamma-1}{\gamma+1}\left(1 - \frac{1}{\mathcal{M}_1^2}\right)\right]^{-1} \\
\frac{P_2}{P_1} &= \frac{2\gamma}{\gamma+1}\,\mathcal{M}_1^2 - \frac{\gamma-1}{\gamma+1} \\
\frac{T_2}{T_1} &= \frac{P_2\,\rho_2}{P_1\,\rho_1} = \frac{\gamma-1}{\gamma+1}\left[\frac{2}{\gamma+1}\left(\gamma\mathcal{M}_1^2 - \frac{1}{\mathcal{M}_1^2}\right) + \frac{4\gamma}{\gamma-1} - \frac{\gamma-1}{\gamma+1}\right]
\end{aligned}$$

Here we have used that for an **ideal gas**

$$P = (\gamma-1)\,\rho\,\varepsilon = \frac{k_{\rm B}\,T}{\mu\,m_{\rm p}}\,\rho$$

Given that $\mathcal{M}_1 > 1$, we see that $\rho_2 > \rho_1$ (shocks **compress**), $u_2 < u_1$ (shocks **decelerate**), $P_2 > P_1$ (shocks **increase pressure**), and $T_2 > T_1$ (shocks **heat**). The latter may seem surprising, given that the shock is considered to be **adiabatic**: although the process has been adiabatic, in that ${\rm d}Q/{\rm d}t = 0$, the gas <u>has</u> changed its adiabat; its entropy has increased as a consequence of the shock converting kinetic energy into thermal, internal energy. In general, in the presence of **viscosity**, a change that is adiabatic does not imply that the states before and after are simply linked by the relation $P = K\,\rho^\gamma$, with K some constant. Shocks are always viscous, which causes K to change across the shock, such that the entropy increases; it is this aspect of the shock that causes **irreversibility**, thus defining an "arrow of time".

Back to the RH jump conditions: in the limit $\mathcal{M}_1 \gg 1$ we have that

$$\rho_2 = \frac{\gamma + 1}{\gamma - 1}\,\rho_1 = 4\,\rho_1$$

where we have used that $\gamma = 5/3$ for a monoatomic gas. Thus, with an **adiabatic shock** you can achieve a maximum compression in density of a factor four! Physically, the reason why there is a maximal compression is that the pressure and temperature of the downstream fluid diverge as $\mathcal{M}_1^2$. This huge increase in downstream pressure inhibits the amount of compression of the downstream gas. However, this is only true under the assumption that the shock is **adiabtic**. The downstream, post-shocked gas is out of thermal equilibrium, and in general will be cooling (i.e., $\mathcal{L} > 0$). At a certain distance past the shock (i.e., when $x = x_3$ in Fig. 10), the fluid will re-establish **thermal equilibrium** (i.e., $\mathcal{L} = 0$). In some special cases, one can obtain the properties of the fluid in the new equilibrium state; one such case is the example of an **isothermal shock**, for which the downstream gas has the same temperature as the upstream gas (i.e., $T_3 = T_1$).

In the case of an **isothermal shock**, the first two **Rankine-Hugoniot jump conditions** are still valid, i.e.,

$$\begin{aligned}
\rho_1\,u_1 &= \rho_3\,u_3 \\
\rho_1\,u_1^2 + P_1 &= \rho_3\,u_3^2 + P_3
\end{aligned}$$

However, the third condition, which derives from the energy equation, is no longer valid. After all, in deriving that one we had assumed that the shock was adiabatic. In the case of an isothermal shock we have to replace the third RH jump condition with $T_1 = T_3$. The latter implies that $c_{\rm s}^2 = P_3/\rho_3 = P_1/\rho_1$, and allows us to rewrite the second RH condition as

$$
\begin{aligned}
& & \rho_1(u_1^2 + c_\mathrm{s}^2) &= \rho_3(u_3^2 + c_\mathrm{s}^2) \\
&\Leftrightarrow & u_1^2 - \tfrac{\rho_3}{\rho_1} u_3^2 &= \tfrac{\rho_3}{\rho_1} c_\mathrm{s}^2 - c_\mathrm{s}^2 \\
&\Leftrightarrow & u_1^2 - u_1 u_3 &= \left(\tfrac{u_1}{u_3} - 1\right) c_\mathrm{s}^2 \\
&\Leftrightarrow & u_1 u_3 (u_1 - u_3) &= (u_1 - u_3)\, c_\mathrm{s}^2 \\
&\Leftrightarrow & c_\mathrm{s}^2 &= u_1 u_3
\end{aligned}
$$

Here the second step follows from using the first RH jump condition. If we now substitute this result back into the first RH jump condition we obtain that

$$\frac{\rho_3}{\rho_1} = \frac{u_1}{u_3} = \left(\frac{u_1}{c_\mathrm{s}}\right)^2 = \mathcal{M}_1^2$$

Hence, in the case of **isothermal shock** (or an adiabatic shock, but sufficiently far behind the shock in the downstream fluid), we have that there is no restriction to how much compression the shock can achieve; depending on the Mach number of the shock, the compression can be huge.